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Finite element approximation of a free boundary plasma problem

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Abstract In this article, we study a finite element approximation for a model free boundary plasma problem. Using a mixed approach (which resembles an optimal control problem with control constraints), we formulate a weak formulation and study the existence and uniqueness of a solution to the continuous model problem. Using the same setting, we formulate and analyze the discrete problem. We derive optimal order energy norm *a priori* error estimates proving the convergence of the method. Further, we derive a reliable and efficient *a posteriori* error estimator for the adaptive mesh refinement algorithm. Finally, we illustrate the theoretical results by some numerical examples.

Keywords Plasma problem · Finite element · Weak formulation · Free boundary · Optimal control

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary denoted by $\partial\Omega$. However the results are applicable for any bounded polyhedral domain in $\mathbb{R}^n$ $(n \geq 1)$, where $\mathbb{R}^n$ denotes the Euclidean space of dimension n. We consider the following model problem which arises in plasma physics, see [5, page 521] :

$$-\Delta u + \lambda u_- = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u|_{\partial\Omega} = c \in \mathbb{R} \quad \text{on} \quad \partial\Omega \quad (\text{i.e., } u \equiv \text{constant on } \partial\Omega), \tag{1.2}$$

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\, ds = I, \tag{1.3}$$

where λ, I are given positive real numbers and $u_-(x) = \max\{0, -u(x)\}$, the negative part of u. Hereafter $\partial u/\partial n$ denotes the normal derivative of u on the boundary in the sense of the trace. A slightly more general but similar problem is studied in [2]. The plasma problem consists of finding u and a constant c such that $u \equiv c$ on $\partial\Omega$ and they satisfy (1.1)–(1.3). It is shown in [5] by some minimization formulation that the above model problem has a solution. Further under an additional assumption that $\lambda < \lambda_2$ it is shown [5, Theorem 11.2] that the solution is unique, where λ_2 is the second smallest eigen-value corresponding to $-\Delta$ on Ω with homogeneous Dirichlet boundary condition. The set $\Omega_p = \{x \in \Omega : u(x) < 0\}$ is called the plasma set and the set $\Omega_v = \{x \in \Omega : u(x) > 0\}$ is called the vacuum set. Further the sets Ω_p and Ω_v are connected sets in Ω, see [5, Theorem 12.1]

For the rest of the discussion, we introduce some notation. Let $L^2(D)$ be the standard space of measurable and square integrable functions defined on the open set $D \subseteq \Omega$. The $L^2(D)$ inner-product and norm are denoted by $(\cdot,\cdot)_D$ and $\|\cdot\|_{L^2(D)}$, respectively. When $D = \Omega$, we denote the $L^2(\Omega)$ inner-product and norm by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. We denote by $|D|$, the 2-dimensional measure of an open subset $D \subseteq \Omega$. The Sobolev-Hilbert space of order $m \geq 0$ on $D \subseteq \Omega$ is denoted by $H^m(D)$ with the norm and semi-norm,

$$\|v\|^2_{H^m(D)} = \sum_{|\alpha| \leq m} \|D^\alpha v\|^2_{L^2(D)} \quad \text{and} \quad |v|^2_{H^m(D)} = \sum_{|\alpha| = m} \|D^\alpha v\|^2_{L^2(D)}, \text{ respectively,}$$

where α is a multi-index and D^α is the distribution derivative of order $|\alpha|$, see [6]. Let $\mathcal{D}(D)$ be the topological vector space consisting of $C^\infty(D)$ functions with compact support in D. Further let $H^1_0(D)$ be the closure of $\mathcal{D}(D)$ in $H^1(D)$. Finally $C(\bar{D})$ denotes the space of continuous function on $\bar{D}$, the closure of $D \subseteq \Omega$.

We briefly discuss the proof of existence of a solution to the model problem (1.1)–(1.3) by the approach in [5]. Let K be a closed and convex subset of $L^2(\Omega)$ defined by

$$\mathrm{K} = \{p \in L^2(\Omega) : p \geq 0 \text{ a.e. in } \Omega, \quad \int_\Omega p\, dx = I\}. \tag{1.4}$$

It is easy to prove that the set K is nonempty. For instance, let $p \in L^2(\Omega)$ with $p > 0$ in Ω. If $t = (p, 1)$, then $\tilde{p} = Ip/t \in \mathrm{K}$. Let $G(x, y)$ denote the Green's function for $-\Delta$ on Ω with homogeneous Dirichlet boundary condition. Define the functional $\tilde{J} : L^2(\Omega) \to \mathbb{R}$ by

$$\tilde{J}(p) = \frac{1}{2\lambda} \int_\Omega p^2 \, dx - \frac{1}{2} \int_\Omega \int_\Omega G(x, y) p(x) p(y) \, dx \, dy. \tag{1.5}$$

It is proved in [5, Theorem 11.1] that the minimization problem

$$\tilde{J}(q) = \min_{p \in \mathrm{K}} \tilde{J}(p).$$

has a solution q. Let $\tilde{J}'$ be the Frechet derivative of $\tilde{J}$ which is given by

$$\tilde{J}'(p) = \frac{1}{\lambda} p - \int_\Omega G(x, y) p(y) \, dy \quad \forall p \in \mathrm{K}.$$

Then the first order optimality condition implies that

$$\tilde{J}'(q)(p - q) \geq 0 \quad \forall p \in \mathrm{K},$$

or in other words,

$$\int_\Omega \left(\frac{1}{\lambda} p(x) - \int_\Omega G(x, y) p(y) \, dy \right) (p - q)(x) \, dx \geq 0 \quad \forall p \in \mathrm{K}. \tag{1.6}$$

If the function u is defined by

$$u(x) = -\int_\Omega G(x, y) q(y) \, dy + c,$$

for some constant $c \in \mathbb{R}$ satisfying

$$\begin{aligned} \tilde{J}'(q) &:= -c \quad \text{in } \{x \in \Omega : q(x) > 0\}, \\ &\geq -c \quad \text{in } \{x \in \Omega : q(x) = 0\}, \end{aligned}$$

then u is a solution of Eqs. 1.1–1.3 and $q = \lambda u_-$, see [5, page 523].

The direct numerical approximation of Eq. 1.5 requires either closed form Green's function (although it is known for Laplace operator, in general it is not available) or requires a discrete version of it. Even if the Green's function is explicitly known, the optimality condition (1.6) appears in a nonlocal fashion which results in a dense matrix in its discrete version. Therefore in the upcoming discussion in this article, we introduce a mixed formulation to study the model problem and its approximation. This approach resembles an optimal control problem with control constraints, see for example [7] for the fundamental literature on optimal control problems.

The contribution of this article can be described as follows:

- Using a mixed formulation, we formulate the continuous model problem as an optimal control problem with control constraints.
- We derive an explicit formula for the constant c which appears in Eq. 1.2 that will be quite useful to design its discrete counterpart.
- We propose a finite element method and study the convergence of the method. Further, we derive a reliable and efficient *a posteriori* error estimator.
- Finally, we illustrate the theoretical results by some numerical examples using a primal dual active-set method.

