Convergence rate analysis of a Dykstra-type projection algorithm

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(Joint work with Xiaozhou Wang)

Motivating applications

- Nearest correlation matrices: (Higham '02)

$$\underset{x \in S^n}{\text{Minimize}} \ \|x - \bar{v}\|_F^2 \ \text{ subject to } \ x \in \cap_{i=1}^3 C_i,$$

where $\bar{v} \in S^n$, $C_1 = S^n_+$, $C_2 = \{x \in S^n : \ L_{ij} \le x_{ij} \le U_{ij} \ \forall i, j\}$ and

$$C_3 := \{x \in S^n : \ x_{ii} = 1 \ \forall i\}.$$

- System identification: (Liu et al. '20)

$$\underset{x_1,\ldots,x_m \in \mathbb{R}^n}{\text{Minimize}} \ \sum_{i=1}^m \|x_i - \bar{v}_i\|^2 \ \text{ subject to } \ x := (x_1, \ldots, x_m) \in D_1 \cap D_2,$$

where $\bar{v}$ is given, and

$$D_1 := \{x \in \mathbb{R}^{mn} : \|\mathcal{H}_{r_{i+1}}(x_i)\|_* \le k_i, i = 1, \ldots, m\},$$
$$D_2 := \{x \in \mathbb{R}^{mn} : \|[\mathcal{H}_{r+1}(x_1) \ \cdots \ \mathcal{H}_{r+1}(x_m)]\|_* \le k\},$$

with $\mathcal{H}_s$ being a linear map that returns a suitable Hankel matrix, $\|\cdot\|_*$ is the nuclear norm, $r_i, \ r \in \mathbb{N}$, and $k_i, \ k > 0$.

Best approximation problems

Consider the following best approximation problem:

$$\operatorname*{Minimize}_{x\in\mathbb{R}^n} \ \frac{1}{2}\|x-\bar{v}\|^2 \ \text{ subject to } \ x\in\bigcap_{i=1}^{\ell} A_i^{-1}C_i,$$

where

- $A_i : \mathbb{R}^n \to \mathbb{R}^{m_i}$ is linear and nonzero;
- each $C_i \subseteq \mathbb{R}^{m_i}$ is closed and convex;
- projection onto C_i can be computed more efficiently than projection onto $A_i^{-1}C_i$;
- For technical succinctness:

$$\bigcap_{i=1}^{\ell} A_i^{-1}C_i \neq \emptyset.$$

When $A_i = I$

Dykstra's projection algorithm: (Boyle, Dykstra '86, Han '88)

Set $x_\ell^0 = \bar{v}$, $y_1^0 = \cdots = y_\ell^0 = 0$. For each $t \geq 0$, set $x_0^{t+1} = x_\ell^t$ and compute

$$x_i^{t+1} = \mathrm{Proj}_{C_i}(y_i^t + x_{i-1}^{t+1}), \quad y_i^{t+1} = y_i^t + x_{i-1}^{t+1} - x_i^{t+1} \text{ for } i = 1, \ldots, \ell.$$

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Known facts:

- $x_\ell^t =: x^t \to \text{Proj}_{\cap_{i=1}^{\ell} C_i}(\bar{v})$.
- Reduces to cyclic projection when each C_i is affine. (Gaffke, Mathar '89).
- Equivalent to CGD applied to

$$\underset{y_1,\ldots,y_\ell}{\text{Minimize}} \ \frac{1}{2}\left\|\sum_{i=1}^{\ell} y_i - \bar{v}\right\|^2 - \frac{1}{2}\|\bar{v}\|^2 + \sum_{i=1}^{\ell} \sigma_{C_i}(y_i)$$

starting from $y_1^0 = \cdots = y_\ell^0 = 0$.

When $A_i = I$ cont.

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Known facts cont.:

- Local linear convergence of $\{x_\ell^t\}$ and all $\{y_i^t\}$ when each C_i is polyhedral. (Luo, Tseng '93)
- Convergence rate unknown for general C_i.

When $A_i = I$ cont.

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Outline:

- Develop Dykstra-type projection algorithm when $A_i \neq I$.
- Identify a class of sets C_i for convergence rate analysis.
- Explicit error bounds and convergence rate.

