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Abstract

We study a hybridizable discontinuous Galerkin method for solving the vorticity-velocity formulation of the Stokes equations in three-space dimensions. We show how to hybridize the method to avoid the construction of the divergence-free approximate velocity spaces, recover an approximation for the pressure and implement the method efficiently. We prove that, when all the unknowns use polynomials of degree $k \geq 0$, the L^2 norm of the errors in the approximate vorticity and pressure converge with order $k + 1/2$ and the error in the approximate velocity converges with order $k + 1$. We achieve this by letting the normal stabilization function go to infinity in the error estimates previously obtained for a hybridizable discontinuous Galerkin method.

Keywords: discontinuous Galerkin methods, hybridization, incompressible fluid flow.

1 Introduction

In this paper, we analyze a hybridizable discontinuous Galerkin (HDG) method for the numerical solution of the Stokes equations:

$$\boldsymbol{w} - \nabla \times \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.1a}$$

$$\nabla \times \boldsymbol{w} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1b}$$

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$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.1c}$$

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \partial\Omega, \tag{1.1d}$$

$$\int_\Omega p = 0. \tag{1.1e}$$

Here we assume that $\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} = 0$ and that $\Omega \subset \mathbb{R}^3$ is a Lipschitz polyhedral domain.

The present study of the HDG method under consideration focuses on the way in which the divergence-free condition is handled. Let us give a brief idea of the main difficulties around this issue. Let us begin by noting, since the approximate velocities in the test space are all taken to be divergence-free, the pressure disappears from the formulation. This poses two problems. The first is how to efficiently implement the method: It is well known that the construction of basis functions for the velocity spaces are almost impossible to construct, especially for high-order approximations in three space dimensions; see the discussion in [1] and the references therein. The second is how to recover an approximate pressure converging as well as the approximate vorticity does. Here we address these questions by extending to our setting the approach taken in [6] for HDG methods based on velocity gradient-velocity formulations.

We proceed as follows. First, we enhance the space of approximate velocities by not requiring that they be elementwise divergence-free and by introducing an approximate pressure in the *interior of the elements*. Then, following [1], we show how the construction of basis functions for the space of approximate velocities is avoided by relaxing the continuity of their normal component across elements and by introducing an approximate pressure defined on the *interelement boundaries*. Finally, we argue that if these two approximate pressures are related by *suitably defined auxiliary unknowns* on each border of the elements, the HDG method under consideration can be formally thought of as a limit of the HDG method introduced in 2009 in [3] and recently analyzed in [2]. More precisely, the HDG method in [3, 2] uses a stabilization function τ_n to control the interelement jumps the normal component of the approximate velocity. In this paper, we let τ_n go to infinity and obtain an HDG method that provides a divergence-conforming and globally divergence-free approximate velocity.

Here we prove that this limiting process does not degrade the already proven convergence properties of the HDG method [2] as the error estimates are *independent* of the normal stabilization function τ_n. This idea was proposed in [6], where the HDG methods based on a velocity gradient-velocity formulation were shown to be the limit as τ_n goes to infinity of the HDG methods introduced in [9] and analyzed in [4]. Thus, the approximate vorticity and pressure, which are piecewise polynomials of degree k, converge with order $k + 1/2$ in L^2 norm for any $k \geq 0$; and, the approximate velocity, which is piecewise polynomial of degree k, converges with order $k + 1$.

These results have to be compared with those corresponding to the method proposed in [1], which hold for the two-dimensional case. Therein, the approximate vorticity and pressure, are taken to be polynomials of degree $k - 1$ converge with order k in L^2 norm for any $k \geq 1$. Moreover, the approximate velocity, which is piecewise polynomial of degree k, converges with order $k+1$ if $k \geq 1$.

Another finite element method for incompressible fluid flow problem, which also uses a vorticity-velocity formulation and $H(\text{div})$-conforming spaces for the velocity approximation was developed in [8]. Therein, the second-type $H(\text{curl})$-conforming edge elements of order k are used to approximate the vorticity, and the $H(\text{div})$-conforming edge elements of order $k - 1$ are used for the approximate velocity. In [8], it was shown that both vorticity and velocity converge with order $k-1$ in the L^2-norm for $k \geq 1$.