There is hardly any literature on numerical approximation of the aforementioned free boundary plasma problem. This article makes an attempt to study a finite element approximation and some of its analysis. The rest of the article is organized as follows. In Section 2, we discuss the continuous problem in a mixed formulation. In Section 3, we study the finite element approximation and derive both *a priori* and *a posteriori* error estimates. We discuss some numerical examples in Section 4 and finally we conclude the article in Section 5.

2 Mixed formulation

In this section, we rewrite the formulation of [5] as in [2] as a mixed formulation which appears like an optimal control problem with control constraints. We begin with defining a cost functional $J : L^2(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$J(p, w) = \frac{1}{2\lambda}\|p\|^2 + \frac{1}{2}(w, p). \tag{2.1}$$

For given $p \in L^2(\Omega)$, let $w \in H_0^1(\Omega)$ be the unique weak solution of

$$(\nabla w, \nabla z) = -(p, z) \quad \forall z \in H_0^1(\Omega). \tag{2.2}$$

We can associate a solution map $S : L^2(\Omega) \to H_0^1(\Omega)$ by assigning for $p \in L^2(\Omega)$ the solution $S(p) = w$ with w satisfying (2.2). Then restricting to such pairs $\{p, w\}$, we can introduce the reduced functional $j : L^2(\Omega) \to \mathbb{R}$ as

$$j(p) = \frac{1}{2\lambda}\|p\|^2 + \frac{1}{2}(S(p), p). \tag{2.3}$$

Then the mathematical model problem is to find $q \in \mathrm{K}$ such that

$$j(q) = \min_{p \in \mathrm{K}} j(p), \tag{2.4}$$

where K is defined as in Eq. 1.4.

The following assumption is made for the rest of the article.

Assumption-L There holds $0 < \lambda < \lambda_1$, where λ_1 is the best constant that appears in the Poincaré inequality (or the smallest eigen-value of $-\Delta$ on Ω with homogeneous Dirichlet boundary condition):

$$\|v\|^2 \le \frac{1}{\lambda_1}\|\nabla v\|^2 \quad \forall v \in H_0^1(\Omega). \tag{2.5}$$

Before proving the existence of a unique solution for Eq. 2.4, we prove the following lemma:

Lemma 2.1 *There holds for all* $p \in L^2(\Omega)$ *that*

$$\|S(p)\| \le \frac{1}{\lambda_1}\|p\| \quad \textit{and} \quad \|\nabla S(p)\| \le \frac{1}{\sqrt{\lambda_1}}\|p\|.$$

Proof Note by the definition of S, the Cauchy-Schwarz inequality and Eq. 2.5 that

$$\|\nabla S(p)\|^2 = |(p, S(p))| \leq \|p\| \, \|S(p)\| \leq \frac{1}{\sqrt{\lambda_1}} \|p\| \|\nabla S(p)\|.$$

The remaining proof follows by applying Eq. 2.5. □

In [5], it is assumed that $0 < \lambda < \lambda_2$ to prove the uniqueness of the solution, where λ_2 is the second smallest eigen-value of $-\Delta$ on Ω with homogeneous Dirichlet boundary condition. However, our assumption is more restrictive than that in [5]. This is due to fact that the arguments in [5] for the continuous problem seems to suggest to use discrete maximum principles of finite element methods in order to analyze the discrete problem. Therefore the subject of investigation for the case $\lambda_1 < \lambda < \lambda_2$ is postponed to the future. However the subsequent derivation of explicit constant c is valid once the solution exists. Further the numerical method is useful in computations without restriction on λ.

Theorem 2.2 *Let $\lambda < \lambda_1$. Then the minimization problem (2.4) has a unique solution denoted by $q \in \mathrm{K}$. Further the following first order optimality condition holds:*

$$(S(q) + \frac{1}{\lambda} q, p - q) \geq 0 \quad \forall p \in \mathrm{K},$$

where $S(q) \in H_0^1(\Omega)$ satisfies

$$(\nabla S(q), \nabla z) = -(q, z) \quad \forall z \in H_0^1(\Omega).$$

Proof Using Eq. 2.3 and Lemma 2.1, we find

$$j(p) = \frac{1}{2\lambda}\|p\|^2 - \frac{1}{2}\|\nabla S(p)\|^2 \geq \left(\frac{1}{2\lambda} - \frac{1}{2\lambda_1}\right)\|p\|^2 \geq 0.$$

The existence and uniqueness of a solution for Eq. 2.4 follow from the standard theory of optimal control problems [7]. The necessary first order optimality condition is given by

$$j'(q)(p-q) = \lim_{t \to 0} \frac{j(q + t(p-q)) - j(q)}{t} \geq 0 \quad \forall p \in \mathrm{K},$$

with

$$j'(q) = S(q) + \frac{1}{\lambda} q.$$

To verify this, consider

$$\begin{aligned}\frac{j(q + t(p-q)) - j(q)}{t} &= \frac{1}{\lambda}(q, p-q) + \frac{1}{2}(S(p-q), q) + \frac{1}{2}(S(q), p-q) \\ &\quad + \frac{t}{2\lambda}(p-q, p-q) + \frac{t}{2}(S(p-q), p-q).\end{aligned}$$

Then

$$j'(q)(p-q) = \frac{1}{\lambda}(q, p-q) + \frac{1}{2}(S(p-q), q) + \frac{1}{2}(S(q), p-q) \quad \forall p \in \mathrm{K}.$$

Now from the definition of S,

$$(S(p-q), q) = -(\nabla S(q), \nabla S(p-q)) = -(S(q), p-q).$$

This completes the proof. $\square$

Note that by introducing $v = S(q)$, we find that the following is satisfied by q and v:

$$(\nabla v, \nabla z) = -(q, z) \quad \forall z \in H_0^1(\Omega), \tag{2.6}$$

$$(v + \frac{1}{\lambda} q, p - q) \geq 0 \quad \forall p \in \mathrm{K}. \tag{2.7}$$

It is easy to deduce that the model problem (2.6)–(2.7) has a unique solution. The following theorem finds a solution for Eqs. 1.1–1.3 with an explicit constant c such that $u \equiv c$ on $\partial\Omega$.

Theorem 2.3 *Let $v \in H_0^1(\Omega)$ and $q \in L^2(\Omega)$ be the solutions of* Eqs. 2.6–2.7. *Define*

$$u(x) = v(x) + c, \tag{2.8}$$

where c is defined by

$$c = \frac{-1}{|\Omega_1|} \int_{\Omega_1} \left(v(y) + \frac{1}{\lambda} q(y) \right) dy, \tag{2.9}$$

and $\Omega_1 := \{x \in \Omega : q(x) > 0\}$. Assume that $q \in C(\bar{\Omega})$. Then, u is the solution of Eqs. 1.1–1.3.

