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Research Article

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Multigrid Methods Based on Hodge Decomposition for a Quad-Curl Problem

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Abstract: In this paper we investigate multigrid methods for a quad-curl problem on graded meshes. The approach is based on the Hodge decomposition. The solution for the quad-curl problem is approximated by solving standard second-order elliptic problems and optimal error estimates are obtained on graded meshes. We prove the uniform convergence of the multigrid algorithm for the resulting discrete problem. The performance of these methods is illustrated by numerical results.

Keywords: Quad-Curl Problem, Hodge Decomposition, Graded Meshes, Multigrid Methods

MSC 2010: 65N30, 65N15, 35Q60

Dedicated to Amiya K. Pani on the occasion of his 60th birthday and part of the special issue “Recent Advances in PDE: Theory, Computations and Applications” in his honour.

1 Introduction

Let Ω be a bounded connected and polygonal domain in $\mathbb{R}^2$ and $\boldsymbol{f} \in [L_2(\Omega)]^2$. We consider the following quad-curl problem, which arises from Maxwell’s transmission eigenvalue problem (cf. [17, 30]) and magnetohydrodynamics models (cf. [5, 18]):

Find $\boldsymbol{u} \in \mathbb{E}$ such that

$$(\mathbf{curl}(\operatorname{curl}\boldsymbol{u}), \mathbf{curl}(\operatorname{curl}\boldsymbol{v})) + \beta(\operatorname{curl}\boldsymbol{u}, \operatorname{curl}\boldsymbol{v}) + \gamma(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \mathbb{E}, \tag{1.1}$$

where $(\cdot,\cdot)$ denotes the L_2 inner product over Ω, β and γ are nonnegative constants with $\gamma > 0$ if Ω is not simply connected, and

$$\mathbb{E} = \{\boldsymbol{v} \in [L_2(\Omega)]^2 : \operatorname{curl}\boldsymbol{v} \in H_0^1(\Omega),\ \operatorname{div}\boldsymbol{v} = 0 \text{ and } \boldsymbol{n} \times \boldsymbol{v} = 0 \text{ on } \partial\Omega\}.$$

Here the curl of a vector field $\boldsymbol{v} = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$ is the scalar function defined by

$$\operatorname{curl}\boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_2}{\partial x_1},$$

and the **curl** of a scalar function ϕ is the vector field defined by

$$\mathbf{curl}\,\phi = \begin{bmatrix} \frac{\partial \phi}{\partial x_2} \\ -\frac{\partial \phi}{\partial x_1} \end{bmatrix}.$$

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The vector $\boldsymbol{n} = \left[\begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix}\right]$ is a unit outer normal along $\partial\Omega$ and $\boldsymbol{n} \times \boldsymbol{v}$ is the tangential component of $\boldsymbol{v}$ given by

$$\boldsymbol{n} \times \boldsymbol{v} = n_1 v_2 - n_2 v_1.$$

Remark 1.1. Note that $\mathbf{curl}\,\phi$ is just the rotation of $\mathbf{grad}\,\phi$ by a right angle and also $\operatorname{curl}\mathbf{curl} = -\Delta$.

Under the assumptions on β and γ, problem (1.1) has a unique solution by the Fredholm theory [37]. Details can be found in [16].

The quad-curl problem has been studied by using a discontinuous Galerkin method [25], a nonconforming finite element method [39] and a mixed finite element method [34] in three-dimensional space. Those approaches are based on Nédélec's elements [28, 33]. The constructions of such high-order edge elements are difficult. Moreover, the error analysis therein is complicated due to the fact that the basis functions are vector polynomials.

Following the ideas in [11, 13, 16], we will solve problem (1.1) based on the Hodge decomposition [22, Section 1.3.1] in two dimensions, where (1.1) can be decomposed into several standard second-order elliptic boundary value problems.

The Hodge decomposition for a function $\boldsymbol{u} \in H(\operatorname{div}^0; \Omega)$ is as follows:

$$\boldsymbol{u} = \mathbf{curl}\,\phi + \sum_{j=1}^{m} c_j\,\mathbf{grad}\,\varphi_j, \tag{1.2}$$

where $\phi \in H^1(\Omega)$ satisfies

$$(\phi, 1) = 0.$$

Here

$$\begin{aligned} H(\operatorname{div}; \Omega) &= \left\{\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in [L_2(\Omega)]^2 : \operatorname{div}\boldsymbol{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \in L_2(\Omega)\right\}, \\ H(\operatorname{div}^0; \Omega) &= \{\boldsymbol{v} \in H(\operatorname{div}; \Omega) : \operatorname{div}\boldsymbol{u} = 0\}. \end{aligned}$$

The nonnegative integer m is the Betti number for Ω ($m = 0$ if Ω is simply connected, cf. Figure 1). Let the outer boundary of Ω be denoted by Γ_0, and the m components of the inner boundary be denoted by $\Gamma_1, \dots, \Gamma_m$. The harmonic functions $\varphi_1, \dots, \varphi_m$ satisfy

$$(\mathbf{grad}\,\varphi_j, \mathbf{grad}\,v) = 0 \qquad \text{for all } v \in H_0^1(\Omega), \tag{1.3a}$$

$$\varphi_j|_{\Gamma_0} = 0, \tag{1.3b}$$

$$\varphi_j|_{\Gamma_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \qquad \text{for } 1 \le k \le m. \tag{1.3c}$$

Let $\boldsymbol{u}$ be the solution of (1.1), $\xi = \operatorname{curl}\boldsymbol{u} \in H_0^1(\Omega)$ and $L_2^0(\Omega) = \{v \in L_2(\Omega) : (v, 1) = 0\}$. Then the function ϕ in the Hodge decomposition (1.2) is determined by

$$(\mathbf{curl}\,\phi, \mathbf{curl}\,\psi) = (\boldsymbol{u}, \mathbf{curl}\,\psi) = (\operatorname{curl}\boldsymbol{u}, \psi) = (\xi, \psi) \quad \text{for all } \psi \in H^1(\Omega). \tag{1.4}$$

Note that the condition $\boldsymbol{n} \times \boldsymbol{u} = 0$ on $\partial\Omega$ implies

$$(\xi, 1) = 0,$$

so the singular Neumann problem (1.4) has a unique solution $\phi \in H^1(\Omega) \cap L_2^0(\Omega)$. An equivalent formulation

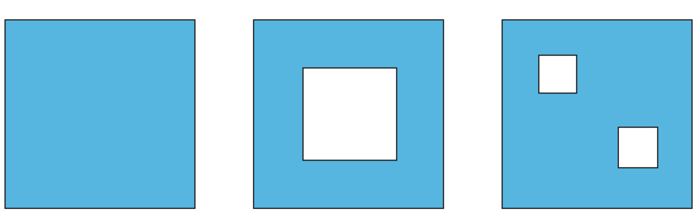

Figure 1: Betti numbers $m = 0$, 1 and 2 (from left to right).

of (1.4) that avoids the zero-mean constraint is to find $\phi \in H^1(\Omega)$ such that

$$(\mathbf{curl}\,\phi, \mathbf{curl}\,\psi) + (\phi, 1)(\psi, 1) = (\xi, \psi) \quad \text{for all } \psi \in H^1(\Omega). \tag{1.5}$$

It is proved in [16] that $\xi \in H_0^1(\Omega) \cap L_2^0(\Omega)$ satisfies the equation

$$(\mathbf{curl}\,\xi, \mathbf{curl}(\operatorname{curl}\boldsymbol{\zeta})) + \beta(\mathbf{curl}\,\xi, \boldsymbol{\zeta}) + \gamma(\boldsymbol{u}, \boldsymbol{\zeta}) = (Q\boldsymbol{f}, \boldsymbol{\zeta}) \quad \text{for all } \boldsymbol{\zeta} \in [C_c^\infty(\Omega)]^2, \tag{1.6}$$

where $Q : [L_2(\Omega)]^2 \to H(\operatorname{div}^0; \Omega)$ is the orthogonal projection.

Moreover, it follows from Remark 1.1 and (1.6) that

$$\mathbf{curl}(-\Delta\xi) = -\beta\,\mathbf{curl}\,\xi - \gamma\boldsymbol{u} + Q\boldsymbol{f} \tag{1.7}$$

in the sense of distributions.