A Dykstra-type algorithm

Dykstra-type projection algorithm:
Set $y_i^0 = 0 \in \mathbb{R}^{m_i}$ and $\gamma_i := \lambda_{\max}(A_i^T A_i)$ for all i, $x_\ell^0 = \bar{v} \in \mathbb{R}^n$.
For each $t \geq 0$, set $x_0^{t+1} = x_\ell^t$ and $x^t = x_\ell^t$. Compute, for $i = 1, \ldots, \ell$,

$$x_i^{t+1} = (I - \gamma_i^{-1} A_i^T A_i) x_{i-1}^{t+1} + \gamma_i^{-1} A_i^T \mathrm{Proj}_{C_i}(\gamma_i y_i^t + A_i x_{i-1}^{t+1}),$$
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For notational simplicity, write $\boldsymbol{y}^t := (y_1^t, \ldots, y_\ell^t)$ for all t.

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- The algorithm is equivalent to a proximal CGD applied to

$$\underset{y_1,\ldots,y_\ell}{\text{Minimize}} \;\; d(\boldsymbol{y}) := \frac{1}{2}\left\| \sum_{i=1}^{\ell} A_i^T y_i - \bar{v} \right\|^2 - \frac{1}{2}\|\bar{v}\|^2 + \sum_{i=1}^{\ell} \sigma_{C_i}(y_i)$$

starting from $y_i^0 = 0$ for all i.

A Dykstra-type algorithm cont.

Key facts:

- It holds that

$$-\inf_{\boldsymbol{y}} d(\boldsymbol{y}) = \inf_{x} \left\{ \frac{1}{2}\|x - \bar{v}\|^2 : \; x \in \cap_{i=1}^{\ell} A_i^{-1} C_i \right\}.$$

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- $\|\boldsymbol{y}^{t+1} - \boldsymbol{y}^t\| \to 0$ and $\operatorname{dist}(\mathbf{0}, \partial d(\boldsymbol{y}^t)) \to 0$.
- Every accumulation point of $\{\boldsymbol{y}^t\}$ minimizes d.

A Dykstra-type algorithm cont.

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- If $\boldsymbol{y}^* \in \operatorname{Arg\,min} d$, then $\mathrm{Proj}_{\cap_{i=1}^{\ell} A_i^{-1} C_i}(\bar{v}) = \bar{v} - \sum_{i=1}^{\ell} A_i^T y_i^*$.

A Dykstra-type algorithm cont.

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- (Auslender, Cominetti, Crouziex '93) Suppose that $\cap_{i=1}^{\ell} A_i^{-1} \operatorname{ri} C_i \neq \emptyset$. Then $\operatorname{Arg\,min} d = E_1 + E_2 \neq \emptyset$, where E_1 is compact and E_2 is a subspace. Moreover, we have
$$d(\boldsymbol{y}^t) \to \inf d, \qquad \operatorname{dist}(\boldsymbol{y}^t, \operatorname{Arg\,min} d) \to 0,$$
and $d(\boldsymbol{y} + \boldsymbol{u}) = d(\boldsymbol{y})$ for all $\boldsymbol{y} \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_\ell}$ and $\boldsymbol{u} \in E_2$.

$C^{1,\alpha}$-cone reducibility

We use the typeface $\mathbb{Z}$ to denote a finite dimensional Hilbert space.

Definition: Let $\alpha \in (0, 1]$. A closed set $\Omega \subseteq \mathbb{X}$ is said to be $C^{1,\alpha}$-cone reducible at $\hat{x} \in \Omega$ if $\exists\ \rho > 0$, a mapping $\Xi : \mathbb{X} \to \mathbb{Y}$ that satisfies $\Xi(\hat{x}) = 0$ and is $C^{1,\alpha}$ in $B(\hat{x}, \rho)$ with $D\Xi(\hat{x})$ being surjective, and a closed convex pointed cone $K \subseteq \mathbb{Y}$ such that

$$\Omega \cap B(\hat{x}, \rho) = \{x : \Xi(x) \in K\} \cap B(\hat{x}, \rho).$$

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- A closed convex pointed cone is $C^{1,1}$-cone reducible at the origin.

$C^{1,\alpha}$-cone reducibility cont.

Examples cont.:

- Let C be closed and convex, and B be the unit ball, then for any $\epsilon > 0$, the set $C + \epsilon B$ is $C^{1,1}$-cone reducible.

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$$C + \epsilon B = \{x : \text{dist}(x, C)^2 - \epsilon^2 \leq 0\};$$

  for boundary points, take $K = \mathbb{R}_-$ and $\Xi(x) := \text{dist}(x, C)^2 - \epsilon^2$.