Proof Let x_0 be a point in Ω_1. As Ω_1 is an open set, there is a neighborhood $B_0 = B_\delta(x_0)$ such that $B_0 \subset \Omega_1$. By the definition of Ω_1, there holds $q > 0$ on B_0. Let $\phi \in \mathcal{D}(B_0)$ with $\max_{B_0} |\phi| \leq 1$ and $(\phi, 1)_{B_0} = 0$. Extend ϕ to Ω by zero outside of B_0. Then for sufficiently small $\epsilon > 0$, the functions $p^\pm = q \pm \epsilon\phi \in \mathrm{K}$. Substituting $p^\pm$ in Eq. 2.7, we find

$$(v + \frac{1}{\lambda} q, \phi)_{\Omega_1} = (v + \frac{1}{\lambda} q, \phi)_{B_0} = 0.$$

Since $(\phi, 1)_{\Omega_1} = (\phi, 1)_{B_0} = 0$, there holds

$$(v + \frac{1}{\lambda} q + c, \phi)_{\Omega_1} = 0,$$

for any $c \in \mathbb{R}$. We take in particular c to be the constant defined in Eq. 2.9. By scaling there holds

$$(v + \frac{1}{\lambda} q + c, \phi)_{\Omega_1} = 0 \quad \forall \phi \in \mathcal{D}(B_0) \text{ with } (\phi, 1)_{\Omega_1} = 0.$$

Using this argument, we conclude that

$$(v + \frac{1}{\lambda}q + c, \phi)_{\Omega_1} = 0 \quad \forall \phi \in \mathcal{D}(\Omega_1) \text{ with } (\phi, 1)_{\Omega_1} = 0,$$

and further

$$(v + \frac{1}{\lambda}q + c, \phi + r)_{\Omega_1} = 0 \quad \forall r \in \mathbb{R} \text{ and } \forall \phi \in \mathcal{D}(\Omega_1) \text{ with } (\phi, 1)_{\Omega_1} = 0.$$

From the fact that the function space $\mathcal{D}_0(\Omega_1) := \{\phi \in \mathcal{D}(\Omega_1) : (\phi, 1)_{\Omega_1} = 0\}$ is dense in the space $L^2_0(\Omega_1) := \{\phi \in L^2(\Omega_1) : \int_{\Omega_1} \phi \, dx = 0\}$ with respect to the L^2-norm, we conclude that

$$(v + \frac{1}{\lambda}q + c, \phi + r)_{\Omega_1} = 0 \quad \forall r \in \mathbb{R} \text{ and } \forall \phi \in L^2_0(\Omega_1).$$

Let $\tilde{\phi} \in L^2(\Omega_1)$ and define the function ϕ by

$$\phi := \tilde{\phi} - \frac{1}{|\Omega_1|} \int_{\Omega_1} \tilde{\phi} \, dx.$$

Then $\phi \in L^2_0(\Omega_1)$ and

$$0 = (v + \frac{1}{\lambda}q + c, \phi)_{\Omega_1} = (u + \frac{1}{\lambda}q, \tilde{\phi})_{\Omega_1},$$

which proves that

$$(u + \frac{1}{\lambda}q, \tilde{\phi})_{\Omega_1} = 0 \quad \forall \tilde{\phi} \in L^2(\Omega_1).$$

Therefore $q = -\lambda u$ on Ω_1. Now let the set $\Omega_0 = \{x \in \Omega : q(x) = 0\}$. Then since $(1, p - q) = 0$ for any $p \in \mathrm{K}$, we find

$$\begin{aligned} 0 \leq \int_\Omega (v + \frac{1}{\lambda}q)(p - q) \, dx &= \int_\Omega (v + \frac{1}{\lambda}q + c)(p - q) \, dx = \int_\Omega (u + \frac{1}{\lambda}q)(p - q) \, dx \\ &= \int_{\Omega_1} (u + \frac{1}{\lambda}q)(p - q) \, dx + \int_{\Omega_0} (u + \frac{1}{\lambda}q)(p - q) \, dx \\ &= \int_{\Omega_0} (u + \frac{1}{\lambda}q)(p - q) \, dx = \int_{\Omega_0} up \, dx. \end{aligned} \tag{2.10}$$

If $p \in L^2(\Omega)$ with $p \not\equiv 0$ and $p \geq 0$ a.e. in Ω, then $\tilde{p} = Ip/(p, 1) \in \mathrm{K}$. By taking $\tilde{p}$ in Eq. 2.10, we conclude that

$$\int_{\Omega_0} up \, dx \geq 0 \quad \forall p \in \{p \in L^2(\Omega) : p \geq 0 \text{ a.e. in } \Omega\}.$$

This implies that $u \geq 0$ on the set Ω_0 and proves that $q = \lambda u_-$ on Ω. Further since $q = \Delta v = \Delta u$ and $q \in \mathrm{K}$, we conclude that

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} \, ds = \int_\Omega \Delta u \, dx = \int_\Omega q \, dx = I.$$

This proves that u is the solution of Eqs. 1.1–1.3. $\square$

Remark 2.4 The characterization of c in Eq. 2.9 is useful in defining its discrete counterpart and in deriving error estimates of finite element approximation.

Note that since $q \in L^2(\Omega)$, the solution $v \in H_0^1(\Omega)$ of Eq. 2.6 will be in $H^{3/2+\epsilon}(\Omega)$ for any $\epsilon > 0$, by the well known shift theorem [6]. As we are in $2D$, the Sobolev imbedding theorem that $H^{1+\epsilon}(\Omega) \subset C(\bar{\Omega})$ for any $\epsilon > 0$ holds true [6]. We conclude that $v \in C(\bar{\Omega})$ and hence $u \in C(\bar{\Omega})$. Since $u \in H^{3/2+\epsilon}(\Omega) \cap C(\bar{\Omega})$, there holds $q = -\lambda u_- \in H^1(\Omega) \cap C(\bar{\Omega})$. If we assume further that the domain Ω is convex, then $v \in H^2(\Omega)$ and hence $u \in H^2(\Omega)$ by the elliptic regularity theory [6]. In general due to the free boundary, the boundary of the set Ω_1, the solution $q \notin H^2(\Omega)$.

3 Finite element approximation

Let $\mathcal{T}_h$ be a regular triangulation of Ω (see [1, 4]) and T be a generic triangle in $\mathcal{T}_h$ with diameter denoted by h_T. Set $h = \max_{T \in \mathcal{T}_h} h_T$.

Let $V_h \subset V$ be a finite element subspace defined by

$$V_h = \{v \in H_0^1(\Omega) : v|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h\},$$

where $\mathbb{P}_r(T)$ is the space of polynomials of degree less than or equal to r. Let Q_h be another finite element space defined by

$$Q_h = \{p \in L^2(\Omega) : p|_T \in \mathbb{P}_0(T) \quad \forall T \in \mathcal{T}_h\}.$$

The discrete closed and convex set K_h is defined as

$$\mathrm{K}_h = \{p \in Q_h : p \geq 0 \text{ a.e. in } \Omega, \quad \int_\Omega p\,dx = I\}.$$

It is obvious that $\mathrm{K}_h \subset \mathrm{K}$ and K_h is nonempty.

Let $\mathcal{E}_h^i$ be the set of all interior edges in $\mathcal{T}_h$ and a generic edge is denoted by e whose length is denoted by h_e. For any $e \in \mathcal{E}_h^i$, there are two triangles T_1^e and T_2^e in $\mathcal{T}_h$ such that $e = \partial T_1^e \cap \partial T_2^e$. Then set $\mathcal{T}_e = T_1^e \cup T_2^e$. For any $v_h \in V_h$, the jump of ∇v_h on any edge $e \in \mathcal{E}_h^i$ is defined as

$$[\![\nabla v_h]\!] = \nabla v_h|_{T_1^e} n_1 + \nabla v_h|_{T_2^e} n_2,$$

where n_i is the unit outward normal to ∂T_i^e on the edge e for $i = 1, 2$.