Let $\rho \in H^1(\Omega) \cap L_2^0(\Omega)$ be defined by

$$(\mathbf{curl}\,\rho, \mathbf{curl}\,\psi) = -\gamma(\boldsymbol{u}, \mathbf{curl}\,\psi) + (Q\boldsymbol{f}, \mathbf{curl}\,\psi) = -\gamma(\xi, \psi) + (\boldsymbol{f}, \mathbf{curl}\,\psi) \quad \text{for all } \psi \in H^1(\Omega). \tag{1.8}$$

It can be observed from (1.7) that $-\Delta\xi + \beta\xi$ is also a solution of (1.8). Therefore

$$-\Delta\xi + \beta\xi = \rho + c$$

for some constant number c, and hence

$$(\mathbf{curl}\,\xi, \mathbf{curl}\,\eta) + \beta(\xi, \eta) = (\rho, \eta) \quad \text{for all } \eta \in H_0^1(\Omega) \cap L_2^0(\Omega). \tag{1.9}$$

In the case where $\gamma = 0$ (and Ω is simply connected), the two equations (1.8) and (1.9) are decoupled; hence we can solve them consecutively for ρ and ξ. Note that in this case, (1.8) becomes a consistent singular Neumann boundary value problem, and its equivalent formulation (without the zero-mean constraint) is to find $\rho \in H^1(\Omega)$ such that

$$(\mathbf{curl}\,\rho, \mathbf{curl}\,\psi) + (\rho, 1)(\psi, 1) = (\boldsymbol{f}, \mathbf{curl}\,\psi) \quad \text{for all } \psi \in H^1(\Omega). \tag{1.10}$$

We can also determine the solution ξ of (1.9) by the following steps without the zero-mean constraint (cf. [16, Lemma 3.3]):

Find

$$\xi = \xi_0 - \frac{(1, \xi_0)}{(1, \xi_1)}\xi_1, \tag{1.11}$$

where $\xi_0, \xi_1 \in H_0^1(\Omega)$ satisfy

$$(\mathbf{curl}\,\xi_0, \mathbf{curl}\,\eta) + \beta(\xi_0, \eta) = (\rho, \eta) \quad \text{for all } \eta \in H_0^1(\Omega), \tag{1.12}$$

$$(\mathbf{curl}\,\xi_1, \mathbf{curl}\,\eta) + \beta(\xi_1, \eta) = (1, \eta) \quad \text{for all } \eta \in H_0^1(\Omega). \tag{1.13}$$

In the case where $\gamma > 0$ and Ω is not simply connected (i.e., $m \geq 1$), the coefficients c_j in (1.2) are determined by the symmetric positive-definite system (cf. [11, 16])

$$\sum_{j=1}^{m} (\mathbf{grad}\,\varphi_j, \mathbf{grad}\,\varphi_i)c_j = \frac{1}{\gamma}(\boldsymbol{f}, \mathbf{grad}\,\varphi_i) \quad \text{for } 1 \leq i \leq m. \tag{1.14}$$

Remark 1.2. Note that $(\mathbf{curl}\,\zeta, \mathbf{curl}\,\psi) = (\mathbf{grad}\,\zeta, \mathbf{grad}\,\psi)$ for all $\zeta, \psi \in H^1(\Omega)$. Therefore the boundary value problems (1.4) and (1.9) for ϕ and ξ are Neumann and Dirichlet problems respectively for the Laplace operator.

The P_k ($k \geq 1$) Lagrange finite element methods for solving (1.1) based on Hodge decomposition were developed in [16] for quasi-uniform triangulations. In that case, the optimal convergence rates cannot be achieved when the domain is nonconvex. In this paper, we use the P_1 finite element method and recover optimal convergence results on a properly graded triangulation. Also note that in this approach an approximate solution for the quad-curl problem is obtained by solving standard second-order scalar elliptic boundary value problems. Therefore we can apply standard results in the convergence analysis for multigrid methods.

The rest of the paper is organized as follows. In Section 2, we briefly recall the numerical procedure introduced in [16], and focus on the P_1 finite element method. The elliptic regularity results in terms of weighted Sobolev spaces are reviewed in Section 3. The analysis of the P_1 finite element method based on graded meshes is carried out in Section 4. Then in Section 5 we introduce multigrid methods for the resulting discrete problems, followed by the convergence results of the W-cycle multigrid algorithm. The full multigrid algorithm is analyzed in Section 6. Finally, some numerical results are reported in Section 7 and we end with some concluding remarks in Section 8.

2 A P_1 Finite Element Method

Let $\mathcal{T}_h$ be a simplicial triangulation of Ω with mesh size h. Let $\widehat{V}_h$ (resp., $\widehat{\mathring{V}}_h$) be the P_1 finite element subspace of $H^1(\Omega) \cap L_2^0(\Omega)$ (resp. $H_0^1(\Omega) \cap L_2^0(\Omega)$) associated with $\mathcal{T}_h$. Moreover, we denote by V_h the P_1 finite element subspace of $H^1(\Omega)$ associated with $\mathcal{T}_h$ and by $\mathring{V}_h$ the subspace of V_h whose members vanish on $\partial\Omega$.

A numerical procedure for solving (1.1) based on the Hodge decomposition (1.2) is described in the rest of the section. More details can be found in [16] for the P_k finite element method.

The case where $\gamma = 0$. According to our assumption, in this case Ω is simply connected and the two equations (1.8) and (1.9) are decoupled. Hence we can first find the approximation of ρ, and then find the approximations of ξ and ϕ. More precisely, we can find $\rho_h \in \widehat{V}_h$, $\xi_h \in \widehat{\mathring{V}}_h$ and $\phi_h \in \widehat{V}_h$ consecutively as follows:

$$(\mathbf{curl}\,\rho_h, \mathbf{curl}\,\psi) = (\boldsymbol{f}, \mathbf{curl}\,\psi) \quad \text{for all } \psi \in \widehat{V}_h, \tag{2.1}$$

$$(\mathbf{curl}\,\xi_h, \mathbf{curl}\,\eta) + \beta(\xi_h, \eta) = (\rho_h, \eta) \quad \text{for all } \eta \in \widehat{\mathring{V}}_h, \tag{2.2}$$

$$(\mathbf{curl}\,\phi_h, \mathbf{curl}\,\psi) = (\xi_h, \psi) \quad \text{for all } \psi \in \widehat{V}_h. \tag{2.3}$$

However, it is difficult to construct natural bases for $\widehat{V}_h$ and $\widehat{\mathring{V}}_h$, due to the fact that they both belong to $L_2^0(\Omega)$. For the simplicity of computation and error analysis, we will consider spaces V_h and $\mathring{V}_h$ instead and take the following three steps, which are equivalent to (2.1)–(2.3), to compute the approximate solutions ρ_h, ξ_h and ϕ_h. The idea is analogous to the one for the continuous problems in Section 1.

Step 1. Find $\rho_h \in V_h$ such that

$$(\mathbf{curl}\,\rho_h, \mathbf{curl}\,\psi) + (\rho_h, 1)(\psi, 1) = (\boldsymbol{f}, \mathbf{curl}\,\psi) \quad \text{for all } \psi \in V_h. \tag{2.4}$$

Step 2. Find $\xi_h \in \mathring{V}_h$ such that

$$\xi_h = \xi_{0,h} - \frac{(1, \xi_{0,h})}{(1, \xi_{1,h})}\xi_{1,h}, \tag{2.5}$$

where $\xi_{0,h}, \xi_{1,h} \in \mathring{V}_h$ satisfy

$$(\mathbf{curl}\,\xi_{0,h}, \mathbf{curl}\,\eta) + \beta(\xi_{0,h}, \eta) = (\rho_h, \eta) \quad \text{for all } \eta \in \mathring{V}_h, \tag{2.6}$$

$$(\mathbf{curl}\,\xi_{1,h}, \mathbf{curl}\,\eta) + \beta(\xi_{1,h}, \eta) = (1, \eta) \quad \text{for all } \eta \in \mathring{V}_h. \tag{2.7}$$

Step 3. Find $\phi_h \in V_h$ such that

$$(\mathbf{curl}\,\phi_h, \mathbf{curl}\,\psi) + (\phi_h, 1)(\psi, 1) = (\xi_h, \psi) \quad \text{for all } \psi \in V_h. \tag{2.8}$$

Finally, we can compute the approximate $\boldsymbol{u}_h$ of $\boldsymbol{u}$ on $\mathcal{T}_h$ by

$$\boldsymbol{u}_h = \mathbf{curl}\,\phi_h. \tag{2.9}$$

The case where $\gamma > 0$ and Ω is not simply connected. By defining $\zeta = \gamma^{-\frac{1}{2}}\rho$, we can reformulate the coupled equations (1.8) and (1.9) as follows:

Find $(\zeta, \xi) \in [H^1(\Omega) \cap L_2^0(\Omega)] \times [H_0^1(\Omega) \cap L_2^0(\Omega)]$ such that

$$(\mathbf{curl}\,\zeta, \mathbf{curl}\,\psi) + \gamma^{\frac{1}{2}}(\psi, \xi) = \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \mathbf{curl}\,\psi) \quad \text{for all } \psi \in H^1(\Omega) \cap L_2^0(\Omega), \tag{2.10a}$$

$$-\gamma^{\frac{1}{2}}(\zeta, \eta) + \big[(\mathbf{curl}\,\xi, \mathbf{curl}\,\eta) + \beta(\xi, \eta)\big] = 0 \quad \text{for all } \eta \in H_0^1(\Omega) \cap L_2^0(\Omega). \tag{2.10b}$$

Equivalently, equations (2.10) can be written in a concise form

$$\mathcal{A}\big((\zeta,\xi),(\psi,\eta)\big) = \gamma^{-\frac{1}{2}}(\boldsymbol{f},\mathbf{curl}\,\psi) \quad \text{for all } (\psi,\eta) \in [H^1(\Omega)\cap L_2^0(\Omega)] \times [H_0^1(\Omega)\cap L_2^0(\Omega)], \tag{2.11}$$

where the bilinear form $\mathcal{A}(\cdot,\cdot)$ is defined by

$$\mathcal{A}\big((\zeta,\xi),(\psi,\eta)\big) = (\mathbf{curl}\,\zeta,\mathbf{curl}\,\psi) + \gamma^{\frac{1}{2}}(\psi,\xi) - \gamma^{\frac{1}{2}}(\zeta,\eta) + \big[(\mathbf{curl}\,\xi,\mathbf{curl}\,\eta) + \beta(\xi,\eta)\big]. \tag{2.12}$$

Note that $\mathcal{A}(\cdot,\cdot)$ is clearly bounded on $H^1(\Omega)\times H_0^1(\Omega)$, and it is also coercive on $[H^1(\Omega)\cap L_2^0(\Omega)]\times H_0^1(\Omega)$ due to a standard Poincaré–Friedrichs inequality (cf. [32]) and the fact that

$$\mathcal{A}\big((\psi,\eta),(\psi,\eta)\big) = (\mathbf{curl}\,\psi,\mathbf{curl}\,\psi) + (\mathbf{curl}\,\eta,\mathbf{curl}\,\eta) + \beta(\eta,\eta).$$

Therefore problem (2.10) has a unique solution by the Lax–Milgram theorem [27].