The organization of the paper is as follows. In Section 2, we introduce the method and show how to formally relate it to the HDG methods introduced in [3] and analyzed in [2]. We also state the main convergence results which are then proven in Section 3.

2 The HDG method

Here, we present the divergence-conforming method based on a vorticity-velocity formulation. We then show how to introduce an approximate pressure and how to render it efficiently implementable by using a hybridization technique. Finally, we relate the resulting method to the HDG method based on a vorticity-velocity-pressure formulation introduced in [3] and analyzed in [2].

2.1 The mesh and the associated spaces

Let $\mathcal{T}_h$ be a shape-regular, conforming triangulation of Ω which consists of tetrahedra T. We denote by $\mathcal{E}_h$ the set of all faces of all tetrahedra T of $\mathcal{T}_h$ and by $\partial\mathcal{T}_h$ the set of boundaries ∂T of $\mathcal{T}_h$.

We associate to the triangulation $\mathcal{T}_h$ the following finite dimensional spaces that will be used to define the HDG method:

$$\boldsymbol{W}_h := \{\boldsymbol{w} \in \boldsymbol{L}_2(\mathcal{T}_h) : \boldsymbol{w}|_T \in \boldsymbol{P}_k(T) \quad \forall\, T \in \mathcal{T}_h\}, \tag{2.1a}$$

$$\boldsymbol{V}_h := \{\boldsymbol{v} \in \boldsymbol{L}_2(\mathcal{T}_h) : \boldsymbol{v}|_T \in \boldsymbol{P}_k(T) \quad \forall\, T \in \mathcal{T}_h\}, \tag{2.1b}$$

$$\boldsymbol{M}_h := \{\boldsymbol{\mu} \in \boldsymbol{L}_2(\mathcal{E}_h) : \boldsymbol{\mu}|_F \in \boldsymbol{P}_k(F) \quad \forall\, F \in \mathcal{E}_h\}, \tag{2.1c}$$

$$M_h^{\partial} := \{\mu \in L_2(\partial\mathcal{T}_h) : \mu|_{\partial T} \in \mathcal{R}_k(\partial T) \quad \forall\, T \in \mathcal{T}_h\}, \tag{2.1d}$$

where $P_k(T)$ is the space of polynomials of total degree at most k defined on T, and $\boldsymbol{P}_k(T) = [P_k(T)]^n$, $P_k(F)$ and $\boldsymbol{P}_k(F)$ are the corresponding polynomial spaces on F, and

$$\mathcal{R}_k(\partial T) := \{\delta \in L_2(\partial T) : \delta|_F \in P_k(F) \ \ \forall F \in \partial T\}.$$

Note that functions in M_h^∂ are allowed to have different values on two sides of an interior face F. We are also going to use the affine manifolds

$$\widehat{\boldsymbol{V}}_h(\boldsymbol{g}) =\{\boldsymbol{v} \in \boldsymbol{V}_h \cap \boldsymbol{H}(\mathrm{div}, \Omega) :\ \langle (\boldsymbol{v} - \boldsymbol{g}) \cdot \boldsymbol{n}, \eta \rangle_{\partial\Omega} = 0 \ \forall \eta \in M_h^\partial\}, \tag{2.2a}$$

$$\widetilde{\boldsymbol{V}}_h(\boldsymbol{g}) =\{\boldsymbol{v} \in \widehat{\boldsymbol{V}}_h(\boldsymbol{g}) :\ \nabla \cdot \boldsymbol{v} = 0\}, \tag{2.2b}$$

as well as the pressure space

$$P_h := P_h^{k-1} \oplus P_h^\perp, \tag{2.3a}$$

$$P_h^{k-1} :=\{q \in L_2(\mathcal{T}_h) : q|_T \in P_{k-1}(T) \ \ \forall\, T \in \mathcal{T}_h\}, \tag{2.3b}$$

$$P_h^\perp :=\{q \in L_2(\mathcal{T}_h) : q|_T \in P_k(T)^\perp \ \ \ \forall\, T \in \mathcal{T}_h\}, \tag{2.3c}$$

where $P_k(T)^\perp$ is the space of polynomials in $P_k(T)$ which are $L^2(K)$-orthogonal to the elements of $P_{k-1}(T)$.