Finite element method Find $v_h \in V_h$ and $q_h \in \mathrm{K}_h$ such that

$$(\nabla v_h, \nabla z_h) = -(q_h, z_h) \quad \forall z_h \in V_h, \tag{3.1}$$

$$(v_h + \frac{1}{\lambda} q_h, p_h - q_h) \geq 0 \quad \forall p_h \in \mathrm{K}_h. \tag{3.2}$$

The discrete solution u_h is then defined to be

$$u_h = v_h + c_h, \tag{3.3}$$

$$c_h = \frac{-1}{|\Omega_{1h}|} \int_{\Omega_{1h}} \left(v_h(y) + \frac{1}{\lambda} q_h(y) \right) dy \tag{3.4}$$

where $\Omega_{1h} = \{x \in \Omega : q_h(x) > 0\}$. Since q_h is piecewise constant, it is easy to find the set Ω_{1h}. As in the case of continuous problem, it can be shown that the discrete problem (3.1)–(3.2) has a unique solution for $\lambda < \lambda_1$. Indeed, If $S_h(p_h)$ is the solution of

$$(\nabla S_h(p_h), \nabla z_h) = -(p_h, z_h) \quad \forall z_h \in V_h,$$

then the minimization problem

$$j_h(q_h) = \min_{p_h \in \mathrm{K}_h} j_h(p_h), \tag{3.5}$$

where

$$j_h(p_h) = \frac{1}{2\lambda}\|p_h\|^2 + \frac{1}{2}(S_h(p_h), p_h), \tag{3.6}$$

has a unique solution q_h. Then by setting $v_h = S_h(q_h)$, we find a unique solution for the discrete problem (3.1)–(3.2).

A priori error analysis For the purpose of error analysis, let $\Pi_h : L^2(\Omega) \to Q_h$ be the $L^2(\Omega)$-projection defined as $\Pi_h p \in Q_h$ for $p \in L^2(\Omega)$ and

$$\int_T \Pi_h p \, dx = \int_T p \, dx \quad \forall T \in \mathcal{T}_h.$$

Note that if $p \in \mathrm{K}$, then it is clearly true that $\Pi_h p \in \mathrm{K}$. Similarly, we define the projection $P_h : H_0^1(\Omega) \to V_h$ as follows: For given $v \in H_0^1(\Omega)$, find $P_h v \in V_h$ such that

$$(\nabla P_h v, \nabla z_h) = (\nabla v, \nabla z_h) \quad \forall z_h \in V_h.$$

From Cea's Lemma [1, 4], it is well-known that

$$\begin{aligned}\|\nabla(v - P_h v)\| &= \min_{z_h \in V_h} \|\nabla(v - z_h)\|, \\ \|q - \Pi_h q\| &= \min_{p_h \in Q_h} \|q - p_h)\|.\end{aligned}$$

These projections play a crucial role in obtaining error estimates under the assumption that $\lambda < \lambda_1$ (the assumption made for the continuous problem). We now prove some *a priori* error estimates.

Theorem 3.1 *There holds*

$$\|\nabla(v - v_h)\| + \|q - q_h\| \le C\left(\|\nabla(v - P_h v)\| + \|q - \Pi_h q\| + \|v - \Pi_h v\|\right).$$

Proof First of all, from the Eqs. 2.6 and 3.1, we find that

$$(\nabla(v - v_h), \nabla z_h) = -(q - q_h, z_h) \quad \forall z_h \in V_h.$$

Then using the definition of P_h, we find

$$\begin{aligned}\|\nabla(P_h v - v_h)\|^2 &= (\nabla(v - v_h), \nabla(P_h v - v_h)) = -(q - q_h, P_h v - v_h) \\ &\le \|q - q_h\| \, \|P_h v - v_h\|.\end{aligned}$$

By applying the Poincaré inequality (2.5), we derive

$$\|\nabla(P_h v - v_h)\| \le \frac{1}{\sqrt{\lambda_1}}\|q - q_h\|. \tag{3.7}$$

From the inequalities Eqs. 2.7 and 3.2, we note since $\mathrm{K}_h \subset \mathrm{K}$ that

$$(v + \frac{1}{\lambda} q, q_h - q) \geq 0,$$
$$(v_h + \frac{1}{\lambda} q_h, q - q_h) \geq (v_h + \frac{1}{\lambda} q_h, q - p_h) \quad \forall p_h \in \mathrm{K}_h.$$

Adding the above two inequalities,

$$(v - v_h + \frac{1}{\lambda}(q - q_h), q_h - q) \geq (v_h + \frac{1}{\lambda} q_h, q - p_h) \quad \forall p_h \in \mathrm{K}_h,$$

which implies with $p_h = \Pi_h q$ that

$$\begin{aligned}
\frac{1}{\lambda}\|q - q_h\|^2 &\leq (v - v_h, q_h - q) - (v_h - v, q - p_h) - (v, q - p_h) \\
&\leq \|P_h v - v_h\| \|q - q_h\| \\
&\quad + |(v - P_h v, q_h - q) - (v_h - v, q - p_h) - (v, q - p_h)|.
\end{aligned}$$

Using Eqs. 3.7 and 2.5, we find

$$\begin{aligned}
\left(\frac{1}{\lambda} - \frac{1}{\lambda_1}\right)\|q - q_h\|^2 &\leq |(v - P_h v, q_h - q) - (v_h - v, q - p_h) - (v, q - p_h)| \\
&= |(v - P_h v, q_h - q) - (v_h - v, q - p_h) - (v - \Pi_h v, q - p_h)| \\
&\leq \|v - P_h v\| \, \|q - q_h\| + \|q - p_h\| \left(\|v - P_h v\| + \|v - \Pi_h v\|\right) \\
&\quad + \|q - p_h\| \, \|P_h v - v_h\| \\
&\leq \|v - P_h v\| \, \|q - q_h\| + \|q - p_h\| \left(\|v - P_h v\| + \|v - \Pi_h v\|\right) \\
&\quad + \frac{1}{\lambda_1}\|q - p_h\| \, \|q - q_h\|,
\end{aligned}$$

which implies

$$\|q - q_h\|^2 \leq C\left(\|v - P_h v\|^2 + \|v - \Pi_h v\|^2 + \|q - p_h\|^2\right).$$

This completes the proof. □

If $\lambda < \lambda_1$, the solution u defined in Eq. 2.8 is the unique solution of the mode problem (1.1)–(1.3). By using [5, Theorem 12.1], we note that $\Omega_p = \Omega$. But since $\Omega_p = \Omega_1$, we can redefine c_h in Eq. 3.4 by

$$c_h = \frac{-1}{|\Omega|} \int_\Omega \left(v_h(y) + \frac{1}{\lambda} q_h(y)\right) dy. \tag{3.8}$$

Further note that c defined in Eq. 2.9 is given by

$$c = \frac{-1}{|\Omega|} \int_\Omega \left(v(y) + \frac{1}{\lambda} q(y)\right) dy. \tag{3.9}$$

The following theorem proves estimates for $u - u_h$.