In this case, the approximate solutions can be obtained by the following steps:

Step 0. Compute the P_1 finite element solutions $\varphi_{j,h}$ $(1\le j\le m)$ of the Dirichlet problems (1.3) and find $c_{1,h},\dots,c_{m,h}$ by solving (1.14) numerically with φ_j replaced by $\varphi_{j,h}$.

Step 1. Find the P_1 finite element solution $(\zeta_h,\xi_h)\in \widehat{V}_h\times \widehat{\mathring{V}}_h$ of the coupled equation (2.10) (or (2.11)) such that

$$\mathcal{A}\big((\zeta_h,\xi_h),(\psi,\eta)\big) = \gamma^{-\frac{1}{2}}(\boldsymbol{f},\mathbf{curl}\,\psi) \quad \text{for all } (\psi,\eta)\in \widehat{V}_h\times\widehat{\mathring{V}}_h. \tag{2.13}$$

Step 2. Solve equation (2.8) (which is equivalent to (2.3)) to find an approximate solution $\phi_h\in\widehat{V}_h$.

Step 3. Compute the approximation $\boldsymbol{u}_h$ of $\boldsymbol{u}$ by

$$\boldsymbol{u}_h = \mathbf{curl}\,\phi_h + \sum_{j=1}^m c_{j,h}\,\mathbf{grad}\,\varphi_{j,h}. \tag{2.14}$$

More precisely, in Step 0, the P_1 finite element approximation $\varphi_{j,h}$ for the harmonic function φ_j in the Hodge decomposition (1.2) is determined by

$$(\mathbf{grad}\,\varphi_{j,h},\mathbf{grad}\,v) = 0 \qquad \text{for all } v\in\mathring{V}_h, \tag{2.15a}$$

$$\varphi_{j,h}\big|_{\Gamma_0} = 0, \tag{2.15b}$$

$$\varphi_{j,h}\big|_{\Gamma_i} = \begin{cases} 1, & j=i,\\ 0, & j\ne i,\end{cases} \qquad \text{for } 1\le i\le m. \tag{2.15c}$$

Then we compute $c_{1,h},\dots,c_{m,h}$ by solving

$$\sum_{j=1}^m(\mathbf{grad}\,\varphi_{j,h},\mathbf{grad}\,\varphi_{i,h})c_{j,h} = \frac{1}{\gamma}(\boldsymbol{f},\mathbf{grad}\,\varphi_{i,h}) \quad \text{for } 1\le i\le m. \tag{2.16}$$

Remark 2.1. In this case, we can also find the solution (ζ,ξ) of (2.10) by solving an equivalent problem that does not involve the zero-mean constraints. Similarly, we can find the P_1 approximate solution (ζ_h,ξ_h) by solving an equivalent problem on $V_h\times\mathring{V}_h$ (see [16, Lemma 3.4 and Section 4.1.1] for details).

Remark 2.2. The computations in Step 0 can be carried out in parallel with those in Step 1 and Step 2, and therefore they do not increase the complexity of the computing process.

The case where $\gamma>0$ and Ω is simply connected. We first find approximations $\xi_h\in\widehat{\mathring{V}}_h$ and $\phi_h\in\widehat{V}_h$ by Step 1 and Step 2 in the previous case. Then an approximate solution $\boldsymbol{u}_h$ can be obtained by (2.9).

3 Elliptic Regularity and Graded Meshes

In view of Remark 1.2, the numerical procedure introduced in Section 2 involves the solution of Neumann and Dirichlet problems for the Laplace operator. In this section we briefly review the elliptic regularity results for these problems in terms of weighted Sobolev spaces and introduce the graded meshes.

Let $\omega_1, \dots, \omega_L$ be the interior angles at the corners $c_1, \dots, c_L$ of Ω. Let the parameters $\mu_1, \dots, \mu_L$ be chosen according to

$$\mu_\ell = 1 \quad \text{if } \omega_\ell \le \pi, \qquad \mu_\ell < \frac{\pi}{\omega_\ell} \quad \text{if } \omega_\ell > \pi, \tag{3.1}$$

and the weight function ϕ_μ, where $\mu = (\mu_1, \dots, \mu_L)$, be defined by

$$\phi_\mu(x) = \prod_{\ell=1}^{L} |x - c_\ell|^{1-\mu_\ell}. \tag{3.2}$$

The weighted Sobolev space $H^r_\mu(\Omega)$ is defined by

$$H^r_\mu(\Omega) = \left\{ z \in L_{2,\mathrm{loc}}(\Omega) : \|z\|^2_{H^r_\mu(\Omega)} = \sum_{|\alpha| \le r} \int_\Omega \prod_{\ell=1}^{L} |x - c_\ell|^{2(1-\mu_\ell+|\alpha|-r)} \left| \frac{\partial^\alpha z}{\partial x^\alpha} \right|^2 dx < \infty \right\}, \tag{3.3}$$

where $x = (x_1, x_2)$ and the vector $\alpha = (\alpha_1, \alpha_2)$. In particular, in view of (3.2) and (3.3), the weighted Sobolev space $L_{2,\mu}(\Omega) = H^0_\mu(\Omega)$ is defined by

$$L_{2,\mu}(\Omega) = \left\{ z \in L_{2,\mathrm{loc}}(\Omega) : \|z\|^2_{L_{2,\mu}(\Omega)} = \int_\Omega \phi^2_\mu(x) z^2(x)\, dx < \infty \right\}.$$

Note that $L_2(\Omega) \subset L_{2,\mu}(\Omega)$ and

$$\|z\|_{L_{2,\mu}(\Omega)} \le C_\Omega \|z\|_{L_2(\Omega)} \quad \text{for all } z \in L_2(\Omega).$$

We first consider the singular Neumann problem of finding $\lambda \in H^1(\Omega)$ such that

$$(\mathbf{grad}\, \lambda, \mathbf{grad}\, \psi) + (\lambda, 1)(\psi, 1) = (f, \psi) \quad \text{for all } \psi \in H^1(\Omega), \tag{3.4}$$

where $f \in L_2(\Omega)$. The model problem (3.4) has a unique solution $\lambda \in H^2_\mu(\Omega)$. Moreover, the following regularity estimate holds:

$$\|\lambda\|_{H^2_\mu(\Omega)} \le C_\Omega \|f\|_{L_{2,\mu}(\Omega)} \le C_\Omega \|f\|_{L_2(\Omega)}. \tag{3.5}$$

The preceding discussion also holds for the Dirichlet problem of finding $v \in H^1_0(\Omega)$ such that

$$(\mathbf{grad}\, v, \mathbf{grad}\, \eta) + \beta(v, \eta) = (f, \eta) \quad \text{for all } \eta \in H^1_0(\Omega).$$

Details for these results can be found for example in [26], [23, Section 4.4], [21, Section 2.5] and [31, Section 2.3].

Note that the solution of the model problem (3.4) has singularities when the bounded polygonal domain Ω is nonconvex. To compensate for the lack of full elliptic regularity, the meshes need to be graded properly.

Definition 3.1. We say that $\mathcal{T}_h$ is a properly graded mesh if there exists a positive constant C_μ independent of h such that

$$C_\mu^{-1} \Phi_\mu(T) h \le h_T = \mathrm{diam}(T) \le C_\mu \Phi_\mu(T) h \quad \text{for all } T \in \mathcal{T}_h, \tag{3.6}$$

where the weight $\Phi_\mu(T)$ associated with $T \in \mathcal{T}_h$ is defined by

$$\Phi_\mu(T) = \prod_{\ell=1}^{L} |c_\ell - c_T|^{1-\mu_\ell},$$

and c_T is the center of T.

The graded meshes will play a crucial role in recovering optimal a priori error estimates for P_1 finite element methods and in proving uniform convergence of multigrid methods.

The polynomial approximation result for a properly graded triangulation is recalled below. Details can be found in [2].

