MULTIGRID METHODS FOR TWO-DIMENSIONAL MAXWELL'S EQUATIONS ON GRADED MESHES

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ABSTRACT. In this paper we investigate the numerical solution for two-dimensional Maxwell's equations on graded meshes. The approach is based on the Hodge decomposition. The solution $\boldsymbol{u}$ of Maxwell's equations is approximated by solving standard second order elliptic problems. Quasi-optimal error estimates for both $\boldsymbol{u}$ and $\nabla\times\boldsymbol{u}$ in the L_2 norm are obtained on graded meshes. We prove the uniform convergence of the W-cycle and full multigrid algorithms for the resulting discrete problem. The performance of these methods is illustrated by numerical results.

Keywords: Maxwell's equations, Hodge decomposition, graded meshes, multigrid methods.

1. Introduction

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ and $\boldsymbol{f}\in[L_2(\Omega)]^2$. We consider the following problem:
Find $\boldsymbol{u}\in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div}^0;\Omega)$ such that

$$(\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v})+\alpha(\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}), \tag{1.1}$$

for all $\boldsymbol{v}\in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div}^0;\Omega)$, where $(\cdot,\cdot)$ denotes the inner product of $[L_2(\Omega)]^2$. Here the function spaces $H_0(\mathrm{curl};\Omega)$ and $H(\mathrm{div}^0;\Omega)$ are defined as follows.

$$H(\mathrm{curl};\Omega)=\Big\{\boldsymbol{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix}\in[L_2(\Omega)]^2:\ \nabla\times\boldsymbol{v}=\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\in L_2(\Omega)\Big\},$$
$$H_0(\mathrm{curl};\Omega)=\{\boldsymbol{v}\in H(\mathrm{curl};\Omega):\ \boldsymbol{n}\times\boldsymbol{v}=0 \text{ on } \partial\Omega\},$$

with $\boldsymbol{n}$ being the unit outer normal, and

$$H(\mathrm{div};\Omega)=\Big\{\boldsymbol{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix}\in[L_2(\Omega)]^2:\ \nabla\cdot\boldsymbol{v}=\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}\in L_2(\Omega)\Big\},$$
$$H(\mathrm{div}^0;\Omega)=\Big\{\boldsymbol{v}\in H(\mathrm{div};\Omega):\nabla\cdot\boldsymbol{u}=0\}.$$

We assume that $-\alpha$ is not a Maxwell eigenvalue so that (1.1) is uniquely solvable. In particular, we assume that $\alpha\neq 0$ when Ω is not simply connected, since in this case, $\alpha=0$ is a Maxwell eigenvalue.

In this paper we follow the numerical approach to solve (1.1) presented in [13], which uses the Hodge decomposition of $\boldsymbol{u}$. The idea is to use two decoupled problems to first compute $\nabla\times\boldsymbol{u}$ and then $\boldsymbol{u}$. We summarize it in the rest of this section. Detailed justifications can be found in [13, Sections 2 and 3].

Let $\xi=\nabla\times\boldsymbol{u}\in H^1(\Omega)$. Then ξ is determined by

$$(\nabla\times\xi,\nabla\times\psi)+\alpha(\xi,\psi)=(\boldsymbol{f},\nabla\times\psi)\qquad\forall\,\psi\in H^1(\Omega) \tag{1.2}$$

when $\alpha \neq 0$, and by (1.2) together with the constraint

$$(\xi, 1) = \int_\Omega \xi \, dx = 0 \tag{1.3}$$

when Ω is simply connected and $\alpha = 0$.

We can write [21]

$$\boldsymbol{u} = \nabla \times \phi + \sum_{j=1}^{m} c_j \nabla \varphi_j, \tag{1.4}$$

where $\phi \in H^1(\Omega)$ satisfies

$$\frac{\partial \phi}{\partial n} = 0 \qquad \text{on } \partial\Omega, \tag{1.5}$$

and the constraint

$$(\phi, 1) = \int_\Omega \phi \, dx = 0. \tag{1.6}$$

The non-negative integer m is the Betti number for Ω ($m = 0$ if Ω is simply connected), and the functions $\varphi_1, \ldots, \varphi_m$ are defined as follows.

Suppose $\partial\Omega$ has $m+1$ components. We denote the outer boundary of Ω by Γ_0, and the m components of the inner boundary by $\Gamma_1, \ldots, \Gamma_m$. Then the functions φ_j are determined by

$$(\nabla \varphi_j, \nabla v) = 0 \qquad \forall\, v \in H_0^1(\Omega), \tag{1.7a}$$

$$\varphi_j\big|_{\Gamma_0} = 0, \tag{1.7b}$$

$$\varphi_j\big|_{\Gamma_i} = \delta_{ji} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \quad \text{for } 1 \leq i \leq m. \tag{1.7c}$$

We refer to (1.4) as the Hodge decomposition of $\boldsymbol{u} \in H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega)$.

The function ϕ is determined by (1.6) and the property that

$$(\nabla \times \phi, \nabla \times \psi) = (\xi, \psi), \quad \forall\, \psi \in H^1(\Omega). \tag{1.8}$$

Finally, in the case where $m \geq 1$, the coefficients c_j in (1.4) are determined by the symmetric positive-definite system (cf. [13, Section 2]):

$$\sum_{j=1}^{m} (\nabla \varphi_j, \nabla \varphi_i) c_j = \frac{1}{\alpha} (\boldsymbol{f}, \nabla \varphi_i), \quad \text{for } 1 \leq i \leq m. \tag{1.9}$$

The numerical procedure for solving (1.1) is as follows.

First we use (1.2) and (1.3) to compute a numerical approximation $\tilde{\xi}$ of ξ. Then we compute an approximation $\tilde{\phi}$ of ϕ by using (1.6) and (1.8), where ξ is replaced by $\tilde{\xi}$. The numerical approximations $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_m$ of $\varphi_1, \ldots, \varphi_m$ are computed by solving (1.7). Then we compute numerical approximations $\tilde{c}_1, \ldots, \tilde{c}_m$ of $c_1, \ldots, c_m$ by solving (1.9), where $\varphi_1, \ldots, \varphi_m$ are replaced with $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_m$. Finally, the numerical approximation $\tilde{\boldsymbol{u}}$ of $\boldsymbol{u}$ is given by

$$\tilde{\boldsymbol{u}} = \nabla \times \tilde{\phi} + \sum_{j=1}^{m} \tilde{c}_j \nabla \tilde{\varphi}_j.$$

Remark 1.1. Note that $(\nabla \times \zeta, \nabla \times \psi) = (\nabla \zeta, \nabla \psi), \ \forall\, \zeta, \psi \in H^1(\Omega)$. Hence the boundary value problems (1.2) and (1.8) for ξ and ϕ are Neumann problems for the Laplace operator.

The above numerical approach was demonstrated by a P_1 finite element method in [13] on quasi-uniform triangulations. In that case, the optimal convergence rates of the discrete errors can not be achieved when the domain is non-convex. In this paper, we extend the results using a properly graded triangulation $\mathcal{T}_h$ and recover optimal convergence rates for a general polygonal domain (cf. Section 3 and 4).

Multigrid methods have been proposed for magnetostatic problems in [25, 27], and for the time-domain Maxwell's equations in [24]. The convergence analysis is based on Nédélec's edge elements [30, 31]. In this paper an approximate solution for Maxwell's equations is obtained by solving standard second order scalar elliptic boundary value problems (cf. Section 2). Hence we can apply standard results in the convergence analysis for multigrid methods. The modified multigrid algorithms are also introduced and analyzed for the singular Neumann problems.

The rest of the paper is organized as follows. In Section 2, we briefly recall the P_1 finite element method introduced in [13] for solving (1.1). The elliptic regularity results in terms of weighted Sobolev space are reviewed in Section 3. The analysis of the numerical method based on graded meshes is carried out in Section 4. Then in Section 5 we introduce multigrid methods for the resulting discrete problems, followed by the convergence analysis of the W-cycle multigrid algorithms in Section 6. The full multigrid algorithms are analyzed in Section 7. Numerical results are reported in Section 8.

2. A P_1 Finite Element Method

Let $\mathcal{T}_h$ be a simplicial triangulation of Ω with mesh size h and $V_h \subset H^1(\Omega)$ be the P_1 finite element space associated with $\mathcal{T}_h$.