Theorem 3.2 *There holds*

$$\|\nabla(u - u_h)\| + \|u - u_h\| + |c - c_h| \leq C\left(\|\nabla(v - P_h v)\| + \|q - \Pi_h q\| + \|v - \Pi_h v\|\right),$$

where c and c_h are defined by Eqs. 3.9 *and* 3.8*, respectively.*

Proof Since $u = v + c$ and $u_h = v_h + c_h$, there holds

$$\|\nabla(u - u_h)\| = \|\nabla(v - v_h)\|.$$

By the definition of c and c_h in Eqs. 3.9 and 3.8, respectively, and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|c - c_h| &\le |\Omega|^{-1/2}\left(\|v - v_h\| + \frac{1}{\lambda}\|q - q_h\|\right) \\
&\le |\Omega|^{-1/2}\left(\frac{1}{\sqrt{\lambda_1}}\|\nabla(v - v_h)\| + \frac{1}{\lambda}\|q - q_h\|\right) \\
&\le C|\Omega|^{-1/2}\left(\|\nabla(v - P_h v)\| + \|q - \Pi_h q\| + \|v - \Pi_h v\|\right).
\end{aligned}$$

This proves the desired estimate since $\|u - u_h\| \le \|v - v_h\| + |\Omega|^{1/2}|c - c_h|$. $\square$

We deduce the following result on the order of convergence:

Corollary 3.3 *There holds*

$$\begin{aligned}
&\|\nabla(v - v_h)\| + \|q - q_h\| \le C\left(h^s|u|_{H^{1+s}(\Omega)} + h\left(|q|_{H^1(\Omega)} + |v|_{H^1(\Omega)}\right)\right), \\
&\|u - u_h\|_{H^1(\Omega)} + |c - c_h| \le C\left(h^s|u|_{H^{1+s}(\Omega)} + h\left(|q|_{H^1(\Omega)} + |v|_{H^1(\Omega)}\right)\right),
\end{aligned}$$

where $s \in (0, 1]$ is the elliptic regularity index of v (or u). If the domain Ω is a convex polygon, then $s = 1$.

A posteriori error analysis Since the model problem can exhibit free boundary, the boundary of the set $\{x \in \Omega : q(x) > 0\}$, and corner singularities, it is appropriate to use adaptive finite element method to resolve the solution around the free boundary. This can be done by deriving a residual based *a posteriori* error estimate and use it for adaptive mesh refinement. In this case *a posteriori* error estimates can be derived efficiently by using suitable auxiliary problems. We define an auxiliary solution denoted by $\bar{v}$ as follows: Find $\bar{v} \in H_0^1(\Omega)$ such that

$$(\nabla\bar{v}, \nabla z) = -(q_h, z) \quad \forall z \in H_0^1(\Omega). \tag{3.10}$$

This auxiliary solution plays a crucial role in deriving an error estimator under the assumption that $\lambda < \lambda_1$ (the assumption made for the continuous problem). Note that v_h is the Galerkin approximation of $\bar{v}$. Therefore the following residual based error estimate for $\|\nabla(\bar{v} - v_h)\|$ is well known [1, 8]:

$$\|\nabla(\bar{v} - v_h)\| \le C\left(\sum_{T\in\mathcal{T}_h} h_T^2\|q_h\|_{L^2(T)}^2 + \sum_{e\in\mathcal{E}_h^i}\int_e h_e[\![\nabla u_h]\!]^2\,ds\right)^{1/2}, \tag{3.11}$$

since $u_h = v_h + c_h$. Define for $T \in \mathcal{T}_h$ and $e \in \mathcal{E}_h^i$,

$$\begin{aligned}
\eta_T &= h_T\|q_h\|_{L^2(T)} + \inf_{w_h\in\mathbb{P}_0(T)}\|\lambda u_h + q_h - w_h\|_{L^2(T)}, \\
\eta_e &= h_e^{1/2}\|[\![\nabla u_h]\!]\|_{L^2(e)},
\end{aligned}$$

and set

$$\eta = \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 + \sum_{e \in \mathcal{E}_h^i} \eta_e^2 \right)^{1/2}.$$

The following theorem derives reliable *a posteriori* error estimates.

Theorem 3.4 *There holds*

$$\begin{aligned} \|\nabla(v - v_h)\| + \|q - q_h\| &\leq C\eta, \\ \|u - u_h\|_{H^1(\Omega)} + |c - c_h| &\leq C\eta, \end{aligned}$$

where c and c_h are defined by Eqs. 3.9 *and* 3.8*, respectively.*

Proof First of all, from the Eqs. 2.6 and 3.10, we find that

$$(\nabla(v - \bar{v}), \nabla z) = -(q - q_h, z) \quad \forall z \in H_0^1(\Omega).$$

Then using the Cauchy-Schwarz inequality, we find

$$\|\nabla(v - \bar{v})\|^2 = -(q - q_h, v - \bar{v}) \leq \|q - q_h\| \, \|v - \bar{v}\|,$$

and using Eq. 2.5,

$$\|\nabla(v - \bar{v})\| \leq \frac{1}{\sqrt{\lambda_1}} \|q - q_h\|, \tag{3.12}$$

which implies that

$$\|v - \bar{v}\| \leq \frac{1}{\lambda_1} \|q - q_h\|. \tag{3.13}$$

As before using the inequalities (2.7), (3.2) and since $\mathrm{K}_h \subset \mathrm{K}$, we note that

$$\begin{aligned} &(v + \frac{1}{\lambda} q, q_h - q) \geq 0, \\ &(v_h + \frac{1}{\lambda} q_h, q - q_h) \geq (v_h + \frac{1}{\lambda} q_h, q - p_h) \quad \forall p_h \in \mathrm{K}_h. \end{aligned}$$

Adding the above two inequalities and using (3.13), (2.5), we obtain for all $p_h \in \mathrm{K}_h$ that

$$\begin{aligned} \frac{1}{\lambda} \|q - q_h\|^2 &\leq (v - v_h, q_h - q) - (v_h + \frac{1}{\lambda} q_h, q - p_h) \\ &\leq \frac{1}{\lambda_1} \|q - q_h\|^2 + |(\bar{v} - v_h, q_h - q) - (v_h + \frac{1}{\lambda} q_h, q - p_h)|. \end{aligned}$$

Choose $p_h = \Pi_h q$. Then since $(w_h, q - p_h) = 0$ for any $w_h \in Q_h$ and $\|q - p_h\| \leq \|q - q_h\|$, we find

$$\begin{aligned} \|q - q_h\|^2 &\leq C|(\bar{v} - v_h, q_h - q) - (v_h + \frac{1}{\lambda} q_h + c_h - w_h, q - p_h)| \\ &\leq C \left(\|\nabla(\bar{v} - v_h)\| + \inf_{w_h \in Q_h} \|\lambda u_h + q_h - w_h\| \right) \|q - q_h\|. \end{aligned}$$