Lemma 3.2. *There exists a positive constant C depending only on the constant C_μ in* (3.6) *such that*

$$\inf_{\psi \in V_h} \|\lambda - \psi\|_{H^1(\Omega)} \le Ch \|\lambda\|_{H^2_\mu(\Omega)} \quad \text{for all } \lambda \in H^2_\mu(\Omega),$$

$$\inf_{\eta \in V_h} \|v - \eta\|_{H^1(\Omega)} \le Ch \|v\|_{H^2_\mu(\Omega)} \quad \text{for all } \mu \in H^2_\mu(\Omega) \cap H^1_0(\Omega).$$

4 Error Analysis

The error estimates for the P_k finite element method are studied in [16] for quasi-uniform triangulations. In this section, we will follow the ideas therein and derive optimal convergence results using P_1 elements on a properly graded triangulation $\mathcal{T}_h$ that satisfies the property (3.6). Since the error analysis for the discrete harmonic functions $\varphi_{1,h}, \dots, \varphi_{m,h}$ and the coefficients $c_{1,h}, \dots, c_{m,h}$ has already been carried out in [11] on uniform meshes, and results have been shown in [20] on graded meshes, we only focus on the error analysis for ξ_h and ϕ_h.

4.1 Error Analysis for ξ_h

We will consider the cases of $\gamma = 0$ and $\gamma > 0$ separately, and derive the error bound for ξ_h in the following lemma.

Lemma 4.1. *There exists a positive constant C independent of h such that*

$$|\xi - \xi_h|_{H^1(\Omega)} \le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{4.1}$$

Proof. There are two cases.

(i) The case where $\gamma = 0$. We first estimate the error for ρ_h in the norm of $[H^1(\Omega)]'$ (cf. (4.7) below) by a duality argument. In view of (1.10), (2.4) and the fact that $\rho, \rho_h \in L_2^0(\Omega)$, we have

$$\|\mathbf{curl}\,\rho\|_{L_2(\Omega)} \le \|\boldsymbol{f}\|_{L_2(\Omega)}, \tag{4.2a}$$
$$\|\mathbf{curl}\,\rho_h\|_{L_2(\Omega)} \le \|\boldsymbol{f}\|_{L_2(\Omega)}, \tag{4.2b}$$

and

$$(\mathbf{curl}(\rho - \rho_h), \mathbf{curl}\,\psi) = 0 \quad \text{for all } \psi \in V_h. \tag{4.3}$$

Let $\lambda \in H^1(\Omega)$ be defined by

$$(\mathbf{curl}\,\psi, \mathbf{curl}\,\lambda) + (\psi, 1)(\lambda, 1) = (\psi, \chi) \quad \text{for all } \psi \in H^1(\Omega), \tag{4.4}$$

where χ is an arbitrary function in $H^1(\Omega)$. Combining (4.3), (4.4) and the fact that $\rho, \rho_h \in L_2^0(\Omega)$, we have

$$(\rho - \rho_h, \chi) = (\mathbf{curl}(\rho - \rho_h), \mathbf{curl}\,\lambda) = \big(\mathbf{curl}(\rho - \rho_h), \mathbf{curl}(\lambda - \psi)\big) \quad \text{for all } \psi \in V_h,$$

which implies

$$|(\rho - \rho_h, \chi)| \le \|\mathbf{curl}(\rho - \rho_h)\|_{L_2(\Omega)} \inf_{\psi \in V_h} |\lambda - \psi|_{H^1(\Omega)}. \tag{4.5}$$

According to (3.4), (3.5) and (4.4), we have $\lambda \in H_\mu^2(\Omega)$ and also $\|\lambda\|_{H_\mu^2(\Omega)} \le C\|\chi\|_{H^1(\Omega)}$. It then follows from Lemma 3.2 that

$$\inf_{\psi \in V_h} |\lambda - \psi|_{H^1(\Omega)} \le Ch\|\lambda\|_{H_\mu^2(\Omega)} \le Ch\|\chi\|_{H^1(\Omega)}. \tag{4.6}$$

Therefore, by combining (4.2), (4.5) and (4.6), we have

$$|(\rho - \rho_h, \chi)| \le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}\|\chi\|_{H^1(\Omega)} \quad \text{for all } \chi \in H^1(\Omega). \tag{4.7}$$

Next we estimate $|\xi - \xi_h|_{H^1(\Omega)}$. Let $\tilde{\xi}_{0,h} \in \mathring{V}_h$ be defined by

$$(\mathbf{curl}\,\tilde{\xi}_{0,h}, \mathbf{curl}\,\eta) + \beta(\tilde{\xi}_{0,h}, \eta) = (\rho, \eta) \quad \text{for all } \eta \in \mathring{V}_h. \tag{4.8}$$

On one hand we have

$$(\mathbf{curl}(\tilde{\xi}_{0,h} - \xi_{0,h}), \mathbf{curl}\,\eta) + \beta(\tilde{\xi}_{0,h} - \xi_{0,h}, \eta) = (\rho - \rho_h, \eta) \quad \text{for all } \eta \in \mathring{V}_h$$

by comparing (2.6) and (4.8). Therefore we have

$$|\tilde{\xi}_{0,h} - \xi_{0,h}|^2_{H^1(\Omega)} \le (\rho - \rho_h, \tilde{\xi}_{0,h} - \xi_{0,h}). \tag{4.9}$$

It then follows from (4.7), (4.9) and a standard Poincaré–Friedrichs inequality that

$$|\tilde{\xi}_{0,h} - \xi_{0,h}|_{H^1(\Omega)} \le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{4.10}$$

On the other hand, we observe from (1.12) and (4.8) that $\tilde{\xi}_{0,h}$ is the Galerkin P_1 finite element approximation of ξ_0. Therefore we have

$$|\xi_0 - \tilde{\xi}_{0,h}|_{H^1(\Omega)} \le C \inf_{\eta \in V_h} |\xi_0 - \eta|_{H^1(\Omega)}$$

by Céa's lemma [15, 19]. Furthermore, according to (1.12) and the regularity results presented in Section 3, we have $\xi_0 \in H^2_\mu(\Omega)$ and also $\|\xi_0\|_{H^2_\mu(\Omega)} \le C\|\rho\|_{H^1(\Omega)}$. It then follows from Lemma 3.2 and (4.2a) that

$$\inf_{\eta \in V_h} |\xi_0 - \eta|_{H^1(\Omega)} \le Ch\|\xi_0\|_{H^2_\mu(\Omega)} \le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{4.11}$$

Combining (4.10)–(4.11), we have

$$|\xi_0 - \xi_{0,h}|_{H^1(\Omega)} \le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{4.12}$$

Similarly, since $\xi_{1,h}$ is the Galerkin finite element approximation of ξ_1 (cf. (1.13) and (2.7)), we have

$$|\xi_1 - \xi_{1,h}|_{H^1(\Omega)} \le Ch. \tag{4.13}$$

Estimate (4.1) follows from (1.11), (2.5), (4.12) and (4.13).

(ii) The case where $\gamma > 0$. In this case, one can prove (4.1) by following the steps in [16, Section 5.1.2] in terms of weighted Sobolev norms. The error analysis follows the ideas of the case where $\gamma = 0$, within the setting of the coupled problem (2.10). We omit the proof and refer the reader to [16] for details. □

4.2 Error Analysis for ϕ_h

The error bound for ϕ_h is obtained in the following lemma for graded meshes $\mathcal{T}_h$. The error analysis follows the ideas in [16, Lemma 5.4] and it is similar to the error analysis for ξ_h in Section 4.1.

Lemma 4.2. *There exists a positive constant C independent of h such that*

$$|\phi - \phi_h|_{H^1(\Omega)} \le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{4.14}$$

Proof. According to (1.5), (3.4) and (3.5), we have $\phi \in H^2_\mu(\Omega)$ and

$$\|\phi\|_{H^2_\mu(\Omega)} \le C\|\xi\|_{H^1(\Omega)}. \tag{4.15}$$

Let the function $\tilde{\phi}_h \in V_h$ be defined by

$$(\mathbf{curl}\, \tilde{\phi}_h, \mathbf{curl}\, \psi) + (\tilde{\phi}_h, 1)(\psi, 1) = (\xi, \psi) \quad \text{for all } \psi \in V_h. \tag{4.16}$$

On one hand, by (2.8), (4.16) and the fact that $(\tilde{\phi}_h - \phi_h, 1) = 0$, we have

$$|\tilde{\phi}_h - \phi_h|_{H^1(\Omega)} \le \|\xi - \xi_h\|_{L_2(\Omega)}. \tag{4.17}$$

On the other hand, we observe from (1.5) and (4.16) that

$$(\mathbf{curl}(\phi - \tilde{\phi}_h), \mathbf{curl}\, \psi) = 0 \quad \text{for all } \psi \in V_h. \tag{4.18}$$

It then follows from (4.15), (4.18) and Lemma 3.2 that

$$|\phi - \tilde{\phi}_h|_{H^1(\Omega)} \le \inf_{\psi \in V_h} |\phi - \psi|_{H^1(\Omega)} \le Ch\|\phi\|_{H^2_\mu(\Omega)} \le Ch\|\xi\|_{H^1(\Omega)}. \tag{4.19}$$

Finally, estimate (4.14) follows from (4.17), (4.19) and the estimate in Lemma 4.1 for $\xi - \xi_h \in H^1_0(\Omega)$. □

4.3 Error Analysis for $\varphi_{h,j}$ and $c_{h,j}$

We compare $\varphi_{j,h}$ and φ_j in the following lemma. Since the function $\varphi_{j,h}$ is obtained by a P_1 finite element method for the Dirichlet problem (2.15), the proof is standard [2, Theorem 5.1].