For $\alpha \neq 0$, the P_1 finite element method for (1.2) is to find $\xi_h \in V_h$ such that

$$(\nabla \times \xi_h, \nabla \times v) + \alpha(\xi_h, v) = (\boldsymbol{f}, \nabla \times v) \qquad \forall\, v \in V_h. \tag{2.1}$$

The problem (2.1) is well-posed for $\alpha > 0$ and also for $\alpha < 0$ provided $-\alpha$ is not a Maxwell eigenvalue and h is sufficiently small. It follows from (2.1) that

$$(\xi_h, 1) = 0. \tag{2.2}$$

When Ω is simply connected and $\alpha = 0$, $\xi_h \in V_h$ is defined by (2.1) together with the constraint (2.2). It is a well-posed problem because of the Poincaré-Friedrichs inequality (cf. [20])

$$\|v\|_{L_2(\Omega)} \leq C\big(|(v,1)| + \|\nabla \times v\|_{L_2(\Omega)}\big) \qquad \forall\, v \in H^1(\Omega). \tag{2.3}$$

The P_1 finite element method for (1.8) is to find $\phi_h \in V_h$ such that

$$(\nabla \times \phi_h, \nabla \times v) = (\xi_h, v) \qquad \forall\, v \in V_h, \tag{2.4a}$$

$$(\phi_h, 1) = 0. \tag{2.4b}$$

In the case where $m \geq 1$, the P_1 finite element approximation $\varphi_{j,h} \in V_h$ for the harmonic function φ_j in the Hodge decomposition (1.4) is determined by

$$(\nabla\varphi_{j,h}, \nabla v) = 0 \qquad \forall\ v \in V_h \cap H_0^1(\Omega), \tag{2.5a}$$

$$\varphi_{j,h}\big|_{\Gamma_0} = 0, \tag{2.5b}$$

$$\varphi_{j,h}\big|_{\Gamma_i} = \delta_{ji} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \quad \text{for } 1 \leq i \leq m. \tag{2.5c}$$

We then compute $c_{1,h}, \ldots, c_{m,h}$ by solving

$$\sum_{j=1}^{m} (\nabla\varphi_{j,h}, \nabla\varphi_{i,h}) c_{j,h} = \frac{1}{\alpha}(\boldsymbol{f}, \nabla\varphi_{i,h}), \quad \text{for } 1 \leq i \leq m. \tag{2.6}$$

Note that we assume $\alpha \neq 0$ when Ω is not simply connected.

Finally we define the piecewise constant approximation $\boldsymbol{u}_h$ of $\boldsymbol{u}$ by

$$\boldsymbol{u}_h = \nabla \times \phi_h + \sum_{j=1}^{m} c_{j,h} \nabla\varphi_{j,h}. \tag{2.7}$$

More details about the numerical schemes presented in this section can be found in [13, Section 4].

3. Elliptic Regularity

In view of Remark 1.1, the P_1 finite element method introduced in Section 2 involves the Neumann and Dirichlet problems for the Laplace operator. In this section we will review the elliptic regularity results for these problems in terms of weighted Sobolev space.

First we consider the Neumann problem of finding $z \in H^1(\Omega)$ such that

$$(\nabla z, \nabla v) + \alpha(z, v) = (f, v) \qquad \forall\, v \in H^1(\Omega), \tag{3.1}$$

where f belongs to the weighted Sobolev space $L_{2,\mu}(\Omega)$ (cf. (3.5) below) and $-\alpha$ is not a Maxwell eigenvalue.

Let $\omega_1, \ldots, \omega_L$ be the interior angles at the corners $c_1, \ldots, c_L$ of Ω. Let the weight $\Phi_\mu(T)$ associated with $T \in \mathcal{T}_h$ be defined by

$$\Phi_\mu(T) = \prod_{\ell=1}^{L} |c_\ell - c_T|^{1-\mu_\ell}, \tag{3.2}$$

where c_T is the center of T, and the grading parameters $\mu_1, \ldots, \mu_L$ are chosen according to the following rule:

$$\begin{aligned} &\mu_\ell = 1 && \text{if} \quad \omega_\ell \leq \pi, \\ &\mu_\ell < \frac{\pi}{\omega_\ell} && \text{if} \quad \omega_\ell > \pi. \end{aligned} \tag{3.3}$$

Definition 3.1. *We say that $\mathcal{T}_h$ is a properly graded mesh if*

$$h_T = diam(T) \approx \Phi_\mu(T)h \qquad \forall\, T \in \mathcal{T}_h. \tag{3.4}$$

Let the weighted Sobolev space $L_{2,\mu}(\Omega)$ be defined by

$$L_{2,\mu}(\Omega) = \{\zeta \in L_{2,\mathrm{loc}}(\Omega) :\ \|\zeta\|^2_{L_{2,\mu}(\Omega)} = \int_\Omega \phi_\mu^2(x)\zeta^2(x)\,dx < \infty\}, \tag{3.5}$$

where the weight function ϕ_μ is defined by

$$\phi_\mu(x) = \prod_{\ell=1}^{L} |x - c_\ell|^{1-\mu_\ell}. \tag{3.6}$$

Note that $L_2(\Omega) \subset L_{2,\mu}(\Omega)$ and

$$\|\zeta\|_{L_{2,\mu}(\Omega)} \leq C_\Omega \|\zeta\|_{L_2(\Omega)} \qquad \forall\, \zeta \in L_2(\Omega). \tag{3.7}$$

The model problem (3.1) has a unique solution z for any $f \in L_{2,\mu}(\Omega)$. The second order weak derivatives of z belong to $L_{2,\mu}$ and they satisfy

$$\|\partial^2 z/\partial x_i \partial x_j\|_{L_{2,\mu}(\Omega)} \leq C_\Omega \|f\|_{L_{2,\mu}(\Omega)} \quad \text{for } 1 \leq i,j \leq 2. \tag{3.8}$$

Moreover, when $\omega_\ell > \pi$ $(1 \leq \ell \leq L)$, we have $z \in H^{1+\mu_\ell}(\Omega_{\ell,\delta})$ and

$$\|z\|_{H^{1+\mu_\ell}(\Omega_{\ell,\delta})} \leq C_\Omega \|f\|_{L_{2,\mu}(\Omega)}, \tag{3.9}$$

where $\Omega_{\ell,\delta} = \{x \in \Omega :\ |x - c_\ell| < \delta\}$ is a small neighborhood around the corner c_ℓ. Details for (3.8) and (3.9) can be found in [26, 19, 29].

Let $\Pi_h : C(\bar{\Omega}) \longrightarrow V_h$ be the nodal interpolation operator for the P_1 finite element. The following interpolation error estimate in terms of weighted Sobolev norms is obtained in [2]. Similar result can be found in [14] for a discontinuous Galerkin method.

$$\|z - \Pi_h z\|_{L_2(\Omega)} + h|z - \Pi_h z|_{H^1(\Omega)} \leq Ch^2 \|f\|_{L_{2,\mu}(\Omega)}. \tag{3.10}$$

The preceding discussion also holds for the singular Neumann problem where $\alpha = 0$, provided that $(f, 1) = 0$ and z is the solution that satisfies the constraint $(z, 1) = 0$.

Next we consider the Dirichlet problem of finding $z \in H^1(\Omega)$ such that

$$(\nabla z, \nabla v) = 0 \qquad \forall\, v \in H^1_0(\Omega), \tag{3.11a}$$

$$z\big|_{\Gamma_j} = \gamma_j \qquad \text{for} \quad 0 \leq j \leq m, \tag{3.11b}$$

where $\gamma_0, \ldots, \gamma_m$ are constants.

According to elliptic regularity, we have the following interpolation error estimate [2]

$$\|z-\Pi_h z\|_{L_2(\Omega)} + h|z-\Pi_h z|_{H^1(\Omega)} \le Ch^2 \sum_{j=0}^{m} |\gamma_j|. \tag{3.12}$$

4. Error Analysis

The error estimates for the P_1 finite element method introduced in Section 2 are studied in [13] on quasi-uniform triangulations. In this section, we extend the results using a properly graded triangulation $\mathcal{T}_h$ and recover optimal convergence rates for a general polygonal domain Ω.

We begin by comparing ξ_h and $\xi = \nabla \times \boldsymbol{u}$ in the norm

$$\|\xi\|^2_{L_{2,-\mu}(\Omega)} = \int_\Omega \phi_\mu^{-2}(x)\xi^2(x)\,dx, \tag{4.1}$$

which is the norm for $L_{2,-\mu}(\Omega)$, the dual space of $L_{2,\mu}(\Omega)$. The triangulation $\mathcal{T}_h$ that is used in the rest of this section satisfies the property (3.4).

Theorem 4.1. *For $\alpha > 0$ (general Ω) and $\alpha = 0$ (simply connected Ω), we have*

$$\|\xi - \xi_h\|_{L_{2,-\mu}(\Omega)} \le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{4.2}$$

Proof. We will prove (4.2) by a duality argument.