2.2 The divergence-free method

The method consists in looking for $(\boldsymbol{w}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) \in \boldsymbol{W}_h \times \widetilde{\boldsymbol{V}}_h(\boldsymbol{g}) \times \boldsymbol{M}_h$ such that

$$(\boldsymbol{w}_h, \boldsymbol{\tau})_{\mathcal{T}_h} - (\boldsymbol{u}_h, \nabla \times \boldsymbol{\tau})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{\tau} \times \boldsymbol{n} \rangle_{\partial\mathcal{T}_h} = 0, \tag{2.4a}$$

$$(\boldsymbol{w}_h, \nabla \times \boldsymbol{v})_{\mathcal{T}_h} + \langle \boldsymbol{n} \times \widehat{\boldsymbol{w}}_h, \boldsymbol{v}_t \rangle_{\partial\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}, \tag{2.4b}$$

$$\langle \boldsymbol{u}_h \cdot \boldsymbol{n} - \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, \boldsymbol{\mu} \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_h} = 0, \tag{2.4c}$$

$$\langle \boldsymbol{n} \times \widehat{\boldsymbol{w}}_h, \boldsymbol{\mu}_t \rangle_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0, \tag{2.4d}$$

$$\langle \widehat{\boldsymbol{u}}_{h,t}, \boldsymbol{\mu}_t \rangle_{\partial\Omega} = \langle \boldsymbol{g}_t, \boldsymbol{\mu}_t \rangle_{\partial\Omega}, \tag{2.4e}$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\mu}) \in \boldsymbol{W}_h \times \widetilde{\boldsymbol{V}}_h(\boldsymbol{0}) \times \boldsymbol{M}_h$, where

$$\widehat{\boldsymbol{w}}_h := \boldsymbol{w}_h + \tau_t(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) \times \boldsymbol{n} \quad \text{on } \partial\mathcal{T}_h \qquad \text{on } \partial\mathcal{T}_h. \tag{2.4f}$$

Here the stabilization function τ_t is taken to be constant on each face on $\partial\mathcal{T}_h$.

Note that $\boldsymbol{v}_t := \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n} = \boldsymbol{n} \times \boldsymbol{v} \times \boldsymbol{n}$. We also use the standard notation $(v, w)_{\mathcal{T}_h} := \sum_{T \in \mathcal{T}_h} (v, w)_T, (\boldsymbol{v}, \boldsymbol{w})_{\mathcal{T}_h} := \sum_{i=1}^n (v_i, w_i)_{\mathcal{T}_h}$, where $(\cdot, \cdot)_T$ denotes the L_2-inner product on $L^2(T)$. Similarly, $\langle v, w \rangle_{\partial\mathcal{T}_h} := \sum_{T \in \mathcal{T}_h} \langle v, w \rangle_{\partial T}$ and $\langle \boldsymbol{v}, \boldsymbol{w} \rangle_{\partial\mathcal{T}_h} := \sum_{i=1}^n \langle v_i, w_i \rangle_{\partial\mathcal{T}_h}$, where $\langle \cdot, \cdot \rangle_{\partial T}$ is the inner product on $L^2(\partial T)$.

Note that the pressure does not appear in the formulation due to the fact that the velocity test functions lie on $\widetilde{\boldsymbol{V}}_h(\boldsymbol{0})$ and are thus divergence-free.