Lemma 4.3. *In the case where Ω is not simply connected, we have*

$$|\varphi_j - \varphi_{j,h}|_{H^1(\Omega)} \le Ch \quad \text{for } 1 \le j \le m.$$

In the next lemma we compare $c_{j,h}$ and c_j. The proof is based on (1.14), (2.16), Lemma 4.3 and is similar to the proof of [11, Lemma 4.7].

Lemma 4.4. *In the case where Ω is not simply connected, we have*

$$|c_j - c_{j,h}| \le Ch\|\boldsymbol{f}\|_{L_2(\Omega)} \quad \text{for } 1 \le j \le m.$$

4.4 Convergence Results

In view of Lemmas 4.1–4.4, (1.2) and (2.14), we immediately have the following convergence result. The proof for $\boldsymbol{u}_h$ is identical with the proof of [11, Theorem 4.9].

Theorem 4.5. *The approximations ξ_h and $\boldsymbol{u}_h$ obtained by the P_1 finite element method on properly graded meshes satisfy*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L_2(\Omega)} + |\operatorname{curl} \boldsymbol{u} - \xi_h|_{H^1(\Omega)} \le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}.$$

5 Multigrid Algorithms

In this section we introduce W-cycle multigrid algorithms [8, 15, 24, 29, 35] for solving the discrete problems (2.1)–(2.3), (2.13) and (2.15) on graded meshes.

5.1 Graded mesh refinements

We first construct properly graded triangulations that satisfy (3.6) (cf. [12, Appendix]). Starting with an initial triangulation $\mathcal{T}_0$, we obtain the k-th level triangulation $\mathcal{T}_k$ for $k \ge 1$ recursively by the following procedure, which is identical to the one in [9].

- If none of the reentrant corners is a vertex of $T \in \mathcal{T}_k$, then we divide T uniformly by connecting the midpoints of the edges of T.
- If a reentrant corner c_ℓ is a vertex of $T \in \mathcal{T}_k$ and the other two vertices of T are denoted by p_1 and p_2, then we divide T by connecting the points m, q_1 and q_2 (cf. Figure 2). Here m is the midpoint of the edge p_1p_2 and q_1 (resp., q_2) is the point on the edge $c_\ell p_1$ (resp., $c_\ell p_2$) such that

$$\frac{|c_\ell - q_i|}{|c_\ell - p_i|} = 2^{-\frac{1}{\mu_\ell}} \quad \text{for } i = 1, 2,$$

  where μ_ℓ is the grading parameter chosen according to (3.1).

The triangulations $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ for an L-shaped domain are depicted in Figure 3, where the grading parameter at the reentrant corner is taken to be $\frac{2}{3}$.

Similar to the notations used in previous sections, we use $\widehat{V}_k$ (resp., $\widehat{\mathring{V}}_k$) to denote the P_1 finite element subspace of $H^1(\Omega) \cap L_2^0(\Omega)$ (resp., $H_0^1(\Omega) \cap L_2^0(\Omega)$) associated with $\mathcal{T}_k$. For ease of implementation, we would like to design W-cycle multigrid algorithms (cf. Algorithm 5.3 below) for discrete problems on spaces V_k and $\mathring{V}_k$, which are the P_1 finite element subspaces of $H^1(\Omega)$ and $H_0^1(\Omega)$ associated with $\mathcal{T}_k$, respectively.

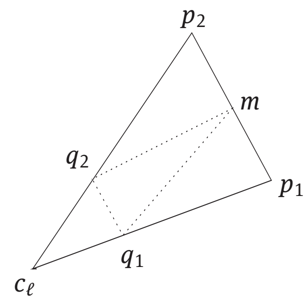

Figure 2: Refinement of a triangle at a reentrant corner.

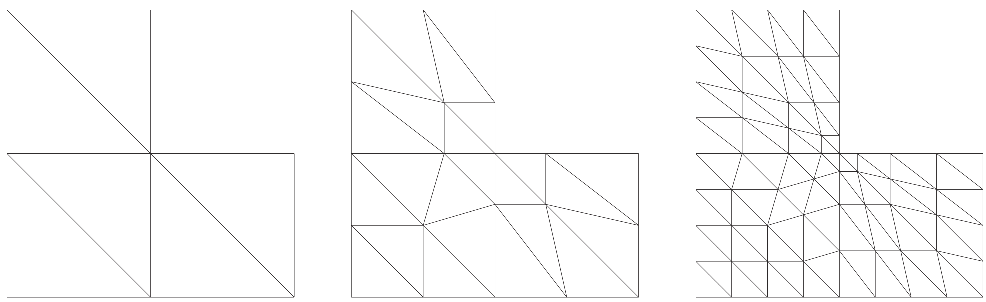

Figure 3: The triangulations $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ for an L-shaped domain.

5.2 The Case Where $\gamma > 0$

The singular Neumann problem (2.8) (or (2.3)) and the Dirichlet problems (1.3) (in the case where Ω is not simply connected) can be solved by standard W-cycle algorithms (cf. [20, Algorithm 5.1] and [20, Algorithm 5.6]). Therefore we will focus on a W-cycle algorithm for (2.13).

Let the operator A_k be defined by

$$\langle A_k(\zeta, \xi), (\psi, \eta)\rangle = \mathcal{A}((\zeta, \xi), (\psi, \eta)) \quad \text{for all } (\zeta, \xi), (\psi, \eta) \in V_k \times \mathring{V}_k, \tag{5.1}$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $(V_k' \times \mathring{V}_k') \times (V_k \times \mathring{V}_k)$. The k-th level P_1 finite element method for (2.13) is:

Find $(\zeta_k, \xi_k) \in (\widehat{V}_k, \widehat{\mathring{V}}_k)$ such that

$$A_k(\zeta_k, \xi_k) = f_k, \tag{5.2}$$

where $f_k \in \widehat{V}_k' \times (\widehat{\mathring{V}}_k)'$ is defined by

$$\langle f_k, (\psi, \eta)\rangle = \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \mathbf{curl}\, \psi) \quad \text{for all } (\psi, \eta) \in \widehat{V}_k \times \widehat{\mathring{V}}_k.$$

Let the operator

$$B_k : V_k \times \mathring{V}_k \to V_k' \times \mathring{V}_k'$$

be defined by

$$\begin{aligned} \langle B_k(\zeta, \xi), (\psi, \eta)\rangle &= h_k^2 \sum_{T \in \mathcal{T}_k} \sum_{p \in \mathcal{N}_T} (\zeta(p)\psi(p) + \xi(p)\eta(p)) \\ &= ((\zeta, \xi), (\psi, \eta))_k \quad \text{for all } (\zeta, \xi), (\psi, \eta) \in V_k \times \mathring{V}_k, \end{aligned} \tag{5.3}$$

where $\mathcal{N}_T$ is the set of the vertices of the triangle T.

Let a projection operator

$$\widehat{P}_k : V_k \times \mathring{V}_k \to \widehat{V}_k \times \widehat{\mathring{V}}_k$$

with respect to $(\cdot, \cdot)_k$ (cf. (5.3)) be given such that for any $(\psi, \eta) \in V_k \times \mathring{V}_k$, $\widehat{P}_k(\psi, \eta) \in \widehat{V}_k \times \widehat{\mathring{V}}_k$ satisfies

$$((\zeta, \xi), \widehat{P}_k(\psi, \eta))_k = ((\zeta, \xi), (\psi, \eta))_k \quad \text{for all } (\zeta, \xi) \in \widehat{V}_k \times \widehat{\mathring{V}}_k.$$

Remark 5.1. One can compute $\widehat{P}_k(\psi, \eta)$ explicitly as

$$\widehat{P}_k(\psi,\eta) = (\psi,\eta) - \left[\frac{((\psi,\eta),(s_k,t_k))_k}{((s_k,t_k),(s_k,t_k))_k}\right](s_k,t_k),$$

where $(s_k, t_k) \in V_k \times \mathring{V}_k$ spans the orthogonal complement of $\widehat{V}_k \times \widehat{\mathring{V}}_k$ with respect to $(\cdot,\cdot)_k$. More precisely, let $\mathcal{N}_k^I$ (resp., $\mathcal{N}_k^B$) be the set of all the interior nodes (resp., boundary nodes) associated with $\mathcal{T}_k$. Then we can take s_k and t_k to be the finite element functions defined by

$$s_k(p) = \frac{1}{3h_k^2 n(\mathcal{T}_p)} \sum_{T\in\mathcal{T}_p} |T| \quad \text{for all } p \in \mathcal{N}_k^I \cup \mathcal{N}_k^B,$$

$$t_k(p) = \begin{cases} \dfrac{1}{3h_k^2 n(\mathcal{T}_p)} \displaystyle\sum_{T\in\mathcal{T}_p} |T| & \text{for all } p \in \mathcal{N}_k^I, \\ 0 & \text{for all } p \in \mathcal{N}_k^B, \end{cases}$$

where $\mathcal{T}_p$ is the set of the triangles in $\mathcal{T}_k$ sharing p as a common vertex, $n(\mathcal{T}_p)$ is the number of triangles in $\mathcal{T}_p$, and $|T|$ is the area of T.