Let $\zeta \in H^1(\Omega)$ be determined by

$$(\nabla\times\zeta, \nabla\times v) + \alpha(\zeta, v) = (\phi_\mu^{-2}(\xi-\xi_h), v) \qquad \forall\, v \in H^1(\Omega). \tag{4.3}$$

Note that (1.3) and (2.2) imply

$$(\xi - \xi_h, 1) = 0. \tag{4.4}$$

We derive from (3.5), (3.6) and (3.10) that

$$\begin{aligned} \|\zeta-\Pi_h\zeta\|_{L_2(\Omega)} + h|\zeta-\Pi_h\zeta|_{H^1(\Omega)} \le& Ch^2\|\phi_\mu^{-2}(\xi-\xi_h)\|_{L_{2,\mu}(\Omega)} \\ =& Ch^2\|\xi-\xi_h\|_{L_{2,-\mu}(\Omega)}. \end{aligned} \tag{4.5}$$

It follows from (4.3), the Galerkin orthogonality (cf. (1.2) and (2.1))

$$\big(\nabla\times(\xi-\xi_h), \nabla\times v\big) + \alpha(\xi-\xi_h, v) = 0 \qquad \forall\, v \in V_h, \tag{4.6}$$

and (4.5) that

$$\begin{aligned} \|\xi-\xi_h\|^2_{L_{2,-\mu}(\Omega)} &= \big(\nabla\times\zeta, \nabla\times(\xi-\xi_h)\big) + \alpha(\zeta, \xi-\xi_h) \\ &= \big(\nabla\times(\zeta-\Pi_h\zeta), \nabla\times(\xi-\xi_h)\big) + \alpha(\zeta-\Pi_h\zeta, \xi-\xi_h) \\ &\le C\big(\|\zeta-\Pi_h\zeta\|_{L_2(\Omega)} + \|\nabla\times(\zeta-\Pi_h\zeta)\|_{L_2(\Omega)}\big) \\ &\qquad \times \big(\|\xi-\xi_h\|_{L_2(\Omega)} + \|\nabla\times(\xi-\xi_h)\|_{L_2(\Omega)}\big) \\ &\le Ch\|\xi-\xi_h\|_{L_{2,-\mu}(\Omega)}\big(\|\xi-\xi_h\|_{L_2(\Omega)} + \|\nabla\times(\xi-\xi_h)\|_{L_2(\Omega)}\big), \end{aligned}$$

which together with (2.3) and (4.4) implies,

$$\|\xi - \xi_h\|_{L_{2,-\mu}(\Omega)} \le Ch|\xi - \xi_h|_{H^1(\Omega)}. \tag{4.7}$$

Now we estimate $|\xi - \xi_h|_{H^1(\Omega)}$. Let $v \in V_h$ satisfy $(v, 1) = 0$. It follows from (2.3), (4.4), and (4.6) that

$$\begin{aligned}|\xi - \xi_h|^2_{H^1(\Omega)} + \alpha\|\xi - \xi_h\|^2_{L_2(\Omega)} &= \big(\nabla \times (\xi - \xi_h), \nabla \times (\xi - v)\big) + \alpha(\xi - \xi_h, \xi - v)\\ &\le C|\xi - \xi_h|_{H^1(\Omega)}|\xi - v|_{H^1(\Omega)},\end{aligned}$$

which implies

$$|\xi - \xi_h|_{H^1(\Omega)} \le C|\xi - v|_{H^1(\Omega)} \qquad \forall\, v \in V_h \ (v, 1) = 0. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\|\xi - \xi_h\|_{L_{2,-\mu}(\Omega)} \le Ch \inf_{v\in V_h} |\xi - v|_{H^1(\Omega)}. \tag{4.9}$$

Under the assumption that $\boldsymbol{f} \in [L_2(\Omega)]^2$, we have the following stability estimate from the well-posedness of the continuous problem:

$$\|\xi\|_{H^1(\Omega)} \le C\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{4.10}$$

Therefore the estimate (4.2) follows from (4.9) and (4.10) □

Theorem 4.2. *The discrete problem* (2.1) *is well-posed for $\alpha < 0$, provided $-\alpha$ is not a Maxwell eigenvalue and h is sufficiently small. Under these conditions the estimate* (4.2) *remains valid.*

Proof. We follow the approach of Schatz (cf. [32]). Assuming that (2.1) has a solution $\xi_h \in V_h$, we can apply the same duality argument in the proof of Theorem 4.1 to obtain the estimate (4.7).

Let $v \in V_h$ satisfy $(v, 1) = 0$. It follows from (1.3), (2.2), (2.3), (4.6) and (4.7) that

$$\begin{aligned}|\xi - \xi_h|^2_{H^1(\Omega)} &= \big(\nabla \times (\xi - \xi_h), \nabla \times (\xi - v)\big) + \alpha(\xi - \xi_h, \xi - v) - \alpha\|\xi - \xi_h\|^2_{L_2(\Omega)}\\ &\le C\|\nabla \times (\xi - \xi_h)\|_{L_2(\Omega)}\|\nabla \times (\xi - v)\|_{L_2(\Omega)} + |\alpha|\|\xi - \xi_h\|^2_{L_{2,-\mu}(\Omega)}\\ &\le C\Big(|\xi - \xi_h|_{H^1(\Omega)}|\xi - v|_{H^1(\Omega)} + h^2|\xi - \xi_h|^2_{H^1(\Omega)}\Big),\end{aligned}$$

and hence, for h sufficiently small,

$$|\xi - \xi_h|_{H^1(\Omega)} \le C|\xi - v|_{H^1(\Omega)} \qquad \forall\, v \in V_h,$$

which again implies (4.8).

In the special case where $\boldsymbol{f} = \boldsymbol{0}$, $\xi = 0$ and $v = 0$, we deduce from (2.2) and (4.8) that the only solution of the homogeneous discrete problem is trivial. Hence the discrete problem (2.1) is well-posed for h sufficient small, and then the estimate (4.2) follows from (4.7), (4.8) and (4.10). □

Corollary 4.3. *Under the conditions in Theorems 4.1 and 4.2, we have*

$$\|\xi - \xi_h\|_{L_2(\Omega)} \le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{4.11}$$

Next we compare ϕ_h and ϕ.

Lemma 4.4. *Under the assumption that $\boldsymbol{f} \in [L_2(\Omega)]^2$, we have*

$$|\phi - \phi_h|_{H^1(\Omega)} \le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{4.12}$$

Proof. Since $(\xi, 1) = 0$, there exists a unique solution $\tilde{\phi}_h$ such that

$$(\nabla \times \tilde{\phi}_h, \nabla \times v) = (\xi, v) \qquad \forall\, v \in V_h, \tag{4.13a}$$

$$(\tilde{\phi}_h, 1) = 0. \tag{4.13b}$$

It follows from (2.4) and (4.13) that

$$\big(\nabla \times (\tilde{\phi}_h - \phi_h), \nabla \times v) = (\xi - \xi_h, v) \qquad \forall\, v \in V_h, \tag{4.14}$$

and $(\tilde{\phi}_h - \phi_h, 1) = 0$. Hence by (2.3), (4.11) and (4.14), we have

$$\begin{aligned} |\tilde{\phi}_h - \phi_h|^2_{H^1(\Omega)} &= \|\nabla \times (\tilde{\phi}_h - \phi_h)\|^2_{L_2(\Omega)} \\ &= (\xi - \xi_h, \tilde{\phi}_h - \phi_h) \\ &\le \|\xi - \xi_h\|_{L_2(\Omega)} \|\tilde{\phi}_h - \phi_h\|_{L_2(\Omega)} \\ &\le Ch\|\boldsymbol{f}\|_{L_2(\Omega)} |\tilde{\phi}_h - \phi_h|_{H^1(\Omega)}, \end{aligned} \tag{4.15}$$

which implies

$$|\tilde{\phi}_h - \phi_h|_{H^1(\Omega)} \le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{4.16}$$

Combining (1.8) and (4.13a), we have the Galerkin relation

$$(\nabla \times (\phi - \tilde{\phi}_h), \nabla \times v) = 0 \qquad \forall\, v \in V_h, \tag{4.17}$$

which together with (3.7) and (3.10) implies

$$|\phi - \tilde{\phi}_h|_{H^1(\Omega)} = \inf_{v \in V_h} |\phi - v|_{H^1(\Omega)} \le |\phi - \Pi_h \phi|_{H^1(\Omega)} \le Ch\|\xi\|_{L_2(\Omega)}. \tag{4.18}$$

The estimate (4.12) follows from (4.10), (4.16) and (4.18). □

We then turn to compare $\varphi_{j,h}$ and φ_j. Since the function $\varphi_{j,h}$ is obtained by a standard P_1 finite element method for the Dirichlet problem (2.5), the following result is standard [2, Theorem 5.1].

Lemma 4.5. *For $1 \le j \le m$, we have*

$$|\varphi_j - \varphi_{j,h}|_{H^1(\Omega)} \le Ch. \tag{4.19}$$

We compare $c_{j,h}$ and c_j in the next lemma. The proof is based on (1.9), (2.6), Lemma 4.5 and similar to the proof of Lemma 4.7 in [13].

Lemma 4.6. *For $1 \le j \le m$, we have*

$$|c_j - c_{j,h}| \le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{4.20}$$

By combining Lemma 4.4, Lemma 4.5 and Lemma 4.6, we have the following convergence theorem for (2.7). The proof is identical with the proof of Theorem 4.9 in [13].

Theorem 4.7.

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L_2(\Omega)} \le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{4.21}$$

5. Multigrid Algorithm

In this section we introduce W-cycle multigrid algorithms for solving the discrete problems (2.1), (2.4) and (2.5) on graded meshes. We start with an initial triangulation $\mathcal{T}_0$ and then obtain the triangulations $\mathcal{T}_k$ for $k \ge 1$ recursively by the following procedure, which is identical to the one in [8].