A similar method was introduced in [6] for HDG methods based on velocity gradient-velocity formulations. Therein, the stabilization function $\widehat{\boldsymbol{u}}_h$ could also be eliminated from the formulation by setting τ_t equal to zero. In our case, the existence and uniqueness of the approximate solution is guaranteed only when τ_t is strictly positive function, as we see in the following result.

Proposition 2.1 Assume that Ω is simply connected and that $\tau_t > 0$ on $\partial\mathcal{T}_h$. Then the approximate solution $(\boldsymbol{w}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)$ defined by the equations (2.4) exists and is unique.

Proof. Since the system defining the approximate solution is square, we only have to show that the only solution in the case $(\boldsymbol{f}, \boldsymbol{g}) := (\boldsymbol{0}, \boldsymbol{0})$ is the trivial solution.

So, assuming that this is the case, taking $\boldsymbol{\tau} := \boldsymbol{w}_h$, $\boldsymbol{v} := \boldsymbol{u}_h$, $\boldsymbol{\mu} := -\widehat{\boldsymbol{u}}_h$ and $\boldsymbol{\mu} := -\boldsymbol{n} \times \widehat{\boldsymbol{w}}_h$ in the first four equations of the weak formulation, respectively, and adding them up, we get

$$(\boldsymbol{w}_h, \boldsymbol{w}_h)_{\mathcal{T}_h} + \langle \boldsymbol{n} \times (\widehat{\boldsymbol{w}}_h - \boldsymbol{w}_h), \boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h \rangle_{\partial\mathcal{T}_h} = 0$$

Since $\tau_t > 0$ on $\partial\mathcal{T}_h$, the equations (2.4c) and (2.4f) allow us to conclude that $\boldsymbol{w}_h = \mathbf{0}$ in $\mathcal{T}_h$ and that $\boldsymbol{u}_h = \widehat{\boldsymbol{u}}_h$ on $\mathcal{E}_h$.

The first equation defining the method implies that $\nabla \times \boldsymbol{u}_h = \mathbf{0}$ in $\mathcal{T}_h$. Since Ω is simply connected, this implies that $\boldsymbol{u}_h = \nabla S$. The fact that S must be a harmonic function follows from the fact that $\nabla \cdot \boldsymbol{u}_h = 0$ in Ω. Finally, since $\boldsymbol{n} \cdot \nabla S|_{\partial\Omega} = \boldsymbol{n} \cdot \widehat{\boldsymbol{u}}_h|_{\partial\Omega} = 0$, we conclude that S is a constant and hence that $\boldsymbol{u}_h = \mathbf{0}$ in $\mathcal{T}_h$ and that $\widehat{\boldsymbol{u}}_h = \mathbf{0}$ on $\mathcal{E}_h$. This completes the proof. □

2.3 The HDG method

Next, we hybridize the method just described in order to define an approximate pressure and in order to render it efficiently implementable. We do this in several steps.

• Step 1: Introduction of the pressure p_h^{k-1} in $\mathcal{T}_h$

We begin by relaxing the condition $\nabla \cdot \boldsymbol{u}_h = 0$. Thus, instead of taking the test velocities $\boldsymbol{v}$ in $\widetilde{\boldsymbol{V}}_h(\mathbf{0})$, we take them in the larger space in $\widehat{\boldsymbol{V}}_h(\mathbf{0})$. This allows us to introduce an approximation for the pressure, $p_h^{k-1} \in P_h^{k-1}$, into the equations. Consequently, instead of taking $\boldsymbol{u}_h$ in $\widetilde{\boldsymbol{V}}_h(\boldsymbol{g})$, we take it in $\widehat{\boldsymbol{V}}_h(\boldsymbol{g})$. We then must force its divergence on each element to be zero by introducing new equations in the formulation.