We will use Richardson relaxation as a smoother. Note that (2.12) and (5.3) imply the spectral radius $\rho(B_k^{-1}A_k)$ is bounded by Ch_k^{-2}. Therefore we can choose a (constant) damping factor λ so that the spectral radius $\rho(\lambda h_k^2 B_k^{-1} A_k)$ satisfies

$$\rho(\lambda h_k^2 B_k^{-1} A_k) < 1 \quad \text{for } k \geq 0.$$

Given any $g \in V_k' \times \mathring{V}_k'$ and $\langle g, 1\rangle = 0$, the smoothing scheme for the linear equation

$$A_k z = g \tag{5.4}$$

is given by

$$z_{\text{new}} = z_{\text{old}} + (\lambda h_k^2)\widehat{P}_k B_k^{-1}(g - A_k z_{\text{old}}). \tag{5.5}$$

Since the finite element spaces are nested, we can take the coarse-to-fine intergrid transfer operator

$$I_{k-1}^k : V_{k-1} \times \mathring{V}_{k-1} \to V_k \times \mathring{V}_k$$

to be the natural injection and define the fine-to-coarse intergrid transfer operator

$$I_k^{k-1} : V_k' \times \mathring{V}_k' \to V_{k-1}' \times \mathring{V}_{k-1}'$$

to be the transpose of I_{k-1}^k with respect to $\langle\cdot,\cdot\rangle$, i.e.,

$$\langle I_k^{k-1}(\zeta,\eta),(\psi,\eta)\rangle = \langle(\zeta,\eta), I_{k-1}^k(\psi,\eta)\rangle \quad \text{for all } (\zeta,\eta) \in V_k' \times \mathring{V}_k',\ (\psi,\eta) \in V_{k-1} \times \mathring{V}_{k-1}.$$

Remark 5.2. Let $\widehat{I}_k : \widehat{V}_k \times \widehat{\mathring{V}}_k \to V_k \times \mathring{V}_k$ be the natural injection. Then the operator $\widehat{A}_k = \widehat{P}_k \circ B_k^{-1} \circ A_k \circ \widehat{I}_k$ satisfies

$$(\widehat{A}_k(\zeta,\xi),(\psi,\eta))_k = \mathcal{A}((\zeta,\xi),(\psi,\eta)) \quad \text{for all } (\zeta,\xi),(\psi,\eta) \in \widehat{V}_k \times \widehat{\mathring{V}}_k.$$

We now define the (modified) W-cycle algorithm for the equation (5.4), where $z \in \widehat{V}_k \times \widehat{\mathring{V}}_k$, and the operator A_k is defined as in (5.1).

Algorithm 5.3 (W-Cycle Algorithm). Let $z_0 \in \widehat{V}_k \times \widehat{\mathring{V}}_k$ be the initial guess. The W-cycle multigrid algorithm for (5.4) produces an approximate solution $MG_W(k, g, z_0, m_1, m_2)$, where m_1 (resp., m_2) is the number of pre-smoothing (resp., post-smoothing) steps.

(i) For $k = 0$, $MG_W(0, g, z_0, m_1, m_2) = \widehat{A}_0^{-1}(\widehat{P}_0 B_0^{-1})g$.

(ii) For $k \geq 1$, $MG_W(k, g, z_0, m_1, m_2)$ is computed recursively as follows.

(1) *Pre-Smoothing.* Apply m_1 steps of (5.5) starting with z_0 to obtain z_{m_1}.

(2) *Coarse-Grid Correction.* Compute $q \in \widehat{V}_{k-1} \times \widehat{\mathring{V}}_{k-1}$ by

$$\begin{aligned} r_{k-1} &= I_k^{k-1}(g - A_k z_{m_1}), \\ q' &= MG_W(k-1, r_{k-1}, 0, m_1, m_2), \\ q &= MG_W(k-1, r_{k-1}, q', m_1, m_2), \end{aligned}$$

and take

$$z_{m_1+1} = z_{m_1} + I_{k-1}^{k} q.$$

(3) *Post-Smoothing.* Apply m_2 steps of (5.5) starting with z_{m_1+1} to obtain $z_{m_1+m_2+1}$.
The final output is

$$MG_W(k, g, z_0, m_1, m_2) = z_{m_1+m_2+1}.$$

Remark 5.4. The zero mean value of functions are preserved by the intergrid transfer operators.

Remark 5.5. By introducing the operators $\widehat{I}_k$ and $\widehat{P}_k$, we can perform all the computations in Algorithm 5.3 in the space $V_k \times \mathring{V}_k$ instead of $\widehat{V}_k \times \widehat{\mathring{V}}_k$. Note that there are natural bases for V_k and $\mathring{V}_k$, hence it is easy to implement Algorithm 5.3 in practice. The idea can be found in [20, Section 5].

5.3 The Case Where $\gamma = 0$

In this case equations (2.1), (2.2) (equivalently (2.6) and (2.7)) and (2.3) can be solved by standard W-cycle algorithms (cf. [20, Algorithm 5.1] and [20, Algorithm 5.6]).

5.4 Convergence Results

We state here the convergence result for the W-cycle multigrid algorithm applied to (5.2). Similar result also holds for the W-cycle algorithm applied to the discrete problems (2.1), (2.2), (2.3) and (2.15). The convergence properties are described in terms of the following mesh-dependent norms for $k \geq 0$:

$$|||(\psi, \eta)|||_{0,k}^2 = \langle B_k(\psi, \eta), (\psi, \eta)\rangle \qquad \text{for all } (\psi, \eta) \in \widehat{V}_k \times \widehat{\mathring{V}}_k, \tag{5.6}$$

$$|||(\psi, \eta)|||_{1,k}^2 = \langle A_k(\psi, \eta), (\psi, \eta)\rangle = \mathcal{A}((\psi, \eta), (\psi, \eta)) \approx \|\psi\|_{H^1(\Omega)}^2 + \|\eta\|_{H^1(\Omega)}^2 \qquad \text{for all } (\psi, \eta) \in \widehat{V}_k \times \widehat{\mathring{V}}_k. \tag{5.7}$$

Theorem 5.6. *For any $0 < \delta < 1$, we have*

$$|||z - MG_W(k, g, z_0, m, m)|||_{0,k} \leq \delta |||z - z_0|||_{0,k}, \tag{5.8}$$

$$|||z - MG_W(k, g, z_0, m, m)|||_{1,k} \leq \delta |||z - z_0|||_{1,k}, \tag{5.9}$$

provided that m is sufficiently large.

Note that the bilinear form $\mathcal{A}(\cdot, \cdot)$ defined by (2.12) is nonsymmetric. One can prove Theorem 5.6 by using the ideas in [3, 6, 7, 14, 36] for solving nonsymmetric problems by multigrid methods. The convergence analysis is similar to that in [10, 12, 20], where multigrid methods are applied to solve second-order elliptic problems and Maxwell's equations on graded meshes. The key is to utilize the connections between the mesh dependent norms and the weighted Sobolev norms [38], which would allow us to analyze the errors within the framework of problems with full elliptic regularity.

6 Full Multigrid Methods

We will consider the two cases ($\gamma > 0$ and $\gamma = 0$) separately.

6.1 The Case Where $\gamma > 0$

First we use the following full multigrid algorithm to solve (5.2), where we apply Algorithm 5.3 r times at each level.

Algorithm 6.1 (Full Multigrid Algorithm for (5.2)). For $k = 0$, $(\zeta_0, \xi_0) \in \widehat{V}_k \times \mathring{\widehat{V}}_k$ is determined by $(A_0\zeta_0, \xi_0) = f_0$. For $k \geq 1$, the approximate solution (ζ_k, ξ_k) is obtained recursively from

$$
\begin{aligned}
(\zeta_0^k, \xi_0^k) &= I_{k-1}^k(\zeta_{k-1}, \xi_{k-1}),\\
(\zeta_\ell^k, \xi_\ell^k) &= MG_W(k, f_k, (\zeta_{\ell-1}^k, \xi_{\ell-1}^k), m, m), \quad 1 \leq \ell \leq r,\\
(\zeta_k, \xi_k) &= (\zeta_r^k, \xi_r^k).
\end{aligned}
$$

The following lemma provides the convergence of ξ_k obtained by the full multigrid method of Algorithm 6.1. The proof is based on (4.1), (5.7) and (5.9), which is similar to that of [10, Theorem 7.2].