- If none of the reentrant corners is a vertex of $T \in \mathcal{T}_k$, then we divide T uniformly by connecting the midpoints of the edges of T.
- If a reentrant corner c_ℓ is a vertex of $T \in \mathcal{T}_k$ and the other two vertices of T are denoted by p_1 and p_2, then we divide T by connecting the points m, q_1 and q_2 (cf. Figure 5.1). Here m is the midpoint of the edge p_1p_2 and q_1 (resp. q_2) is the point on the edge $c_\ell p_1$ (resp. $c_\ell p_2$) such that
$$\frac{|c_\ell - q_i|}{|c_\ell - p_i|} = 2^{-(1/\mu_\ell)} \qquad \text{for} \quad i = 1, 2,$$
where μ_ℓ is the grading factor chosen according to (3.3).

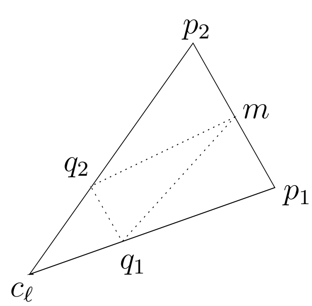

Figure 5.1. Refinement of a triangle at a reentrant corner

The triangulations $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ for an L-shaped domain are depicted in Figure 5.2, where the grading factor at the reentrant corner is taken to be $2/3$.

The resulting family of triangulations $\{\mathcal{T}_k\}_{k\ge 0}$ satisfies the mesh condition (3.4) (cf. the Appendix of [14]). Without loss of generality we may also assume that

$$h_k = \frac{1}{2}\, h_{k-1} \qquad \text{for } k \ge 1. \tag{5.1}$$

Let V_k be the P_1 finite element space associated with $\mathcal{T}_k$ and define

$$a(w, v) = (\nabla \times w, \nabla \times v) + \alpha(w, v). \tag{5.2}$$

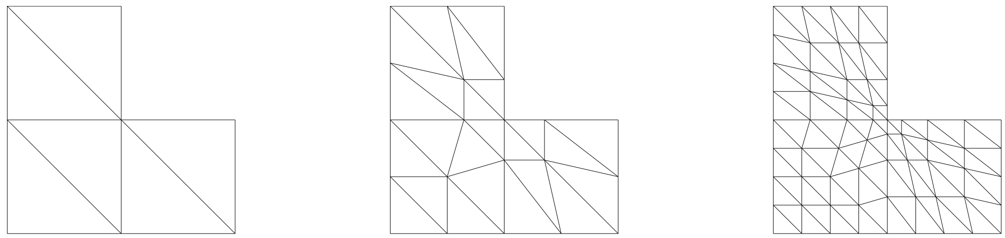

FIGURE 5.2. The triangulations $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ for an L-shaped domain

Let the operators A_k be defined by

$$\langle A_k w, v\rangle = a(w,v) \quad \forall\, v, w \in V_k, \tag{5.3}$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $V_k'\times V_k$. The k-th level P_1 finite element method for (1.2) ($\alpha\neq 0$) is:
Find $\xi_k\in V_k$ such that

$$A_k\xi_k = f_k, \tag{5.4}$$

where $f_k\in V_k'$ is defined by

$$\langle f_k, v\rangle = (\boldsymbol{f}, \nabla\times v) \qquad \forall\, v\in V_k.$$

The equation (5.4) then can be solved by multigrid algorithms [23, 28, 7, 33, 16].

Since the finite element spaces are nested, we can take the coarse-to-fine intergrid transfer operator $I_{k-1}^k : V_{k-1}\longrightarrow V_k$ to be the natural injection and define the fine-to-coarse intergrid transfer operator $I_k^{k-1} : V_k' \longrightarrow V_{k-1}'$ to be the transpose of I_{k-1}^k with respect to $\langle\cdot,\cdot\rangle$, i.e.,

$$\langle I_k^{k-1} w, v\rangle = \langle w, I_{k-1}^k v\rangle \qquad \forall\, w\in V_k',\ v\in V_{k-1}. \tag{5.5}$$

In order to define the smoother, we first introduce an operator $B_k : V_k\longrightarrow V_k'$ defined by

$$\langle B_k w, v\rangle = h_k^2 \sum_{T\in\mathcal{T}_k}\sum_{p\in\mathcal{N}_T} w(p)v(p) = (v,w)_k \qquad \forall\, v, w\in V_k, \tag{5.6}$$

where $\mathcal{N}_T$ is the set of the vertices of the triangle T.

We are now ready to define a W-cycle algorithm for the equation

$$A_k z = g, \tag{5.7}$$

where $z\in V_k$ and $g\in V_k'$.

Algorithm 5.1. W-cycle Algorithm.

The output of the algorithm is denoted by $MG_W(k, g, z_0, m_1, m_2)$, where $z_0\in V_k$ is the initial guess and m_1 (resp. m_2) is the number of pre-smoothing (resp. post-smoothing) steps.

For $k=0$, $MG_W(0, g, z_0, m_1, m_2) = A_0^{-1} g$.

For $k\geq 1$, $MG_W(k, g, z_0, m_1, m_2)$ is computed recursively as follows.

Pre-Smoothing. Compute $z_l\in V_k$ for $1\leq l\leq m_1$ recursively by

$$z_l = z_{l-1} + (\lambda h_k^2) B_k^{-1}(g - A_k z_{l-1}),$$

where λ is a (constant) damping factor such that the spectral radius $\rho(\lambda h_k^2 B_k^{-1} A_k)$ satisfies

$$\rho(\lambda h_k^2 B_k^{-1} A_k) < 1 \qquad \text{for } k \geq 0. \tag{5.8}$$

Coarse-Grid Correction. Compute $q \in V_{k-1}$ by

$$\begin{aligned} r_{k-1} =& I_k^{k-1}(g - A_k z_{m_1}), \\ q' =& MG_W(k-1, r_{k-1}, 0, m_1, m_2), \\ q =& MG_W(k-1, r_{k-1}, q', m_1, m_2), \end{aligned}$$

and take

$$z_{m_1+1} = z_{m_1} + I_{k-1}^k q.$$

Post-Smoothing. Compute $z_l \in V_k$ for $m_1 + 2 \leq l \leq m_1 + m_2 + 1$ recursively by

$$z_l = z_{l-1} + (\lambda h_k^2) B_k^{-1}(g - A_k z_{l-1}).$$

The final output is

$$MG_W(k, g, z_0, m_1, m_2) = z_{m_1+m_2+1}.$$

Remark 5.2. We can also apply Algorithm 5.1 to solve the k-th level discrete problem (2.5) for the Dirichlet boundary value problem (1.7).

The multigrid algorithm can be modified to solve (1.2) ($\alpha = 0$), which is a singular Neumann problem. Let $\widehat{V}_k = \{v \in V_k : (v, 1) = 0\}$. The k-th level P_1 discrete problem for (1.2) ($\alpha = 0$) is as follows:
Find $\xi_k \in \widehat{V}_k$ such that

$$A_k \xi_k = f_k. \tag{5.9}$$

Remark 5.3. The unique solution $\xi_k \in \widehat{V}_k$ of (5.9) is determined by

$$(\nabla \times \xi_k, \nabla \times v) = (\boldsymbol{f}, \nabla \times v) \qquad \forall\, v \in \widehat{V}_k.$$

Let $\widehat{P}_k$ be the orthogonal projection from V_k onto $\widehat{V}_k$ with respect to the inner product $(\cdot,\cdot)_k$ (cf. (5.6)), i.e., given any $v \in V_k$, $\widehat{P}_k v \in \widehat{V}_k$ satisfies

$$(w, \widehat{P}_k v)_k = (w, v)_k \qquad \forall\, w \in \widehat{V}_k. \tag{5.10}$$

Remark 5.4. One can compute $\widehat{P}_k v$ explicitly as

$$\widehat{P}_k v = v - [(v, s_k)_k / (s_k, s_k)_k] s_k,$$

where $s_k \in V_k$ spans the orthogonal complement of $\widehat{V}_k$ with respect to $(\cdot,\cdot)_k$. More precisely, let $\mathcal{N}_k$ be the set of all the nodes associated with V_k. We can take s_k to be the finite element function defined by

$$s_k(p) = \frac{1}{3h_k^2 n(\mathcal{T}_p)} \sum_{T \in \mathcal{T}_p} |T| \qquad \forall\, p \in \mathcal{N}_k,$$

where $\mathcal{T}_p$ is the set of the triangles in $\mathcal{T}_k$ sharing p as a common vertx, $n(\mathcal{T}_p)$ is the number of triangles in $\mathcal{T}_p$, and $|T|$ is the area of T.