Thus, we now define the approximation $(\boldsymbol{w}_h, \boldsymbol{u}_h, p_h^{k-1}, \widehat{\boldsymbol{u}}_h) \in \boldsymbol{W}_h \times \widehat{\boldsymbol{V}}_h(\boldsymbol{g}) \times P_h^{k-1} \times \boldsymbol{M}_h$ by requiring that

$$(\boldsymbol{w}_h, \boldsymbol{\tau})_{\mathcal{T}_h} - (\boldsymbol{u}_h, \nabla \times \boldsymbol{\tau})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{\tau} \times \boldsymbol{n} \rangle_{\partial\mathcal{T}_h} = 0, \tag{2.5a}$$

$$(\boldsymbol{w}_h, \nabla \times \boldsymbol{v})_{\mathcal{T}_h} - (p_h^{k-1}, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \langle \boldsymbol{n} \times \widehat{\boldsymbol{w}}_h, \boldsymbol{v}_t \rangle_{\partial\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}, \tag{2.5b}$$

$$(\nabla \cdot \boldsymbol{u}_h, q)_{\mathcal{T}_h} = 0, \tag{2.5c}$$

$$\langle \boldsymbol{u}_h \cdot \boldsymbol{n} - \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, \boldsymbol{\mu} \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_h} = 0, \tag{2.5d}$$

$$\langle \boldsymbol{n} \times \widehat{\boldsymbol{w}}_h, \boldsymbol{\mu}_t \rangle_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0, \tag{2.5e}$$

$$\langle \widehat{\boldsymbol{u}}_{h,t}, \boldsymbol{\mu}_t \rangle_{\partial\Omega} = \langle \boldsymbol{g}_t, \boldsymbol{\mu}_t \rangle_{\partial\Omega}, \tag{2.5f}$$

$$(p_h^{k-1}, 1)_{\mathcal{T}_h} = 0, \tag{2.5g}$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \boldsymbol{W}_h \times \widehat{\boldsymbol{V}}_h(\boldsymbol{0}) \times P_h^{k-1} \times \boldsymbol{M}_h$, where

$$\widehat{\boldsymbol{w}}_h := \boldsymbol{w}_h + \tau_t(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) \times \boldsymbol{n} \quad \text{on } \partial\mathcal{T}_h \qquad \text{on } \partial\mathcal{T}_h. \tag{2.5h}$$

• Step 2: Introduction of the pressure $\widehat{p}_h$ on $\mathcal{E}_h$

Now, we relax the continuity constraint of the normal component of the approximate velocity on interelement boundaries. Thus, instead of taking the test velocities $\boldsymbol{v}$ in $\widehat{\boldsymbol{V}}_h(\boldsymbol{0})$, we take them in the larger space in $\boldsymbol{V}_h$. This allows us to introduce an approximation for the pressure, $\widehat{p}_h \in M_h^{\partial}$, into the equations. Consequently, instead of taking $\boldsymbol{u}_h$ in $\widehat{\boldsymbol{V}}_h(\boldsymbol{g})$, we take it in $\boldsymbol{V}_h$. We then must enforce the continuity of its normal component across interelement boundaries by introducing new equations in the formulation.

So, we look for $(\boldsymbol{w}_h, \boldsymbol{u}_h, p_h^{k-1}, \widehat{\boldsymbol{u}}_h, \widehat{p}_h)$ in the space $\boldsymbol{W}_h \times \boldsymbol{V}_h \times P_h^{k-1} \times \boldsymbol{M}_h \times M_h^{\partial}$ and determine it by requiring that

$$(\boldsymbol{w}_h, \boldsymbol{\tau})_{\mathcal{T}_h} - (\boldsymbol{u}_h, \nabla \times \boldsymbol{\tau})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \boldsymbol{\tau} \times \boldsymbol{n} \rangle_{\partial\mathcal{T}_h} = 0, \tag{2.6a}$$

$$(\boldsymbol{w}_h, \nabla \times \boldsymbol{v})_{\mathcal{T}_h} - (p_h^{k-1}, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \langle \boldsymbol{n} \times \widehat{\boldsymbol{w}}_h + \widehat{p}_h \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}, \tag{2.6b}$$

