Splitting methods for nonconvex feasibility problems

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Feasibility Problem

- Given closed sets D_i, $i = 1, \ldots, m$, find a point

$$x \in \bigcap_{i=1}^{m} D_i.$$

- Example: Finding a solution of $Ax = b$ with $\|x\|_0 \leq r$.

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- Example: Finding a solution of $Ax = b$ with $\|x\|_0 \leq r$.
- The general problem can be reformulated as finding a point in

$$\{(x_1, \ldots, x_m) : \; x_1 = \cdots = x_m\} \cap (D_1 \times D_2 \times \cdots \times D_m).$$

- Only need to consider the intersection of a closed *convex* set C and a closed set D.

When D is convex

- Alternating projection:

$$x^{t+1} = P_D(P_C(x^t)).$$

- Splitting methods ($0 < \alpha \leq 2$):

$$\begin{cases} y^{t+1} = \arg\min_{y \in C} \left\{ \|y - x^t\| \right\}, \\ z^{t+1} = \arg\min_{z \in D} \left\{ \|2y^{t+1} - x^t - z\| \right\}, \\ x^{t+1} = x^t + \alpha(z^{t+1} - y^{t+1}). \end{cases}$$

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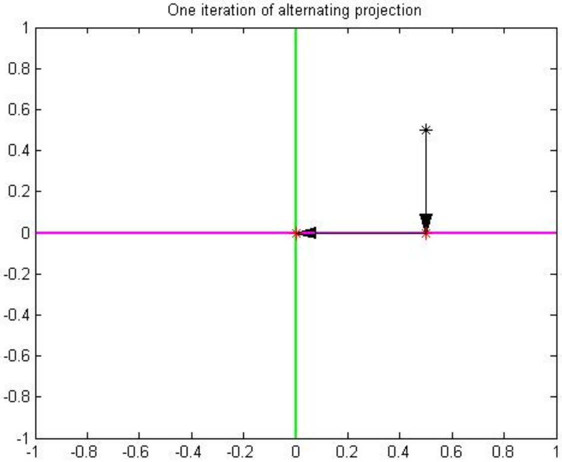
- Douglas-Rachford (DR): $\alpha = 1$.
- Peaceman-Rachford (PR): $\alpha = 2$.

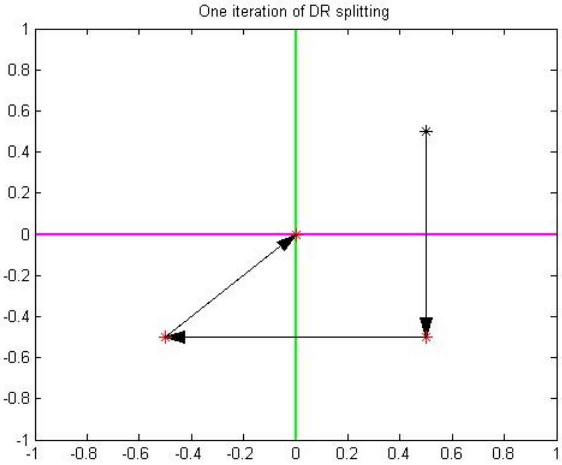
Behavior in convex case: AP

Finding the intersection of the axes, starting from $(0.5, 0.5)$.



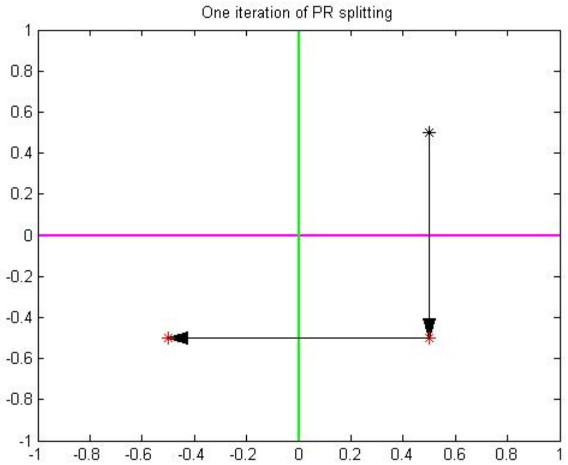
Behavior in convex case: DR

Showing the x-iterates: average after two successive reflections.
$P_C(x^t)$ will converge to the intersection.



Behavior in convex case: PR

Showing the x-iterate; not convergent.



When D is nonconvex

For the convergence of DR splitting:

- Mainly local convergence results.
- Require various regularity conditions on the sets.
- Local convergence for finding intersection of $Ax = b$ and $\|x\|_0 \leq r$. (Hesse, Luke, Neumann '13).