Remark 5.5. Let $\widehat{I}_k : \widehat{V}_k \longrightarrow V_k$ be the natural injection. Then the operator $\widehat{A}_k = \widehat{P}_k \circ B_k^{-1} \circ A_k \circ \widehat{I}_k$, satisfies

$$(\widehat{A}_k w, v)_k = (\nabla \times w, \nabla \times v) \qquad \forall\, v, w \in \widehat{V}_k.$$

We consider the W-cycle algorithm for the following equation:

$$A_k z = g, \tag{5.11}$$

where $z \in \widehat{V}_k$, $g \in V_k'$ and $\langle g, 1\rangle = 0$.

Algorithm 5.6. The output of the algorithm is denoted by $\overline{MG}_W(k, g, z_0, m_1, m_2)$, where $z_0 \in \widehat{V}_k$ is the initial guess and m_1 (resp. m_2) is the number of pre-smoothing (resp. post-smoothing) steps.

For $k = 0$, $\overline{MG}_W(k, g, z_0, m_1, m_2) = \widehat{A}_0^{-1}(\widehat{P}_0 B_0^{-1} g)$.

For $k \geq 1$, $\overline{MG}_W(k, g, z_0, m_1, m_2)$ is computed recursively as follows.

Pre-Smoothing. Compute $z_l \in \widehat{V}_k$ for $1 \leq l \leq m_1$ recursively by

$$z_l = z_{l-1} + (\lambda h_k^2)\widehat{P}_k B_k^{-1}(g - A_k z_{l-1}), \tag{5.12}$$

where λ is a (constant) damping factor such that λh_k^2 dominates the spectral radius of $B_k^{-1} A_k$.

Coarse-Grid Correction. Compute

$$\begin{aligned} r_{k-1} =& I_k^{k-1}(g - A_k z_{m_1}), \\ q' =& \overline{MG}_W(k-1, r_{k-1}, 0, m_1, m_2), \\ q =& \overline{MG}_W(k-1, r_{k-1}, q', m_1, m_2), \end{aligned}$$

and take

$$z_{m_1+1} = z_{m_1} + I_{k-1}^k q.$$

Post-Smoothing. Compute $z_l \in \widehat{V}_k$ for $m_1 + 2 \leq l \leq m_1 + m_2 + 1$ recursively by

$$z_l = z_{l-1} + (\lambda h_k^2)\widehat{P}_k B_k^{-1}(g - A_k z_{l-1}). \tag{5.13}$$

The final output is

$$\overline{MG}_W(k, g, z_0, m_1, m_2) = z_{m_1+m_2+1}.$$

Remark 5.7. By introducing the operators $\widehat{I}_k$ and $\widehat{P}_k$, we can perform all the computations in Algorithm 5.6 in the space V_k instead of $\widehat{V}_k$. Since there is a natrual basis for V_k, it is easy to implement Algorithm 5.6 in practice [4]. In view of Remark 5.5, we can rewrite (5.12) and (5.13) as

$$z_l = z_{l-1} + (\lambda h_k^2)(\widehat{P}_k B_k^{-1} g - \widehat{A}_k z_{l-1}).$$

Hence Algorithm 5.6 is essentially the same as Algorithm 5.1.

Remark 5.8. We can also apply Algorithm 5.6 to solve the singular Neumann problem (1.8).

6. Convergence of the W-Cycle Algorithm

We follow the ideas in [34, 6, 3, 5, 15] to analyze the W-cycle multigrid algorithm for the discrete problem (5.7) in this section.

Let $E_k : V_k \longrightarrow V_k$ be the k-th level error propagation operator for Algorithm 5.1. We have the following well-known recursive relation [23, 16]:

$$E_k = R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k E_{k-1}^2 P_k^{k-1}) R_k^{m_1}, \tag{6.1}$$

where Id_k is the identity operator on V_k. The operator $R_k : V_k \longrightarrow V_k$ which measures the effect of one smoothing step is defined by

$$R_k = Id_k - (\lambda h_k^2) B_k^{-1} A_k, \tag{6.2}$$

and the operator $P_k^{k-1} : V_k \longrightarrow V_{k-1}$ is the transpose of I_{k-1}^k with respect to the variational forms, i.e.,

$$a(P_k^{k-1} w, v) = a(w, I_{k-1}^k v) \qquad \forall\, v \in V_{k-1},\ w \in V_k. \tag{6.3}$$

Let the operators $M_k, N_k : V_k \longrightarrow V_k'$ be defined by

$$\langle M_k w, v\rangle = (\nabla \times w, \nabla \times v) + |\alpha|(w, v) \qquad \forall\, v, w \in V_k, \tag{6.4}$$

$$\langle N_k w, v\rangle = (|\alpha| - \alpha)(w, v) \qquad \forall\, v, w \in V_k. \tag{6.5}$$

It is clear that the operators have the following relation

$$A_k = M_k - N_k \qquad \forall\ k \geq 0. \tag{6.6}$$

For $\alpha \neq 0$, let the mesh-dependent norms $|\!|\!| v |\!|\!|_{j,k}$ for $j = 0, 1, 2$ and $k \geq 1$ be defined by

$$|\!|\!| v |\!|\!|_{j,k} = \sqrt{\langle B_k (B_k^{-1} M_k)^j v, v\rangle} \qquad \forall\, v \in V_k,\ k \geq 1. \tag{6.7}$$

In view of (6.4), we have

$$|\!|\!| v |\!|\!|_{0,k}^2 = \langle B_k v, v\rangle \qquad \forall\, v \in V_k, \tag{6.8}$$

$$|\!|\!| v |\!|\!|_{1,k}^2 = \langle M_k v, v\rangle \approx \|v\|_{H^1(\Omega)}^2 \qquad \forall\, v \in V_k. \tag{6.9}$$

Also the Cauchy-Schwarz inequality implies that

$$|\!|\!| v |\!|\!|_{2,k} = \max_{w \in V_k \setminus \{0\}} \frac{\langle M_k v, w\rangle}{|\!|\!| w |\!|\!|_{0,k}} \qquad \forall\, v \in V_k. \tag{6.10}$$

There is an important connection between the mesh-dependent norm $|\!|\!| \cdot |\!|\!|_{0,k}$ and the norm $\| \cdot \|_{L_{2,-\mu}(\Omega)}$:

$$|\!|\!| v |\!|\!|_{0,k}^2 = h_k^2 \sum_{T \in \mathcal{T}_k} \sum_{p \in \mathcal{N}_T} [v(p)]^2 \approx \|v\|_{L_{2,-\mu}(\Omega)}^2 \qquad \forall\, v \in V_k, \tag{6.11}$$

where the positive constants in the equivalence depend only on the shape regularity of $\mathcal{T}_h$. From (6.7) and (6.9), we immediately have

$$|\!|\!| v |\!|\!|_{t,k} \leq C h_k^{s-t} |\!|\!| v |\!|\!|_{s,k} \qquad \forall\, v \in V_k,\ k \geq 1,\ t \geq s. \tag{6.12}$$

Lemma 6.1. *For $\alpha < 0$, we have*

$$\|B_k^{-1}N_k v\|_{0,k} \le C\|v\|_{0,k}, \quad \forall\ k \ge 1. \tag{6.13}$$

Proof. It follows from (6.5) and (6.11) that

$$\langle N_k w, v\rangle \le C\int_\Omega wv\,dx \le C\|w\|_{0,k}\|v\|_{0,k} \quad \forall\, v, w \in V_k,\ k \ge 1. \tag{6.14}$$

In view of (6.8) and (6.14), we have

$$\begin{aligned}\|B_k^{-1}N_k v\|_{0,k}^2 \le&\langle B_k B_k^{-1}N_k v, B_k^{-1}N_k v\rangle \\ \le& C\|v\|_{0,k}\|B_k^{-1}N_k v\|_{0,k},\end{aligned} \tag{6.15}$$

which implies (6.13). □

In view of (6.2) and (6.6), we can write

$$\begin{aligned}R_k =& Id_k - (\lambda h_k^2)B_k^{-1}(M_k - N_k) \\ =& \widetilde{R}_k + \widetilde{N}_k,\end{aligned} \tag{6.16}$$

where

$$\widetilde{R}_k = Id_k - (\lambda h_k^2)B_k^{-1}M_k \tag{6.17}$$

$$\widetilde{N}_k = (\lambda h_k^2)B_k^{-1}N_k. \tag{6.18}$$

The following smoothing properties of $\widetilde{R}_k$ are standard [4, 23, 16]:

$$\|\widetilde{R}_k v\|_{j,k} \le C\|v\|_{j,k} \qquad \forall\, v \in V_k,\ k \ge 1,\ j = 0, 1. \tag{6.19}$$

$$\|\widetilde{R}_k^\ell v\|_{j+1,k} \le Ch_k^{-1}\ell^{-1/2}\|v\|_{j,k} \qquad \forall\, v \in V_k,\ \ell, k \ge 1,\ j = 0, 1. \tag{6.20}$$

Combining (6.13) and (6.18), we immediately have

$$\|\widetilde{N}_k v\|_{0,k} \le Ch_k^2\|v\|_{0,k} \qquad \forall\ v \in V_k,\ k \ge 1. \tag{6.21}$$

We now follow the ideas in [15] to prove a smoothing property for Algorithm 5.1.