Using Eq. 3.11, we find the estimate for $q - q_h$. Then using Eqs. 3.12 and 3.11, we find the estimate for $v - v_h$. The definitions of c, c_h together with the estimates for v and q imply the estimate for $c - c_h$. This completes the proof. $\square$

The following theorem proves the local-efficiency estimates of the error estimators:

Theorem 3.5 *There holds*

$$\begin{aligned}
h_T \|q_h\|_{L^2(T)} &\leq C \left(\|\nabla(v - v_h)\|_{L^2(T)} + h_T \|q - q_h\|_{L^2(T)} \right), \\
h_e^{1/2} \|[\![\nabla u_h]\!]\|_{L^2(e)} &\leq C \left(\|\nabla(v - v_h)\|_{L^2(\mathcal{T}_e)} + h_e \|q - q_h\|_{L^2(\mathcal{T}_e)} \right), \\
\inf_{w_h \in \mathbb{P}_0(T)} \|\lambda u_h + q_h - w_h\|_{L^2(T)} &\leq C \left(\|u - u_h\|_{L^2(T)} + \|q - q_h\|_{L^2(T)} \right) \\
&\quad + C \left(\|u - \Pi_h u\|_{L^2(T)} + \|q - \Pi_h q\|_{L^2(T)} \right).
\end{aligned}$$

Proof First note that

$$\begin{aligned}
\inf_{w_h \in \mathbb{P}_0(T)} \|\lambda u_h + q_h - w_h\|_{L^2(T)} &\leq \inf_{w_h \in \mathbb{P}_0(T)} \|\lambda(u_h - u) + (q_h - q) - w_h\|_{L^2(T)} \\
&\quad + \inf_{w_h \in \mathbb{P}_0(T)} \|\lambda u + q - w_h\|_{L^2(T)} \\
&\leq \left(\|u - u_h\|_{L^2(T)} + \|q - q_h\|_{L^2(T)} \right) \\
&\quad + C \left(\|u - \Pi_h u\|_{L^2(T)} + \|q - \Pi_h q\|_{L^2(T)} \right),
\end{aligned}$$

by choosing $w_h = \Pi_h(\lambda u + q)$. The first two inequalities follow from the standard bubble function techniques [8]. $\square$

4 Numerical examples

In this section, we discuss the modification of the primal-dual active set strategy introduced in [7, Section 2.12.4] and present some numerical experiments illustrating the theoretical results.

4.1 Primal-dual active set strategy

We begin by describing the numerical procedure to solve the discrete problem by a primal-dual active set method [7]. To this end, we introduce the Lagrangian $\mathcal{L}$: $\mathrm{K}_{0h} \times \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{L}(p_h, r) = j_h(p_h) + r \left(\int_\Omega p_h \, dy - I \right) \quad p_h \in \mathrm{K}_{0h}, \; r \in \mathbb{R},$$

and find the solution through the critical point of $\mathcal{L}$, where

$$\mathrm{K}_{0h} := \{p_h \in Q_h : p_h \geq 0 \text{ a.e. in } \Omega\}.$$

The optimality system then leads to the following system: find $v_h \in V_h$, $q_h \in \mathrm{K}_{0h}$ and $c_h \in \mathbb{R}$ such that

$$\begin{aligned}(\nabla v_h, \nabla z_h) &= -(q_h, z_h) \quad \forall z_h \in V_h,\\ (v_h + \frac{1}{\lambda} q_h + c_h, p_h - q_h) &\geq 0 \quad \forall p_h \in \mathrm{K}_{0h},\\ (1, q_h - I) &= 0.\end{aligned}$$

We solve the above system by using the primal-dual active set strategy described in [7, Section 2.12.4]. For completeness, we describe the method here since we have the additional unknown c_h in the system:

Let the triangles in $\mathcal{T}_h$ be enumerated by $\{T_j\}_{\{1\leq j\leq N_1\}}$. Let N_2 be the dimension of V_h and $\{\phi_i\}_{\{1\leq i\leq N_2\}}$ be its canonical Lagrange basis functions. The basis of Q_h denoted by $\{\psi_j\}_{\{1\leq j\leq N_1\}}$ is given by the characteristic functions of T_j, $1 \leq j \leq N_1$, i.e.,

$$\psi_j(x) := \begin{cases} 1 & \text{if } x \in T_j,\\ 0 & \text{if } x \notin T_j.\end{cases}$$

Denote by $K = [K_{ij}]_{\{1\leq i,j\leq N_2\}}$ the stiffness matrix, where

$$K_{ij} = (\nabla\phi_j, \nabla\phi_i).$$

Define the matrices $B = [B_{ij}]_{\{1\leq i\leq N_1, 1\leq j\leq N_2\}}$ and $D = [D_{ij}]_{\{1\leq i,j\leq N_1\}}$, where

$$B_{ij} = \int_{T_j} \phi_i \, dx \quad \text{and} \quad D_{ij} = \int_\Omega \psi_i \psi_j \, dx.$$

Further define $M = [M_j]_{\{1\leq j\leq N_1\}}$, with

$$M_j = \int_\Omega \psi_j \, dx = |T_j|.$$

Let $X_a = [X_{ij}]_{\{1\leq i,j\leq N_1\}}$ be the matrix defined by

$$X_{ij} := \begin{cases} 1 & \text{if } i = j,\ j \in A_a,\\ 0 & \text{if } i = j, j \notin A_a,\\ 0 & \text{if } i \neq j.\end{cases} \tag{4.1}$$

where A_a is the index set of the active set in the iteration which will be defined later. Further define the matrix E by

$$E = \left(\frac{1}{\lambda} D\right)^{-1} (Id - X_a),$$

with Id is the identity matrix of size $N_1 \times N_1$. Now, consider the problem of finding $\vec{\alpha} \in \mathbb{R}^{N_2}$, $\vec{\beta} \in \mathbb{R}^{N_1}$ and $c \in \mathbb{R}$ such that

$$K\vec{\alpha} + B\vec{\beta} = 0, \tag{4.2}$$

$$EB^T\vec{\alpha} + \vec{\beta} + EMc = 0, \tag{4.3}$$

$$M^T\vec{\beta} = I. \tag{4.4}$$

Since $v_h \in V_h$ and $q_h \in Q_h$, we write

$$v_h = \sum_{i=1}^{N_2} \alpha_i \phi_i \quad \text{and} \quad q_h = \sum_{i=1}^{N_1} \beta_i \phi_i,$$

for scalars α_i $(i = 1, 2, \cdots, N_2)$ and β_i $(i = 1, 2, \cdots, N_1)$. Set $\vec{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_{N_2})$ and $\vec{\beta} = (\beta_1, \beta_2, \cdots, \beta_{N_1})$. Define the multiplier $\vec{\mu} = [\mu_j]_{\{1 \le j \le N_1\}}$ by

$$\vec{\mu} = -(EB^T\vec{\alpha} + cEM + \vec{\beta}).$$

The primal-dual active set strategy is defined as follows: Let $\vec{\mu}^0$ and $\vec{\beta}^0$ are given. Let $r > 0$ be some fixed real number and $k = 1$.