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- Mainly local convergence results.
- Require various regularity conditions on the sets.
- Local convergence for finding intersection of $Ax = b$ and $\|x\|_0 \leq r$. (Hesse, Luke, Neumann '13).
- Global convergence shown for the intersection of a circle and a hyperplane/line. (Artacho, Borwein '12, Benoist '15)
- Global convergence shown for the intersection of a halfspace and a compact set. (Artacho, Borwein, Tam '15)

Our DR splitting

- DR splitting: ($\gamma > 0$)

$$\begin{cases} y^{t+1} = \arg\min_{y} \left\{ \frac{1}{2} d_C^2(y) + \frac{1}{2\gamma}\|y - x^t\|^2 \right\}, \\ z^{t+1} \in \operatorname{Arg}\min_{z \in D} \left\{ \|2y^{t+1} - x^t - z\|^2 \right\}, \\ x^{t+1} = x^t + (z^{t+1} - y^{t+1}). \end{cases}$$

- The y-update is $\frac{1}{1+\gamma}(x^t + \gamma P_C(x^t))$.

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- DR splitting applied to minimizing $\frac{1}{2} d_C^2 + \delta_D$.

DR Convergence result I

Fact 1 (Li, P '15): [Global convergence]

Suppose that $0 < \gamma < \sqrt{\frac{3}{2}} - 1$, and either C or D is compact.

Then the sequence $\{(y^t, z^t, x^t)\}$ generated from DR splitting is bounded, and any cluster point (y^*, z^*, x^*) satisfies $z^* = y^*$. Moreover, y^* is a stationary point of

$$\min_{u \in D} \frac{1}{2} d_C^2(u),$$

i.e., $0 \in y^* - P_C(y^*) + N_D(y^*)$.

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i.e., $0 \in y^* - P_C(y^*) + N_D(y^*)$.

- Clearly, if $d_C(y^*) = 0$, then y^* solves the feasibility problem.

DR Convergence result II

Fact 2 (Li, P '15): [Convergence of the whole sequence]
Suppose that $0 < \gamma < \sqrt{\frac{3}{2}} - 1$, C and D are semi-algebraic, and one of them is compact.

Then the sequence $\{(y^t, z^t, x^t)\}$ generated from DR splitting is bounded, and is convergent to some (y^*, z^*, x^*) satisfying $z^* = y^*$, with y^* being a stationary point of the problem $\min_{u \in D} \frac{1}{2} d_C^2(u)$. Furthermore,

$$\sum_{t=1}^{\infty} \|y^{t+1} - y^t\| < \infty.$$

DR Convergence result III

Fact 3 (Li, P '15): [Local convergence]
Let $C = \{x : \ Ax = b\}$ and D be a closed semi-algebraic set, $0 < \gamma < \sqrt{\frac{3}{2}} - 1$ and $\lim(y^t, z^t, x^t) = (y^*, z^*, x^*)$.
Suppose that $z^* \in C \cap D$ with

$$N_C\big(z^*\big) \cap -N_D(z^*) = \{0\}.$$

Then there exist $\eta \in (0, 1)$ and $\kappa > 0$ such that for all large t,

$$\mathrm{dist}\big(0, z^t - P_C(z^t) + N_D(z^t)\big) \leq \kappa\, \eta^t.$$

For PR splitting

- Does not converge in general even if D is convex.
- Modifying as follows also cannot guarantee convergence even if both sets are convex:

$$\begin{cases} y^{t+1} = \arg\min_{y} \left\{ \frac{1}{2} d_C^2(y) + \frac{1}{2\gamma}\|y - x^t\|^2 \right\}, \\ z^{t+1} \in \operatorname{Arg}\min_{z \in D} \left\{ \|2y^{t+1} - x^t - z\|^2 \right\}, \\ x^{t+1} = x^t + 2(z^{t+1} - y^{t+1}). \end{cases}$$

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x^{t+1} = x^t + 2(z^{t+1} - y^{t+1}).
\end{cases}
$$

- Indeed, PR splitting applied to minimizing sum of convex functions $f + g$ converges when f is continuous and strictly convex. (Lions, Mercier '79)

Our PR splitting

- PR splitting: ($\gamma > 0$)

$$\begin{cases} y^{t+1} = \arg\min_{y} \left\{ \frac{1}{2} d_C^2(y) + \frac{5}{2}\|y\|^2 + \frac{1}{2\gamma}\|y - x^t\|^2 \right\}, \\ z^{t+1} \in \operatorname{Arg}\min_{z \in D} \left\{ -\frac{5}{2}\|z\|^2 + \frac{1}{2\gamma}\|2y^{t+1} - x^t - z\|^2 \right\}, \\ x^{t+1} = x^t + 2(z^{t+1} - y^{t+1}). \end{cases}$$

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- Closed form updates for $\gamma \in (0, \frac{1}{5})$:

$$y^{t+1} = \frac{1}{6\gamma + 1}\left[x^t + \gamma P_C\left(\frac{x^t}{5\gamma + 1}\right)\right], \quad z^{t+1} \in P_D\left(\frac{2y^{t+1} - x^t}{1 - 5\gamma}\right).$$

PR Convergence result I

Fact 4 (Li, P '15): [Global convergence]
Suppose that $0 < \gamma < \frac{1}{12}$, and either C or D is compact.