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- Let $p \in (1, \infty)$ and let $C = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : \ \|x\|_p \leq r\}$. Then C is $C^{1,\alpha}$-cone reducible with $\alpha = \min\{1, p - 1\}$.

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$$\|x\|_p^{1-p} \cdot \begin{bmatrix} \operatorname{sgn}(x_1)|x_1|^{p-1} & \cdots & \operatorname{sgn}(x_n)|x_n|^{p-1} & -\|x\|_p^{p-1} \end{bmatrix}^T,$$

  which is nonzero at any nonzero boundary points, and is $\min\{1, p-1\}$- Hölder continuous.

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  which is nonzero at any nonzero boundary points, and is $\min\{1, p - 1\}$- Hölder continuous.
- Let $p \in (1, \infty)$ and let $C = \{x \in \mathbb{R}^n : \ \|x\|_p \leq 1\}$. Then C is $C^{1,\alpha}$-cone reducible with $\alpha = \min\{1, p - 1\}$.

Error bound

Theorem 1. (Wang, P. '22)
For the best approximation problem

$$\underset{x\in\mathbb{R}^n}{\text{Minimize}}\ 0.5\|x-\bar{v}\|^2 \ \text{ subject to } \ x\in\cap_{i=1}^{\ell} A_i^{-1}C_i,$$

suppose that

(i) Each C_i is $C^{1,\alpha}$-cone reducible with $\alpha\in(0,1]$, closed & convex;

(ii) $\cap_{i=1}^{\ell} A_i^{-1}\operatorname{ri} C_i \neq \emptyset$;

(iii) $0\in x^*-\bar{v}+\operatorname{ri}\partial(\sum_{i=1}^{\ell}\delta_{A_i^{-1}C_i})(x^*)$, where $x^*=\operatorname{Proj}_{\cap_{i=1}^{\ell}A_i^{-1}C_i}(\bar{v})$.

Then there exist $\epsilon>0$ and $c>0$ such that

$$\operatorname{dist}(\boldsymbol{y},\operatorname{Arg\,min} d)\le c\,(d(\boldsymbol{y})-\inf d)^{1-\frac{1}{1+\alpha}}$$

whenever $\boldsymbol{y}$ satisfies $\operatorname{dist}(\boldsymbol{y},\operatorname{Arg\,min} d)\le\epsilon$ & $\inf d\le d(\boldsymbol{y})\le\inf d+\epsilon$.

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whenever $\boldsymbol{y}$ satisfies $\operatorname{dist}(\boldsymbol{y},\operatorname{Arg\,min}d)\leq\epsilon$ & $\inf d\leq d(\boldsymbol{y})\leq\inf d+\epsilon$.

Note: The exponent is the BEST possible.

Convergence rate

Theorem 2. (Wang, P. '22)
For the best approximation problem, suppose that

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Let $\{x^t\}$ and $\{\boldsymbol{y}^t\}$ be generated by the Dykstra-type algorithm. Then there exist $c > 0$ and $\bar{t} \in \mathbb{N}$ such that

- If $\alpha = 1$, then $\exists \boldsymbol{y}^* \in \mathrm{Arg\,min}\, d$, $\tau \in (0,1)$ such that for any $t \geq \bar{t}$,

$$\|x^t - x^*\| \leq c\tau^t, \quad \|\boldsymbol{y}^t - \boldsymbol{y}^*\| \leq c\tau^t.$$

- If $\alpha \in (0,1)$, then for any $t \geq \bar{t}$,

$$\|x^t - x^*\| \leq ct^{-\frac{\alpha^2}{2(1-\alpha)}}, \; d(\boldsymbol{y}^t) - \inf d \leq ct^{-\frac{\alpha}{1-\alpha}}, \; \mathrm{dist}(\boldsymbol{y}^t, \mathrm{Arg\,min}\, d) \leq ct^{-\frac{\alpha^2}{2(1-\alpha)}}.$$

Conclusion

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- Explicit convergence rate of a Dykstra-type method is deduced for $C^{1,\alpha}$-cone reducible sets, $\alpha \in (0, 1]$.
- An example was constructed to illustrate the tightness of an essential error bound condition.

Reference:

- X. Wang and T. K. Pong.
  Convergence rate analysis of a Dykstra-type projection algorithm.
  Preprint. Available at https://arxiv.org/abs/2301.03026.

Thanks for coming! :)