Step 1. Find the set A_a^k by

$$A_a^k = \{1 \le j \le N_1 : \beta_j^{k-1} + r\mu_j^{k-1} < 0\}.$$

Step 2. Compute X_a^k using the index set A_a^k and Eq. 4.1.
Step 3. Solve the system (4.2)–(4.4) for finding $\vec{\alpha}^k \in \mathbb{R}^{N_2}$, $\vec{\beta}^k \in \mathbb{R}^{N_1}$ and $c^k \in \mathbb{R}$.
Step 4. Set $k = k + 1$. Go to *Step 1* and compute A_a^k. If $A_a^k = A_a^{k-1}$ stop the iteration otherwise continue.

4.2 Numerical experiments

We present numerical experiments for a few model examples to illustrate the theoretical results and to draw some conclusions. In all the examples below, we take the constant $r = 1$ in the *Step 1* of the above algorithm.

Example 1 In this example, we consider the domain $\Omega = (0, 1)^2$, $\lambda = 19$ and $I = 4$. Note that the eigen-values of

$$\begin{aligned} -\Delta\phi &= \lambda\phi \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

are given by $\lambda_n = 2n^2\pi^2$ for $n = 1, 2, 3, \cdots$. Therefore we have the hypothesis $\lambda < \lambda_1 = 2\pi^2$ fulfilled. We consider a sequence of uniform triangulations of Ω with mesh sizes $h_k = 1/2^k$, for $k = 2, 3, \cdots, 7$. On this sequence of meshes, we solve

Table 1 Errors and orders of convergence for Example 1

h	$\|\nabla(u_h^k - u_h^{k-1})\|$	Order	$\|\nabla(v_h^k - v_h^{k-1})\|$	Order	$\|q_h^k - q_h^{k-1}\|$	Order	$\|c_h^k - c_h^{k-1}\|$	Order
1/8	7.1776e-01	–	7.1776e-01	–	2.2080e+00	–	3.5548e-02	–
1/16	4.0720e-01	0.8178	4.0720e-01	0.8178	2.2080e+00	0.9056	1.3097e-02	1.4405
1/32	2.0979e-01	0.9568	2.0979e-01	0.9568	1.1786e+00	1.0179	3.7055e-03	1.8215
1/64	1.0575e-01	0.9883	1.0575e-01	0.9883	2.8817e-01	1.0142	9.6097e-04	1.9471
1/128	5.3003e-02	0.9965	5.3003e-02	0.9965	1.4362e-01	1.0047	2.4271e-04	1.9853

Table 2 Errors and orders of convergence for Example 2

h	$\|\nabla(u_h^k - u_h^{k-1})\|$	Order	$\|\nabla(v_h^k - v_h^{k-1})\|$	Order	$\|q_h^k - q_h^{k-1}\|$	Order	$\|c_h^k - c_h^{k-1}\|$	Order
$\sqrt{2}/8$	8.0816e-01	–	8.0816e-01	–	1.5461e+00	–	4.1412e-02	–
$\sqrt{2}/16$	4.8976e-01	0.7226	4.8976e-01	0.7226	9.3499e-01	0.7256	1.6775e-02	1.3037
$\sqrt{2}/32$	2.7059e-01	0.8559	2.7059e-01	0.8559	4.6154e-01	1.0185	5.4893e-03	1.6117
$\sqrt{2}/64$	1.4982e-01	0.8529	1.4982e-01	0.8529	2.2375e-01	1.0446	1.6913e-03	1.6985
$\sqrt{2}/128$	8.5003e-02	0.8177	8.5003e-02	0.8177	1.1009e-01	1.0232	5.2949e-04	1.6755

the discrete solutions v_h^k, u_h^k and c_h^k. Since we do not have the exact solution, we compute the errors of discrete solutions at successive levels of meshes by computing $\|\nabla(u_h^k - u_h^{k-1})\|$, $\|\nabla(v_h^k - v_h^{k-1})\|$, $\|q_h^k - q_h^{k-1}\|$ and $|c_h^k - c_h^{k-1}|$ and report them for $k = 3, 4, \cdots, 7$. in the Table 1. The numerical results show clearly the expected order of convergence. However, we can observe from the table that the orders of convergence in approximating the constant c is close to 2, which seems to be a super-convergence.

Example 2 In this example, we consider an L-shaped domain $\Omega = (-1, 1) \times (0, 1) \cup (-1, 0) \times (-1, 0)$. Set $\lambda = 9$ and $I = 4$. Since the first eigen-values of

$$\begin{aligned} -\Delta\phi &= \lambda\phi \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

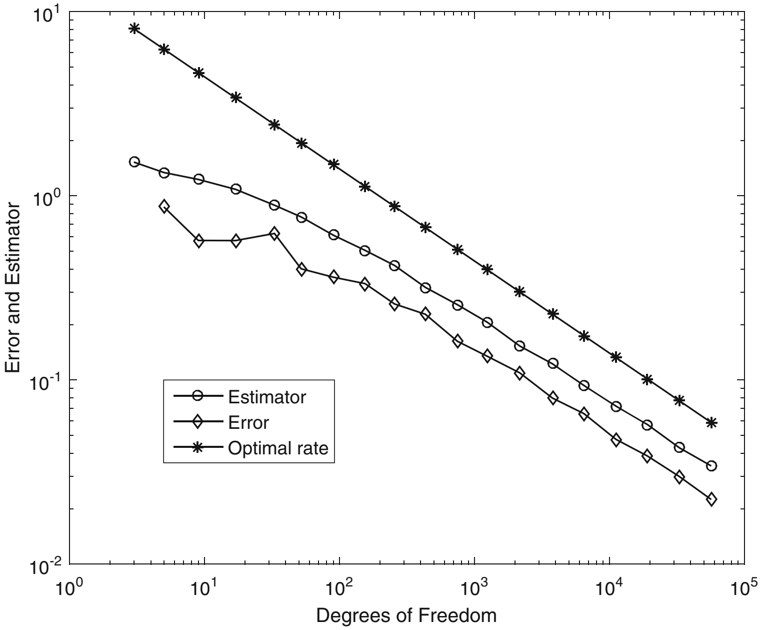

Fig. 1 Estimator converges at optimal rate

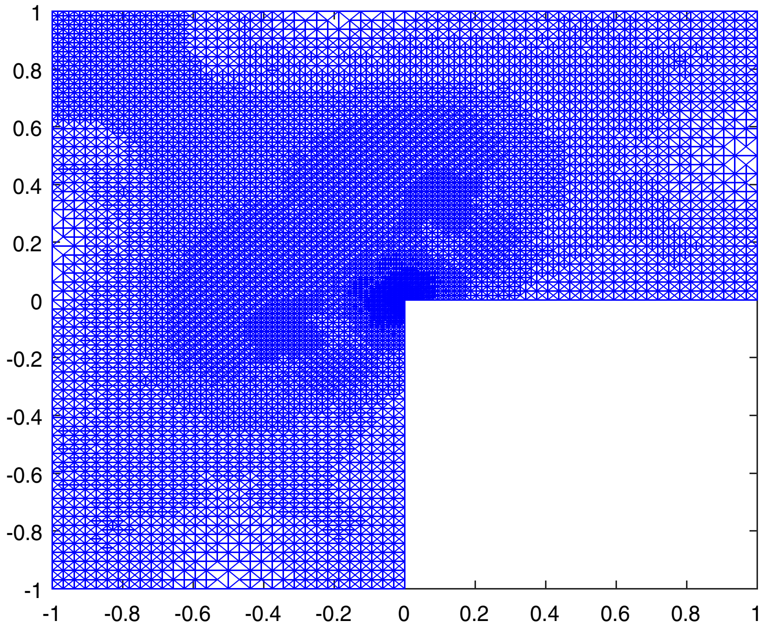

Fig. 2 Mesh refinement at intermediate level

is $\lambda_1 \approx 9.639723844$, [3, Page 421]. Therefore the assumption $\lambda < \lambda_1$ is satisfied. In the first case, we consider a sequence of uniform triangulations of Ω with mesh size $h_k = 1/(2^k\sqrt{2})$, for $k = 2, 3, \cdots, 7$. We compute the discrete solutions and the errors as in example 1 and report them in Table 2. The numerical results show the expected orders of convergence. The order of convergence in u_h (or v_h) is not optimal since the solution on L-shaped domain can have corner singularities.