Then the sequence $\{(y^t, z^t, x^t)\}$ generated from PR splitting is bounded, and any cluster point (y^*, z^*, x^*) satisfies $z^* = y^*$. Moreover, y^* is a stationary point of

$$\min_{u \in D} \frac{1}{2} d_C^2(u),$$

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If both sets are in addition semi-algebraic, then the whole sequence is convergent.

Convergence proof?

- KEY: Makes use of $\mathfrak{P}_\gamma(y, z, x)$ defined as

$$\frac{1}{2}d_C^2(y) + \delta_D(z) + \frac{5}{2}(\|y\|^2 - \|z\|^2) + \frac{1}{2\gamma}\|x - z\|^2 - \frac{1}{2\gamma}\|x - 2z + y\|^2.$$

- Can show that for some $k_1, k_2 > 0$:

$$\begin{aligned} \mathfrak{P}_\gamma(y^t, z^t, x^t) - \mathfrak{P}_\gamma(y^{t+1}, z^{t+1}, x^{t+1}) &\geq k_1\|y^{t+1} - y^t\|^2; \\ \mathrm{dist}(0, \partial\mathfrak{P}_\gamma(y^t, z^t, x^t)) &\leq k_2\|y^{t+1} - y^t\|. \end{aligned}$$

Numerical simulations

- Find a point in $Ax = b$ with $\|x\|_0 \leq r$ and $\|x\|_\infty \leq 10^6$.
- Consider random instances: generate an r-sparse vector $\tilde{x}$, an $m \times n$ matrix A, and set $b = A\tilde{x}$.
- Compare with alternating projection. Initialize all three algorithms at $x^0 = 0$.
- Terminate when successive changes are less than 10^{-8}.

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- Compare with alternating projection. Initialize all three algorithms at $x^0 = 0$.
- Terminate when successive changes are less than 10^{-8}.
- For the splitting methods, start with a γ larger than the threshold, decrease γ if $\|y^{t+1} - y^t\|$ does not deteriorate quickly enough or $\|y^t\|$ becomes too large.

Numerical simulations

Over 50 trials for each m, n; sparsity is $\lceil \frac{m}{5} \rceil$; succ means $\mathrm{fval} < 10^{-12}$.

Data	DR: $fval = \frac{1}{2} d_C^2(z^t)$			PR: $fval = \frac{1}{2} d_C^2(z^t)$			Alt Proj: $fval = \frac{1}{2} d_C^2(x^t)$		
m, n	iter	$\mathrm{fval}_{\max}$	succ	iter	$\mathrm{fval}_{\max}$	succ	iter	$\mathrm{fval}_{\max}$	succ
100, 4000	1967	3e-02	30	491	7e-2	0	1694	8e-2	0
100, 5000	2599	2e-02	18	586	7e-2	0	1978	7e-2	0
100, 6000	2046	1e-02	12	684	5e-2	0	2350	5e-2	0
200, 4000	836	2e-15	50	310	2e-1	14	1076	3e-1	0
200, 5000	1080	3e-15	50	364	1e-1	2	1223	2e-1	0
200, 6000	1279	7e-02	43	431	1e-1	5	1510	2e-1	1
300, 4000	600	3e-15	50	223	2e-1	35	872	4e-1	3
300, 5000	710	4e-15	50	295	2e-1	25	1068	3e-1	3
300, 6000	812	3e-15	50	350	2e-1	21	1252	3e-1	1
400, 4000	520	2e-15	50	156	3e-1	47	818	6e-1	30
400, 5000	579	3e-15	50	213	3e-1	42	946	4e-1	12
400, 6000	646	4e-15	50	288	2e-1	38	1108	3e-1	4

Conclusion

- The DR splitting applied to $\min_{u \in D} \frac{1}{2} d_C^2(u)$ with a compact C or D generates a sequence that clusters at a stationary point.
- The PR splitting *suitably* applied to $\min_{u \in D} \frac{1}{2} d_C^2(u)$ with a compact C or D generates a sequence that clusters at a stationary point.
- Under semi-algebraicity assumption, the whole sequence converges.

Reference:

- G. Li and T. K. Pong.
  Douglas-Rachford splitting for nonconvex optimization with application to nonconvex feasibility problems.
  To appear in Math. Program., DOI:10.1007/s10107-015-0963-5.
- G. Li and T. K. Pong.
  Peaceman-Rachford splitting for a class of nonconvex optimization problems.
  Available at *http://arxiv.org/abs/1507.00887*.

Thanks for coming! :-)