Lemma 6.2. *For $\alpha \neq 0$, there exist positive constants γ, C_0, C_1, C_2 and C_3 independent of k such that*

$$\|R_k^m v\|_{0,k} \le C_0(1 + C_1\gamma)^m\|v\|_{0,k} \qquad \forall\, v \in V_k,\ k \ge 1, \tag{6.22}$$

$$\|R_k^m v\|_{2,k} \le C_2 h_k^{-1}m^{-1/2}(1 + C_3\gamma)^m\|v\|_{1,k} \qquad \forall\, v \in V_k,\ m, k \ge 1, \tag{6.23}$$

provided that $h_1^2 \le \gamma$.

Proof. In the case where $\alpha > 0$, (6.22) and (6.23) are direct consequences of (6.19) and (6.20), since $R_k = \widetilde{R}_k$.

In the case where $\alpha < 0$, it is easy to show by mathematical induction that

$$R_k^m = (\widetilde{R}_k + \widetilde{N}_k)^m \tag{6.24}$$

$$=\widetilde{R}_k^m + \sum_{\ell=1}^{m} \widetilde{R}_k^{\ell-1}\widetilde{N}_k(\widetilde{R}_k+\widetilde{N}_k)^{m-\ell} \qquad \forall\ m \geq 1.$$

Therefore, it follows from (6.19), (6.21) and (6.22) that

$$\begin{aligned}
\|R_k^m v\|_{0,k} \leq & C\|\widetilde{R}_k^m v\|_{0,k} + \sum_{\ell=1}^{m} \|\widetilde{R}_k^{\ell-1}\widetilde{N}_k(\widetilde{R}_k+\widetilde{N}_k)^{m-\ell} v\|_{0,k} \\
\leq & C\|v\|_{0,k} + \sum_{\ell=1}^{m} h_k^2 \|(\widetilde{R}_k+\widetilde{N}_k)^{m-\ell} v\|_{0,k} \\
\leq & C\|v\|_{0,k} + Ch_k^2\big((1+Ch_k^2)^{m-1} + (1+Ch_k^2)^{m-2} \\
& \qquad + (1+Ch_k^2)^{m-3} + \cdots + 1\big)\|v\|_{0,k} \\
\leq & C_0\big(1+mh_k^2(1+Ch_k^2)^{m-1}\big)\|v\|_{0,k} \\
\leq & C_0(1+Ch_k^2)^{2m}\|v\|_{0,k} \\
\leq & C_0(1+2Ch_k^2+C^2h_k^4)^m\|v\|_{0,k} \\
\leq & C_0(1+C_1h_k^2)^m\|v\|_{0,k},
\end{aligned} \tag{6.25}$$

which implies (6.22).

The estimate (6.23) can be found in [3] for the analysis of the W-cycle multigrid algorithm with a standard Richardson iteration as smoother. Note that by using weighted Sobolev spaces and graded meshes, we can treat the convergence of the multigrid algorithm with full elliptic regularity (i.e., the index of elliptic regularity equals 1). ☐

The approximation property for the convergence analysis of the multigrid algorithms is provided by the next lemma. It can be proved using (3.10), the fact that

$$a((Id_k - I_{k-1}^k P_k^{k-1})v, w) = 0 \qquad \forall\, v \in V_k,\ w \in V_{k-1},$$

and a duality argument [12, Lemma 5.3]. The proof is similar to that of [34, Theorem 2.5].

Lemma 6.3. *For $\alpha \neq 0$, there exists a positive constant C independent of k such that*

$$\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{0,k} \leq Ch_k^2\|v\|_{2,k} \qquad \forall\, v \in V_k,\ k \geq 1. \tag{6.26}$$

In the case where $\alpha \neq 0$, we have the following theorem on the two-grid algorithm with m pre-smoothing and post-smoothing steps.

Theorem 6.4. *There exists a positive number γ and a positive constant C_* independent of k such that*

$$\|R_k^m(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|_{0,k} \leq C_* m^{-1/2}\|v\|_{0,k} \qquad \forall\, v \in V_k,\ k \geq 1, \tag{6.27}$$

provided that $h_1^2 \leq \gamma$ and m is sufficiently large.

Proof. The estimate (6.27) follows from (6.12), (6.22), (6.23) and (6.26):

$$\begin{aligned}
\|R_k^m(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|_{0,k} &\le C_0(1+C_1\gamma)^m \|(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|_{0,k} \\
&\le C_4(1+C_1\gamma)^m h_k^2 \|R_k^m v\|_{2,k} \\
&\le C_5 m^{-1/2}[(1+C_1\gamma)(1+C_3\gamma)]^m \|v\|_{0,k}.
\end{aligned}$$

Let $C_* = 2C_5$, we can choose m and then γ such that $C_5 m^{-1/2}[(1+C_1\gamma)(1+C_3\gamma)]^m \le C_* m^{-1/2}$, which implies (6.27). □

We have the operator bounds in the following lemma.

Lemma 6.5. *There exists a positive constant C independent of k such that for $\alpha \neq 0$,*

$$\|I_{k-1}^k v\|_{0,k} \le C\|v\|_{0,k-1} \qquad \forall\, v \in V_{k-1}, \tag{6.28}$$

$$\|P_k^{k-1} v\|_{0,k-1} \le C\|v\|_{0,k} \qquad \forall\, v \in V_k, \tag{6.29}$$

and for $\alpha > 0$,

$$\|I_{k-1}^k v\|_{1,k} \le C\|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}, \tag{6.30}$$

$$\|P_k^{k-1} v\|_{1,k-1} \le C\|v\|_{1,k} \qquad \forall\, v \in V_k. \tag{6.31}$$

Proof. The estimate (6.28) follows immediately from (6.11) and the fact that $I_{k-1}^k : V_{k-1} \to V_k$ is natural injection.

By combining (5.2), (6.3), (6.9), (6.11) and duality, we have (6.29), (6.30) and (6.31). □

Finally, we obtain the following convergence theorem for the W-cycle algorithm for the P_1 finite element method. The proof is based on Theorem 6.4, Lemma 6.5 and a perturbation argument [14]. An estimate similar to (6.33) below can be found in [3, 34]. Similar results are also obtained in [6, 5] for W-cycle and V-cycle multigrid methods applied to nonsymmetric and indefinite problems.

Theorem 6.6. *For any $0 < \delta < 1$, there exists a positive number γ independent of k, such that for $\alpha \neq 0$,*

$$\|z - MG_W(k, g, z_0, m, m)\|_{0,k} \le \delta \|z - z_0\|_{0,k}, \tag{6.32}$$

provided that $h_1^2 \le \gamma$ and m is large enough, and for $\alpha > 0$,

$$\|z - MG_W(k, g, z_0, m, m)\|_{1,k} \le \delta \|z - z_0\|_{1,k}, \tag{6.33}$$

provided that m is large enough.

Remark 6.7. For $\alpha = 0$, the equation (6.9) becomes

$$\|v\|_{1,k}^2 = \langle A_k v, v\rangle = |v|_{H^1(\Omega)}^2 \qquad \forall\, v \in \widehat{V}_k. \tag{6.34}$$

Therefore,

$$|\xi - \xi_k|_{H^1(\Omega)} = \inf_{v \in \widehat{V}_k} |\xi - v|_{H^1(\Omega)}$$

$$\leq \big|\xi - \big(\Pi_k\xi - \frac{1}{|\Omega|}\int_\Omega \Pi_k\xi\big)\big|_{H^1(\Omega)} \tag{6.35}$$
$$= |\xi - \Pi_k\xi|_{H^1(\Omega)}.$$

Hence the convergence results obtained in Theorem 6.6 are valid if we replace V_k by $\widehat{V}_k$.

7. Full Multigrid Methods

In the application of k-th level iteration to (5.4), we use the following full multigrid algorithm, where we apply multigrid algorithm r times at each level.

Algorithm 7.1. Full Multigrid Algorithm for (5.4).
For $k = 0$, $\widehat{\xi}_0 = A_0^{-1} f_0$.
For $k \geq 1$, the approximation solution $\widehat{\xi}_k$ is obtained recursively from

$$\begin{aligned}
\xi_0^k &= I_{k-1}^k \widehat{\xi}_{k-1},\\
\xi_\ell^k &= MG_W(k, f_k, \xi_{\ell-1}^k, m, m), \quad 1 \leq \ell \leq r,\\
\widehat{\xi}_k &= \xi_r^k.
\end{aligned}$$

The following theorem shows that the convergence of the full multigrid method is a simple consequence of the convergence of the k-th level iteration.