We now test the performance of *a posteriori* error estimator by using the successive mesh refinement adaptive algorithm consisting of the steps

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}$$

with Dörfler's bulk marking strategy with parameter 0.3 and the longest edge bisection algorithm in the refinement step. The the errors (sum of the errors in the discrete

Table 3 Errors and orders of convergence for Example 3 with $\lambda = 40$

h	$\|\nabla(u_h^k - u_h^{k-1})\|$	Order	$\|\nabla(v_h^k - v_h^{k-1})\|$	Order	$\|q_h^k - q_h^{k-1}\|$	Order	$\lvert c_h^k - c_h^{k-1}\rvert$	Order
1/8	1.0768e+00	–	1.0768e+00	–	3.5479e+00	–	8.5863e-02	–
1/16	6.3641e-01	0.7587	6.3641e-01	0.7587	2.5827e+00	0.4581	3.1001e-02	1.4697
1/32	3.3720e-01	0.9164	3.3720e-01	0.9164	1.3304e+00	0.9570	9.3145e-03	1.7348
1/64	1.7105e-01	0.9792	1.7105e-01	0.9792	6.6645e-01	0.9972	2.4852e-03	1.9061
1/128	8.5920e-02	0.9933	8.5920e-02	0.9933	3.3007e-01	1.0137	6.3005e-04	1.9798

Table 4 Errors and orders of convergence for Example 3 with $\lambda = 78$

h	$\\|\nabla(u_h^k - u_h^{k-1})\\|$	Order	$\\|\nabla(v_h^k - v_h^{k-1})\\|$	Order	$\\|q_h^k - q_h^{k-1}\\|$	Order	$\|c_h^k - c_h^{k-1}\|$	Order
1/8	1.6588e+00	–	1.6588e+00	–	2.0943e+01	–	1.8353e-01	–
1/16	6.2581e-01	1.4063	6.2581e-01	1.4063	9.4903e+00	1.1419	2.6873e-02	2.7718
1/32	5.9125e-01	0.0820	5.9125e-01	0.0820	7.9485e+00	0.2558	3.2962e-02	-0.2946
1/64	1.8561e-01	1.6715	1.8561e-01	1.6715	2.9408e+00	1.4345	1.6807e-03	4.2937
1/128	9.9283e-02	0.9027	9.9283e-02	0.9027	2.9709e+00	-0.0147	2.2620e-03	-0.4285

solutions for u, v and q) which is computed using discrete solutions at two consecutive mesh levels and the computed estimator at each mesh are reported in Fig. 1. The numerical results clearly show the optimal order of convergence and the estimator has effectively captured the corner singularity, see Fig. 2.

Example 3 Similar to the example 1, we consider the domain $\Omega = (0, 1)^2$ and $I = 4$. But we take two cases with $\lambda = 40$ and $\lambda = 78$ which lie between the first two least eigen-values λ_1 and λ_2 of

$$\begin{aligned} -\Delta\phi &= \lambda\phi \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Although the hypothesis $\lambda < \lambda_1$ is not satisfied, we would like to check the performance of the method. As in the example 1, we consider a sequence of uniformly refined meshes and compute the discrete solutions and the errors. In the first case when $\lambda = 40$, we observe the same performance as in the case of example 1, see

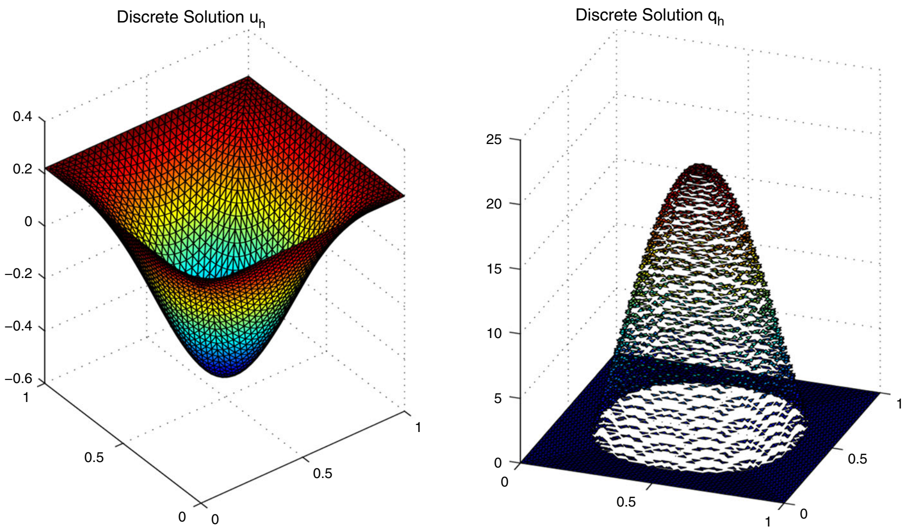

Fig. 3 Discrete solutions u_h (*left*) and q_h (*right*) for Example 3 with $\lambda = 40$

Table 3. But in the case when $\lambda = 78$, we observe deterioration in the performance which can be seen in Table 4. The discrete solutions u_h and q_h are shown in Fig. 3 for the example with $\lambda = 40$.

5 Conclusions

In this paper, we have studied a mixed formulation of a model free boundary plasma problem involving two positive constants λ and I. If $\lambda < \lambda_1$, where λ_1 is the smallest eigen-value of $-\Delta$ on the domain Ω with homogeneous Dirichlet boundary condition, it is shown that the model problem has a unique solution. A result on obtaining explicit form of the boundary value of the solution is derived. Using the same mixed formulation setting, we have proposed a finite element method and studied both *a priori* and *a posteriori* error estimates. Using a primal-dual active set strategy, we have performed numerical experiments which illustrate the theoretical results. The existence and uniqueness of the solution for the model problem is known when $\lambda < \lambda_2$, where λ_2 the second smallest eigen-value of $-\Delta$ on the domain Ω with homogeneous Dirichlet boundary condition, see [5, Theorem 11.2]. The discrete problem and its analysis for the case $\lambda_1 < \lambda < \lambda_2$ may require discrete maximum principles and we investigate this in the future despite our numerical experiments for this case seems to be inconclusive.

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