Lemma 6.2. *If r is sufficiently large, then there exists a positive constant C independent of h_k such that*

$$|\xi - \xi_k|_{H^1(\Omega)} \leq Ch_k.$$

Next we consider the following k-th level P_1 finite element method for (2.3):

Find $\widetilde{\phi}_k \in \widehat{V}_k$ such that

$$(\mathbf{curl}\, \widetilde{\phi}_k, \mathbf{curl}\, \psi) = (\xi_k, \psi) \quad \text{for all } \psi \in \widehat{V}_k, \tag{6.1}$$

where ξ_k is the approximate solution of (5.2). According to Lemma 4.2, we have

$$|\phi - \widetilde{\phi}_k|_{H^1(\Omega)} \leq Ch_k. \tag{6.2}$$

Problem (6.1) can be solved by a W-cycle algorithm (cf. [20, Algorithm 5.6]) and we can apply a full multigrid method (with r iterations of the W-cycle algorithm, cf. [20, Algorithm 7.5]) to compute an approximate solution ϕ_k of $\widetilde{\phi}_k$. The following result can be established by combining (6.2) and the techniques in [4].

Lemma 6.3. *If r is sufficiently large, then there exists a positive constant C independent of h_k such that*

$$|\phi - \phi_k|_{H^1(\Omega)} \leq Ch_k.$$

In the case where Ω is not simply connected (i.e., $m \geq 1$), we can apply a full multigrid method (with r iterations of a standard W-cycle algorithm) to obtain an approximate solution $\varphi_{j,k}$ for the Dirichlet boundary value problem (1.3). The following result is standard.

Lemma 6.4. *If r is sufficiently large, then there exists a positive constant C independent of h_k such that for $1 \leq j \leq m$,*

$$|\varphi - \varphi_{j,k}|_{H^1(\Omega)} \leq Ch_k.$$

For each level k, we compute $c_{1,k}, \dots, c_{m,k}$ by solving

$$\sum_{j=1}^{m} (\mathbf{grad}\, \varphi_{j,k}, \mathbf{grad}\, \varphi_{i,k}) c_{j,k} = \frac{1}{\alpha}(\boldsymbol{f}, \mathbf{grad}\, \varphi_{i,k}) \quad \text{for } 1 \leq i \leq m.$$

We compare the solution c_j of (1.14) and $c_{j,k}$ in the next lemma (cf. [11, Lemma 4.6]).

Lemma 6.5. *If r is sufficiently large, then there exists a positive constant C independent of h_k such that*

$$|c_j - c_{j,k}| \leq Ch_k \quad \textit{for } 1 \leq j \leq m.$$

Finally, we define the approximation $\boldsymbol{u}_k$ of $\boldsymbol{u}$ by

$$\boldsymbol{u}_k = \mathbf{curl}\, \phi_k + \sum_{j=1}^{m} c_{j,k}\, \mathbf{grad}\, \varphi_{j,k}.$$

6.2 The Case Where $\gamma = 0$

First we apply a full multigrid method (with r iterations of a W-cycle algorithm, cf. [20, Algorithm 7.5]) to obtain an approximate solution $\rho_k \in \widehat{V}_k$ for (2.1). We have the following standard result.

Lemma 6.6. *If r is sufficiently large, then there exists a positive constant C independent of h_k such that*

$$|\rho - \rho_k|_{H^1(\Omega)} \le Ch_k.$$

Next we consider the k-th level P_1 finite element method for (2.2):

Find $\tilde{\xi}_k \in \hat{\mathring{V}}_k$ such that

$$(\mathbf{curl}\,\tilde{\xi}_k, \mathbf{curl}\,\eta) = (\rho_k, \eta) \quad \text{for all } \eta \in \hat{\mathring{V}}_k.$$

It follows from Lemma 4.1 that

$$|\xi - \tilde{\xi}_k|_{H^1(\Omega)} \le Ch_k. \tag{6.3}$$

We now apply a full multigrid method (with r iterations of a W-cycle algorithm, cf. [20, Algorithm 7.5]) to obtain an approximate solution ξ_k of $\tilde{\xi}_k$. Combining (6.3) and the theory in [4], we can show that Lemma 6.2 is also valid here.

Finally, we solve (6.1) by a full multigrid method to obtain an approximate solution ϕ_k and we again obtain Lemma 6.3 by the same arguments.

The approximation $\boldsymbol{u}_k$ of $\boldsymbol{u}$ is defined by $\boldsymbol{u}_k = \mathbf{curl}\,\phi_k$.

6.3 Convergence of the Full Multigrid Methods

By combining Lemmas 6.2–6.6, we can show the uniform convergence of multigrid algorithms based on graded meshes for both $\gamma > 0$ and $\gamma = 0$. The proof is similar to the proof of [11, Theorem 4.9].

Theorem 6.7. *If r is sufficiently large, then there exists a positive constant C independent of h_k such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_k\|_{L_2(\Omega)} + |\mathrm{curl}\,\boldsymbol{u} - \xi_k|_{H^1(\Omega)} \le Ch_k.$$

7 Numerical Experiments

In this section we report numerical results for the P_1 finite element method introduced in Section 2. The numerical experiments are performed on three different domains: the unit square, a nonconvex but simply connected domain and a domain whose Betti number is 1. The triangulations $\mathcal{T}_k$ ($k \ge 0$) for each domain are generated by the refinement procedure described in Section 5. The numerical solutions are obtained by full multigrid algorithms, where $\lambda = \frac{1}{10}$, $r = 2$, and the numbers of smoothing steps m_1 and m_2 are taken to be 10.

Experiment 7.1. In the first experiment the domain Ω is the unit square $(0, 1) \times (0, 1)$, so we just need to use uniform meshes. We take $\beta = \gamma = 0$ and the exact solution to be $\boldsymbol{u} = \mathbf{curl}\,\phi$, where

$$\phi(x) = \sin^3(\pi x_1)\sin^3(\pi x_2).$$

We solve (1.1) by the full multigrid algorithm for the P_1 finite element method. The results are presented in Table 1, where Π_k is the nodal interpolation operator for the P_1 finite element space at level k.

h_k	$\|\Pi_k \boldsymbol{u} - \boldsymbol{u}_k\|_{L_2(\Omega)}$	Order	$|\Pi_k\,\mathrm{curl}\,\boldsymbol{u} - \xi_k|_{H^1(\Omega)}$	Order
1/16	7.9690E−01	0.517	4.3451E+01	0.487
1/32	4.1128E−01	0.954	2.2589E+01	0.944
1/64	1.4981E−01	1.457	7.6587E+00	1.560
1/128	6.1158E−02	1.293	2.6138E+00	1.551

Table 1: Results of Experiment 7.1 on uniform meshes.

h_k	$\frac{\|u_k-u_{k+1}\|_{L_2(\Omega)}}{\|u_{k+1}\|_{L_2(\Omega)}}$	Order	$\frac{\|\xi_k-\xi_{k+1}\|_{H^1(\Omega)}}{\|\xi_{k+1}\|_{H^1(\Omega)}}$	Order
1/16	1.0155E−01	0.800	1.6139E−01	0.875
1/32	4.2303E−02	1.223	8.0973E−02	0.968
1/64	1.6907E−02	1.313	4.0316E−02	0.999
1/128	7.6816E−03	1.136	2.0118E−02	1.000

Table 2: Results of Experiment 7.2 on uniform meshes.

Experiment 7.2. In the second experiment the domain Ω is also the unit square $(0,1)\times(0,1)$. We take $\beta=\gamma=0$ and

$$\boldsymbol{f}=\begin{bmatrix}(x_1^2+1)\sin x_1+x_1x_2^3+2\\(x_2^2+1)\cos x_1+x_1^3x_2^2-1\end{bmatrix}.$$

Since the exact solution is not known, the relative errors are estimated on consecutive refinement levels.

Experiment 7.3. In the third experiment we solve (1.1) on the nonconvex domain (cf. Figure 4) whose vertices are

$$(0,0),(.5,0),(.5,.7),(1,.7),(1,1),(0,1),(1,.75),(.25,.75),(.25,.625)\text{ and }(0,.625).$$

The meshes are graded near the reentrant corners $(.5,.7)$, $(.25,.75)$ and $(.25,.625)$, and the grading parameters are taken to be $\frac{2}{3}$. Three consecutive levels of graded triangulations are depicted in Figure 4.

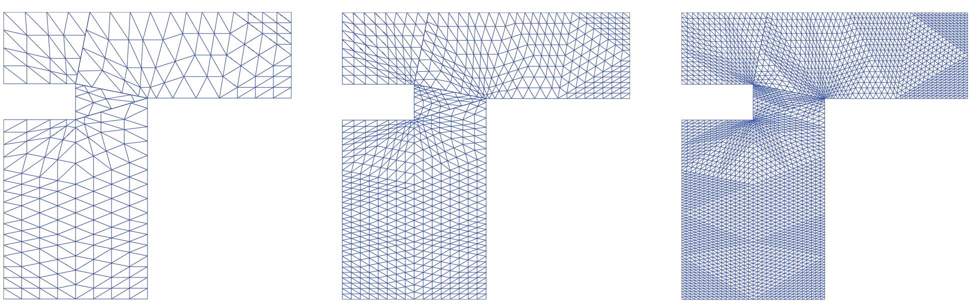

Figure 4: Graded meshes for the domain in Experiment 7.3.