Theorem 7.2. *If r is large enough, then there exists a positive constant C such that*

$$\|\xi - \widehat{\xi}_k\|_{L_2(\Omega)} \leq C h_k \|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{7.1}$$

Proof. Define $\widehat{e}_k = \xi_k - \widehat{\xi}_k$. In particular, $\hat{e}_0 = 0$. By combining (4.2), (5.1), (6.11) and (6.32), we have

$$\begin{aligned}
\|\widehat{e}_k\|_{L_{2,-\mu}(\Omega)} &\approx |\!|\!|\widehat{e}_k|\!|\!|_{0,k}\\
&\leq \delta^r |\!|\!|\xi_k - \widehat{\xi}_{k-1}|\!|\!|_{0,k}\\
&\leq C\delta^r [\|\xi_k - \xi\|_{L_{2,-\mu}(\Omega)} + \|\xi - \xi_{k-1}\|_{L_{2,-\mu}(\Omega)} + |\!|\!|\xi_{k-1} - \widehat{\xi}_{k-1}|\!|\!|_{0,k}]\\
&\leq C\delta^r [h_k \|\boldsymbol{f}\|_{L_2(\Omega)} + \|\widehat{e}_{k-1}\|_{L_{2,-\mu}(\Omega)}].
\end{aligned} \tag{7.2}$$

By iterating (7.2) we have

$$\begin{aligned}
\|\widehat{e}_k\|_{L_{2,-\mu}(\Omega)} &\leq C h_k \delta^r \|\boldsymbol{f}\|_{L_2(\Omega)} + C^2 h_{k-1}\delta^{2r}\|\boldsymbol{f}\|_{L_2(\Omega)} + \ldots + C^{k+1} h_0 \delta^{(k+1)r}\|\boldsymbol{f}\|_{L_2(\Omega)}\\
&\leq C h_k \|\boldsymbol{f}\|_{L_2(\Omega)} \frac{\delta^r}{1 - 2\delta^r},
\end{aligned}$$

if $2\delta^{(r+1)} < 1$. For such choice of r,

$$\|\widehat{e}_k\|_{L_{2,-\mu}(\Omega)} \leq C h_k \|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{7.3}$$

Hence the estimate (7.1) follows directly from (4.2), (7.3) and triangle inequality. □

In the case where $m \geq 1$, we can apply the full multigrid algorithm to obtain an approximate solution for the Dirichlet boundary value problem (1.7). It can be shown that the convergence analysis of multigrid algorithm developed in Section 6 is still valid in this case. Hence we have the following lemma that compares the solution φ_j of (1.7) and $\widehat{\varphi}_{j,k}$. The proof, which uses (6.33), is similar to the proof of Theorem 7.2.

Lemma 7.3. *If r is large enough, then there exists a positive constant C such that for $1 \leq j \leq m$,*

$$|\varphi - \widehat{\varphi}_{j,k}|_{H^1(\Omega)} \leq Ch_k. \tag{7.4}$$

When Ω is not simply connected, for each level k, we compute $\widehat{c}_{1,k}, \ldots, \widehat{c}_{m,k}$ by solving

$$\sum_{j=1}^{m} (\nabla\widehat{\varphi}_{j,k}, \nabla\widehat{\varphi}_{i,k})\widehat{c}_{j,k} = \frac{1}{\alpha}(\boldsymbol{f}, \nabla\widehat{\varphi}_{i,k}), \quad \text{for } 1 \leq i \leq m. \tag{7.5}$$

We compare the solution c_j of (1.9) and $\widehat{c}_{j,k}$ in the next lemma, whose proof is similar to the proof of Lemma 4.6 in [13].

Lemma 7.4. *If r is large enough and h is sufficiently small, then there exists a positive constant C such that*

$$|c_j - \widehat{c}_{j,k}| \leq Ch_k \|\boldsymbol{f}\|_{L_2(\Omega)} \qquad \textit{for} \quad 1 \leq j \leq m. \tag{7.6}$$

In the rest of this section we show multigrid methods developed in Section 5 can be applied to yield approximate solutions for (1.2) ($\alpha = 0$) and (1.8), which are singular Neumann problems. The analysis is similar to that in [4, 9].

We apply the following full multigrid algorithm to solve (5.9).

Algorithm 7.5. Full Multigrid Algorithm for (5.9).
For $k = 0$, $\widehat{\xi}_k \in \widehat{V}_k$ is determined by $A_0\widehat{\xi}_0 = f_0$, where $\alpha = 0$.
For $k \geq 1$, the approximation solution $\widehat{\xi}_k$ is obtained recursively from

$$\begin{aligned} \xi_0^k &= I_{k-1}^k \widehat{\xi}_{k-1}, \\ \xi_\ell^k &= \overline{MG}_W(k, f_k, \xi_{\ell-1}^k, m, m), \quad 1 \leq \ell \leq r, \\ \widehat{\xi}_k &= \xi_r^k. \end{aligned}$$

The solution $\widehat{\xi}_k$ is in $\widehat{V}_k$ since the zero mean value is preserved by the intergrid transfer operators, and the estimate (7.1) still holds in this case, i.e.,

$$\|\xi - \widehat{\xi}_k\|_{L_2(\Omega)} \leq Ch_k \|\boldsymbol{f}\|_{L_2(\Omega)}.$$

In practice, we consider the following k-th level P_1 finite element method for (1.8):
Find $\phi_k \in \widehat{V}_k$ such that

$$A_k\phi_k = g_k, \tag{7.7}$$

where $\alpha = 0$ in the definition of A_k, and $g_k \in V_k'$ is defined by

$$\langle g_k, v\rangle = (\widehat{\xi}_k, v) \qquad \forall\, v \in V_k.$$

Here $\widehat{\xi}_k$ is the approximate solution of (5.4) ($\alpha \neq 0$) or (5.9) ($\alpha = 0$) obtained by the Algorithm 7.5. Moreover, we have the following lemma, whose proof is the analog of the proof of Lemma 4.4.

Lemma 7.6. *Let ϕ be the exact solution of* (1.8) *and ϕ_k be the solution of the discrete problem* (7.7)*. Then*

$$|\phi - \phi_k|_{H^1(\Omega)} \leq C h_k \|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{7.8}$$

We now apply the following full multigrid algorithm to solve (7.7):

Algorithm 7.7. Full Multigrid Algorithm for (7.7).
For $k = 0$, $\widehat{\phi}_0 \in \widehat{V}_0$ is determined by $A_0\widehat{\phi}_0 = g_0$, where $\alpha = 0$.
For $k \geq 1$, the approximation solution $\widehat{\phi}_k$ is obtained recursively from

$$\begin{aligned} \phi_0^k &= I_{k-1}^k \widehat{\phi}_{k-1}, \\ \phi_\ell^k &= \overline{MG}_W(k, g_k, \phi_{\ell-1}^k, m, m), \quad 1 \leq \ell \leq r, \\ \widehat{\phi}_k &= \phi_r^k. \end{aligned}$$

We compare the solution ϕ of (1.8) and $\widehat{\phi}_k$ in the following theorem. The proof is similar to the proof of Theorem 7.2.

Theorem 7.8. *If r is large enough, then there exists a positive constant C such that*

$$|\phi - \widehat{\phi}_k|_{H^1(\Omega)} \leq C h_k \|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{7.9}$$

Finally we define the approximation $\widehat{\boldsymbol{u}}_k$ of $\boldsymbol{u}$ for each level k by

$$\widehat{\boldsymbol{u}}_k = \nabla \times \widehat{\phi}_k + \sum_{j=1}^{m} \widehat{c}_{j,k} \nabla \widehat{\varphi}_{j,k}. \tag{7.10}$$

We can now compare $\widehat{\boldsymbol{u}}_k$ and $\boldsymbol{u}$ in the following theorem by combining Theorem 7.2, Lemma 7.3, Lemma 7.4 and Theorem 7.8. The proof is similar to the proof of Theorem 4.9 in [13].

Theorem 7.9. *If r is large enough and h_1 is sufficiently small, then there exists a positive constant C such that*

$$\|\boldsymbol{u} - \widehat{\boldsymbol{u}}_k\|_{L_2(\Omega)} \leq C h_k \|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{7.11}$$

8. Numerical Experiments

In this section we first report the contraction numbers of the W-cycle algorithms for the P_1 finite element method on the L-shaped domain $(-1,1)^2 \setminus [0,1]^2$. The triangulations $\mathcal{T}_0, \mathcal{T}_1, \ldots,$ are generated by the refinement procedure described in Section 5, where the grading parameter at the reentrant corner $(0,0)$ is taken to be $2/3$.

We used $\lambda = 1/2$ and tabulated the contraction numbers with respect to $\|\!|\cdot|\!\|_{0,k}$ (resp. $\|\!|\cdot|\!\|_{1,k}$) in Table 8.1 (resp. Table 8.2). In both cases the W-cycle algorithms is a contraction for $m = 1$. The numerical results confirm the theoretical results given in Theorem 6.6.