$$(\nabla \cdot \boldsymbol{u}_h, q)_{\mathcal{T}_h} = 0 \tag{2.6c}$$

$$\langle \widehat{\boldsymbol{u}}_h, \boldsymbol{\mu} \rangle_{\partial\Omega} = \langle \boldsymbol{g}, \boldsymbol{\mu} \rangle_{\partial\Omega}, \tag{2.6d}$$

$$\langle \boldsymbol{n} \times \widehat{\boldsymbol{w}}_h + \widehat{p}_h \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0, \tag{2.6e}$$

$$\langle (\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n}, \eta \rangle_{\partial\mathcal{T}_h} = 0, \tag{2.6f}$$

$$(p_h^{k-1}, 1)_{\mathcal{T}_h} = 0, \tag{2.6g}$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}, q, \boldsymbol{\mu}, \eta) \in \boldsymbol{W}_h \times \boldsymbol{V}_h \times P_h^{k-1} \times \boldsymbol{M}_h \times M_h^{\partial}$, where

$$\widehat{\boldsymbol{w}}_h := \boldsymbol{w}_h + \tau_t(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) \times \boldsymbol{n} \quad \text{on } \partial\mathcal{T}_h \quad \text{on } \partial\mathcal{T}_h. \tag{2.6h}$$

Note that the divergence-conforming constraint on $\boldsymbol{u}_h$ is imposed by equation (2.6f) in the above formulation.

• Step 3: The form of $\widehat{p}_h$ and the introduction of the pressure p_h

Finally, we give a particular form to the numerical trace of the pressure $\widehat{p}_h$ which will allow us to improve the quality of the approximation of the pressure in the interior of the elements.

So, for each element $T \in \mathcal{T}_h$, we take

$$\begin{aligned} \widehat{p}_h &:= p_h + \delta_h && \text{on } \partial T, \\ p_h &:= p_h^{k-1} + p_h^{\perp} && \text{in } T, \end{aligned}$$

where $p_h^{\perp}|_T \in P_k^{\perp}(T)$ and $\delta_h|_{\partial T} \in \mathcal{R}_k(\partial T)$ is such that

$$\langle \delta_h, q^{\perp} \rangle_{\partial\mathcal{T}_h} = 0 \qquad \forall\, q^{\perp} \in P_h^{\perp}.$$

Now, note that, since $p_h^{\perp}|_T \in P_k^{\perp}(T)$, we can write that

$$(p_h^{k-1}, 1)_{\Omega} = (p_h^{k-1} + p_h^{\perp}, 1)_{\Omega} = (p_h, 1)_{\Omega},$$

and that, for any $\boldsymbol{v} \in \boldsymbol{P}_k(T)$,

$$\begin{aligned} -(p_h^{k-1}, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} &= -(p_h^{k-1} + p_h^{\perp}, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} \\ &= -(p_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} \\ &= -\langle p_h \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial\mathcal{T}_h} + (\nabla p_h, \boldsymbol{v})_{\mathcal{T}_h}. \end{aligned}$$

Note also that, by the definitions of $\widehat{\boldsymbol{w}}_h$ and $\widehat{p}_h$, the equations (2.6b) and (2.6e) read

$$
\begin{aligned}
(\nabla\times\boldsymbol{w}_h+\nabla p_h,\boldsymbol{v})_{\mathcal{T}_h}+\langle\tau_t(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h)_t+\delta_h\boldsymbol{n},\boldsymbol{v}\rangle_{\partial\mathcal{T}_h}&=(\boldsymbol{f},\boldsymbol{v})_{\mathcal{T}_h},\\
\langle\boldsymbol{n}\times\boldsymbol{w}_h+p_h\boldsymbol{n}+\tau_t(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h)_t+\delta_h\boldsymbol{n},\boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega}&=0.
\end{aligned}
$$