We take $\beta=\gamma=0$ and use a piecewise constant vector field $\boldsymbol{f}$ defined by

$$\boldsymbol{f}=\begin{cases}\left[\frac{1}{4},\frac{5}{4}\right]^t, & |x|<2^{-\frac{1}{2}},\\ \left[\frac{1}{2},\frac{3}{2}\right]^t, & 2^{-\frac{1}{2}}\le|x|<1,\\ \left[1,2\right]^t, & |x|\le 1.\end{cases}$$

The estimated relative errors are presented in Table 3.

h_k	$\frac{\|u_k-u_{k+1}\|_{L_2(\Omega)}}{\|u_{k+1}\|_{L_2(\Omega)}}$	Order	$\frac{\|\xi_k-\xi_{k+1}\|_{H^1(\Omega)}}{\|\xi_{k+1}\|_{H^1(\Omega)}}$	Order
1/32	1.4770E−01	0.625	2.7070E−01	0.856
1/64	8.6300E−02	0.684	1.4000E−01	0.889
1/128	4.3600E−02	0.951	7.0700E−02	0.961
1/256	2.0900E−02	1.054	3.5200E−02	0.999

Table 3: Results of Experiment 7.3 on graded meshes.

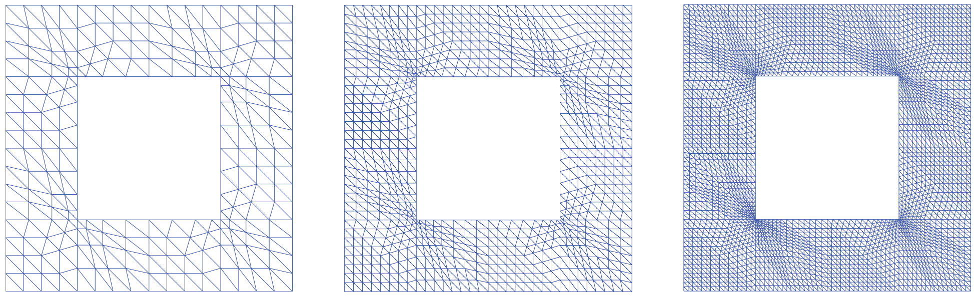

Figure 5: Graded meshes for the domain in Experiment 7.4 and Experiment 7.5.

The last two sets of experiments are performed on a doubly connected domain

$$\Omega = [(0,1)\times(0,1)] \setminus [(\tfrac{1}{4},\tfrac{3}{4})\times(\tfrac{1}{4},\tfrac{3}{4})].$$

In Experiment 7.4, we take the exact solution $\boldsymbol{u} = \mathbf{curl}\,\phi$ for some smooth function ϕ, and the results are obtained on uniform meshes. In Experiment 7.5, we report the results for a given right-hand side function $\boldsymbol{f}$. The meshes are graded near the reentrant corners $(\frac{1}{4},\frac{1}{4})$, $(\frac{1}{4},\frac{3}{4})$, $(\frac{3}{4},\frac{1}{4})$, and $(\frac{3}{4},\frac{3}{4})$ with grading parameters taken to be $\frac{2}{3}$. Three consecutive levels of graded triangulations for the doubly connected domain Ω are depicted in Figure 5.

Note that the Betti number of the doubly connected domain is 1. Hence the solution $\boldsymbol{u}$ of (1.1) is given by

$$\boldsymbol{u} = \mathbf{curl}\,\phi + c\,\mathbf{grad}\,\varphi, \tag{7.1}$$

where the harmonic function φ vanishes on the outer boundary of Ω and equals 1 on the inner boundary of Ω. The approximation of $\boldsymbol{u}$ on each level of k is

$$\boldsymbol{u}_k = \mathbf{curl}\,\phi_k + c_k\,\mathbf{grad}\,\varphi_k.$$

Experiment 7.4. We take $\beta = \gamma = 1$ and the exact solution to be $\boldsymbol{u} = \mathbf{curl}\,\phi$, where

$$\phi = \sin^3(4\pi x_1)\sin^3(4\pi x_2).$$

In this case,

$$\boldsymbol{u} = \mathbf{curl}\,\phi = \begin{bmatrix} 12\pi\sin^3(4\pi x_1)\cos(4\pi x_2)\sin^2(4\pi x_2) \\ -12\pi\cos(4\pi x_1)\sin^2(4\pi x_1)\sin^3(4\pi x_2) \end{bmatrix}. \tag{7.2}$$

Then we have

$$\operatorname{curl}\boldsymbol{u} = -96\pi^2\sin(4\pi x_1)\sin(4\pi x_2)(-3\sin^2(4\pi x_1)\sin^2(4\pi x_2) + \sin^2(4\pi x_1) + \sin^2(4\pi x_2))$$

and $\operatorname{curl}\boldsymbol{u} = 0$ on $\partial\Omega$. In addition, we have $\operatorname{div}\boldsymbol{u} = 0$ and $\boldsymbol{n}\times\boldsymbol{u} = 0$ on $\partial\Omega$. The relative errors obtained on uniform meshes are tabulated in Table 4. Note that in this case $\boldsymbol{u}$ is the **curl** of a smooth function and $c = 0$ in (7.1). It is observed that the values of c_k are very close to 0.

h_k	$\frac{\lVert\Pi_k\boldsymbol{u}-\boldsymbol{u}_k\rVert_{L_2(\Omega)}}{\lVert\Pi_k\boldsymbol{u}\rVert_{L_2(\Omega)}}$	Order	$\lvert c_k\rvert$	$\frac{\lvert\Pi_k\operatorname{curl}\boldsymbol{u}-\xi_k\rvert_{H^1(\Omega)}}{\lvert\Pi_k\operatorname{curl}\boldsymbol{u}\rvert_{H^1(\Omega)}}$	Order
1/32	6.1841E−01	–	1.3675E−10	5.0137E−01	–
1/64	3.4728E−01	0.832	1.3431E−10	2.4546E−01	1.030
1/128	1.5476E−01	1.166	1.6085E−10	6.9475E−02	1.821
1/256	6.3354E−02	1.288	1.5176E−10	1.4356E−02	2.275

Table 4: Results for doubly connected domain with uniform meshes and exact solution given by (7.2)

h_k	$\frac{\|\boldsymbol{u}_k-\boldsymbol{u}_{k+1}\|_{L_2(\Omega)}}{\|\boldsymbol{u}_{k+1}\|_{L_2(\Omega)}}$	Order	c_k	$\frac{\|c_k-c_{k+1}\|}{\|c_{k+1}\|}$	Order	$\frac{\|\xi_k-\xi_{k+1}\|_{H^1(\Omega)}}{\|\xi_{k+1}\|_{H^1(\Omega)}}$	Order
1/32	1.0723E−01	0.840	−0.15127	7.6301E−03	1.627	2.6690E−01	0.731
1/64	5.9115E−02	0.858	−0.15160	2.1866E−03	1.800	1.3582E−01	0.929
1/128	3.2332E−02	0.870	−0.15170	6.6287E−04	1.721	6.8799E−02	0.971
1/256	1.7713E−02	0.878	−0.15173	2.0110E−04	1.721	3.4959E−02	0.975

Table 5: Results for $\boldsymbol{f}$ on doubly connected domain with graded meshes.

Experiment 7.5. In this experiment we report the numerical results for the same $\boldsymbol{f}$ as in Experiment 7.2. We take $\beta = \gamma = 1$ and display the estimated relative errors obtained on graded meshes in Table 5.

Note that the orders of convergence for $\boldsymbol{u}_k$ for all experiments (except Experiment 7.5 where the asymptotic convergence rate has not been reached) are 1 as predicted by Theorem 6.7. The orders of convergence for ξ_k in Table 1 and Table 4 are actually higher than 1. We believe this is the superclose phenomenon due to the higher regularity of $\boldsymbol{u}$ and uniform meshes. The order of convergence for c_k in Table 5 is better than 1, which is due to the fact that $\boldsymbol{f}$ is smooth (cf. [11, Remark 4.8]).

On the nonconvex (but simply connected) domain and doubly connected domain, we compute the numerical solutions by using full (W-cycle) multigrid algorithms for the P_1 finite element method on graded meshes. The convergence results for both $\boldsymbol{u}$ and $\operatorname{curl} \boldsymbol{u}$ have been improved as compared to the results obtained by the P_1 finite element method on uniform meshes (cf. [16, Experiments 3–4]).

8 Concluding Remarks

Following the ideas in [16], we studied P_1 finite element method for a quad-curl problem on properly graded meshes. The approach is based on the Hodge decomposition, and optimal error estimates are obtained. By using properly graded meshes, we improved the convergence results in the presence of reentrant corners. We also developed W-cycle and full multigrid algorithms for the resulting discrete problems.

The Hodge decomposition approach can be applied to quad-curl eigenvalue problems on two-dimensional domains. It can also be applied to quad-curl problems in three dimensions, where the quad-curl problem is reduced to problems that can be solved by standard $H(\text{curl})$, $H(\text{div})$ and H^1 conforming finite element methods. Moreover, these problems have already been analyzed in [1]. These are ongoing projects.

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