Table 8.1. W-cycle contraction numbers on the L-shaped domain with respect to $\|\!|\cdot|\!\|_{0,k}$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$\alpha=-1$								
$m=1$	0.70	0.70	0.73	0.77	0.77	0.77	0.78	0.79
$m=2$	0.62	0.56	0.56	0.61	0.66	0.65	0.65	0.65
$m=3$	0.48	0.45	0.46	0.50	0.50	0.53	0.53	0.54
$m=4$	0.38	0.33	0.35	0.39	0.42	0.44	0.44	0.44
$\alpha=0$								
$m=1$	0.72	0.71	0.70	0.71	0.72	0.70	0.71	0.70
$m=2$	0.63	0.61	0.61	0.61	0.62	0.62	0.63	0.62
$m=3$	0.53	0.52	0.51	0.51	0.52	0.53	0.53	0.54
$m=4$	0.46	0.45	0.42	0.40	0.40	0.42	0.43	0.42
$\alpha=1$								
$m=1$	0.69	0.69	0.72	0.77	0.77	0.77	0.78	0.78
$m=2$	0.60	0.55	0.56	0.61	0.63	0.64	0.65	0.65
$m=3$	0.48	0.42	0.46	0.49	0.50	0.53	0.53	0.54
$m=4$	0.38	0.33	0.35	0.39	0.42	0.43	0.44	0.44

Remark 8.1. Multigrid methods were studied for time-domain Maxwell's equations in [24, 1], where the $H(\mathrm{curl};\Omega)$-conforming Nédélec's edge elements were used. Uniform convergence of the V-cycle multigrid algorithm was obtained. Therein, the variational problem is based $H_0(\mathrm{curl};\Omega)$; hence the problem is non-elliptic. Therefore separate treatments are required for the kernel of the **curl** operator and its complement. For comparison, multigrid methods for Maxwell's equations solved by nonconforming finite element methods have been studied in [17]. Therein, the variational problem is based $H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$, thus the problem becomes elliptic. The convergence results of the W-cycle multigrid algorithm on convex domains using uniform meshes were presented.

The above two approaches for solving Maxwell's equations are based on variation forms involving multi-dimensional vectors. In the present paper we solve Maxwell's equations by solving standard second order scalar elliptic boundary value problems. Hence the standard results in the convergence analysis for multigrid methods can be applied. Due to the above

TABLE 8.2. W-cycle contraction numbers on the L-shaped domain with respect to $\|\cdot\|_{1,k}$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$\alpha=0$								
$m=1$	0.71	0.72	0.73	0.78	0.78	0.78	0.79	0.79
$m=2$	0.53	0.56	0.58	0.61	0.63	0.64	0.65	0.65
$m=3$	0.40	0.43	0.47	0.48	0.51	0.53	0.53	0.54
$m=4$	0.32	0.33	0.37	0.40	0.42	0.43	0.44	0.44
$\alpha=1$								
$m=1$	0.68	0.69	0.72	0.77	0.77	0.77	0.78	0.78
$m=2$	0.48	0.55	0.56	0.61	0.63	0.64	0.65	0.65
$m=3$	0.34	0.42	0.46	0.49	0.50	0.53	0.53	0.54
$m=4$	0.25	0.33	0.35	0.39	0.42	0.43	0.44	0.44

reasons, it is more reasonable to compare the edge element multigrid methods [24, 1] with nonconforming multigrid methods [17]. The convergence analysis and numerical examination for V-cycle multigrid algorithm, and for general polygonal domains using graded meshes constitute future work [18].

In the rest of the section we report some numerical results for the P_1 finite element method introduced in Section 2. The numerical solutions presented in Table 8.3–Table 8.4 are obtained by full multigrid algorithms, where r is taken to be 2, and number of smoothing steps m is taken to be 5.

The first experiment is performed on the L-shaped domain $(-1,1)^2 \setminus [0,1]^2$ with graded meshes. The exact solution is chosen to be

$$\boldsymbol{u} = \nabla \times \left(r^{2/3} \cos\left(\frac{2}{3}\theta - \frac{\pi}{3}\right)\phi(x)\right), \tag{8.1}$$

where (r,θ) are the polar coordinates at the origin and $\phi(x) = (1-x_1^2)^2(1-x_2^2)^2$. The grading parameter is taken to be $2/3$ at the reentrant corner $(0,0)$. The results are tabulated in Table 8.3 for $\alpha = -1$, 0 and 1. Note that the order of convergence for $\widehat{\boldsymbol{u}}_k$ is 1 as predicted by Theorem 7.9, which has improved as compared to the results on uniform meshes in [13]. The order of convergence for $\widehat{\xi_k}$ is higher than the order predicted by (7.1). This is due to the fact that $\xi = \nabla \times \boldsymbol{u}$ behaves like $r^{2/3}$, which is more regular than $\boldsymbol{u}$.

The goal of the second set of experiments is to exam the convergence behavior of the numerical methods on a doubly connected domain

$$\Omega = (0,4)^2 \setminus [1,3]^2.$$

The solution $\boldsymbol{u}$ of (1.1) can be written as

$$\boldsymbol{u} = \nabla \times \phi + c\nabla\varphi, \tag{8.2}$$

TABLE 8.3. Results for (1.1) on the L-shaped domain and exact solution given by (8.1)

h_k	$\frac{\|\nabla\times\boldsymbol{u}-\widehat{\xi}_k\|_{L_2}}{\|\boldsymbol{f}\|_{L_2}}$	Order	h_k	$\frac{\|\boldsymbol{u}-\widehat{\boldsymbol{u}}_k\|_{L_2}}{\|\boldsymbol{f}\|_{L_2}}$	Order
$\alpha=-1$					
1/16	4.95E−03	1.86	1/16	7.34E−03	1.55
1/32	1.37E−03	1.88	1/32	2.97E−03	1.30
1/64	3.63E−04	1.89	1/64	1.38E−03	1.11
1/128	9.75E−05	1.90	1/128	6.77E−04	1.02
1/256	2.60E−05	1.90	1/256	3.40E−04	0.99
$\alpha=0$					
1/16	2.03E−03	1.84	1/16	5.21E−03	1.13
1/32	5.55E−04	1.87	1/32	2.55E−03	1.02
1/64	1.50E−04	1.88	1/64	1.28E−03	0.99
1/128	4.04E−05	1.89	1/128	6.49E−04	0.98
1/256	1.08E−05	1.90	1/256	3.29E−04	0.98
$\alpha=1$					
1/16	1.43E−03	1.85	1/16	4.88E−03	1.03
1/32	3.87E−04	1.89	1/32	2.45E−03	0.99
1/64	1.03E−04	1.91	1/64	1.24E−03	0.98
1/128	2.74E−05	1.91	1/128	6.29E−04	0.98
1/256	7.25E−06	1.92	1/256	3.19E−04	0.98

where c is a constant number and the harmonic function φ satisfies the following boundary conditions:

$$\varphi\big|_{\Gamma_0} = 0 \quad \text{and} \quad \varphi\big|_{\Gamma_1} = 1.$$

Here Γ_0 (resp. Γ_1) is the boundary of $(0,4)^2$ (resp. $(1,3)^2$).

We take the right-hand side function to be

$$\boldsymbol{f} = \begin{cases} \begin{bmatrix} 1+x_1 \\ 0 \end{bmatrix} & \text{if } x_1 \le x_2 \text{ and } 3 \le x_1 \le 4, \\ \begin{bmatrix} 0 \\ 1+x_2 \end{bmatrix} & \text{otherwise.} \end{cases} \tag{8.3}$$

The results are presented in Table 8.4 for $\alpha = -1$ and 1. The order of convergence for $\widehat{\boldsymbol{u}}_k$ is 1 as predicted by Theorem 7.9, whereas its order of convergence is only 2/3 on uniform meshes as shown in [13]. The orders of convergence for $\widehat{\xi}_k$ and $\widehat{c}_k$ are higher than the orders predicted by (7.1) and (7.6). This is probably due to the fact that the mesh size h is not small enough and the asymptotic behavior has not been reached.

TABLE 8.4. Results for (1.1) on the doubly connected domain and right-hand side given by (8.3)

h_k	$\frac{\Vert\nabla\times\boldsymbol{u}-\widehat{\xi}_k\Vert_{L_2}}{\Vert\boldsymbol{f}\Vert_{L_2}}$	Order	$\widehat{c}_k$	Order	$\frac{\Vert\boldsymbol{u}-\widehat{\boldsymbol{u}}_k\Vert_{L_2}}{\Vert\boldsymbol{f}\Vert_{L_2}}$	Order
$\alpha=-1$						
1/4	7.18E−01	0.91	0.764157	0.91	8.36E−01	0.86
1/8	1.95E−01	1.88	0.765826	1.58	3.01E−01	1.48
1/16	4.01E−02	2.28	0.766367	1.62	1.24E−01	1.28
1/32	1.03E−02	1.97	0.766528	1.75	6.07E−02	1.03
1/64	2.79E−03	1.88	0.766570	1.79	3.09E−02	0.98
$\alpha=1$						
1/4	7.33E−03	1.69	-0.764157	0.91	9.03E−02	0.83
1/8	2.22E−03	1.72	-0.765826	1.58	4.97E−02	0.86
1/16	6.60E−04	1.75	-0.766367	1.62	2.71E−02	0.87
1/32	1.99E−04	1.70	-0.766528	1.75	1.48E−02	0.87
1/64	6.63E−05	1.58	-0.766570	1.79	7.98E−03	0.89

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