Thus, we arrive at a HDG method which seeks the approximate solution $(\boldsymbol{w}_h,\boldsymbol{u}_h,p_h,\widehat{\boldsymbol{u}}_h,\delta_h)$ in the space $\boldsymbol{W}_h\times\boldsymbol{V}_h\times P_h\times\boldsymbol{M}_h\times M_h^{\partial}$ such that

$$
\begin{aligned}
(\boldsymbol{w}_h,\boldsymbol{\tau})_{\mathcal{T}_h}-(\boldsymbol{u}_h,\nabla\times\boldsymbol{\tau})_{\mathcal{T}_h}-\langle\widehat{\boldsymbol{u}}_h,\boldsymbol{\tau}\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}&=0, &\text{(2.7a)}\\
(\nabla\times\boldsymbol{w}_h+\nabla p_h,\boldsymbol{v})_{\mathcal{T}_h}+\langle\tau_t(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h)_t+\delta_h\boldsymbol{n},\boldsymbol{v}\rangle_{\partial\mathcal{T}_h}&=(\boldsymbol{f},\boldsymbol{v})_{\mathcal{T}_h}, &\text{(2.7b)}\\
(\nabla\cdot\boldsymbol{u}_h,q)_{\mathcal{T}_h}&=0, &\text{(2.7c)}\\
\langle\delta_h,q^{\perp}\rangle_{\partial\mathcal{T}_h}&=0, &\text{(2.7d)}\\
\langle\widehat{\boldsymbol{u}}_h,\boldsymbol{\mu}\rangle_{\partial\Omega}&=\langle\boldsymbol{g},\boldsymbol{\mu}\rangle_{\partial\Omega}, &\text{(2.7e)}\\
\langle\boldsymbol{n}\times\boldsymbol{w}_h+p_h\boldsymbol{n}+\tau_t(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h)_t+\delta_h\boldsymbol{n},\boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega}&=0, &\text{(2.7f)}\\
\langle(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n},\eta\rangle_{\partial\mathcal{T}_h}&=0, &\text{(2.7g)}\\
(p_h,1)_{\mathcal{T}_h}&=0. &\text{(2.7h)}
\end{aligned}
$$

for all $(\boldsymbol{\tau},\boldsymbol{v},q,q^{\perp},\boldsymbol{\mu},\eta)\in\boldsymbol{W}_h\times\boldsymbol{V}_h\times P_h^{k-1}\times P_h^{\perp}\times\boldsymbol{M}_h\times M_h^{\partial}$.

Note that, due to the last equation of this formulation, the number of unknowns seems to be exactly one less than the number of linearly independent equations. However, there is *one* equation that is linearly dependent, namely the one obtained by taking $q:=1$ in the equation (2.7c). Indeed, we have

$$
\begin{aligned}
(\nabla\cdot\boldsymbol{u}_h,1)_{\mathcal{T}_h}&=\langle\boldsymbol{u}_h\cdot\boldsymbol{n},1\rangle_{\partial\mathcal{T}_h}\\
&=\langle\widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n},1\rangle_{\partial\mathcal{T}_h} \quad \text{by (2.7g) with } \eta:=1,\\
&=\langle\widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n},1\rangle_{\partial\Omega}\\
&=\langle\boldsymbol{g}\cdot\boldsymbol{n},1\rangle_{\partial\Omega} \quad \text{by (2.7e) with } \boldsymbol{\mu}:=\boldsymbol{n},
\end{aligned}
$$

$$= 0.$$

Therefore, by excluding the choice $q := 1$ as a test function in (2.7c), we can consider that the above system of equations defined by (2.7) is actually square.

Moreover, the method is actually well defined, as we see in the next result.

Proposition 2.2 Assume that Ω is simply connected and that $\tau_t > 0$ on $\partial\mathcal{T}_h$. Then the approximate solution $(\boldsymbol{w}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h, \delta_h)$ given by the HDG method (2.7) exists and is unique. Moreover, the component $(\boldsymbol{w}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)$ is the solution of the equations (2.4).

This result allows us to say that we have successfully introduced an approximation for the pressure, p_h, in the original formulation of the method, as wanted. Moreover, as exactly as for the HDG method developed in [6], we can verify that the method can be implemented in such a way that the only globally coupled unknown are $\widehat{u}_h$ and the elementwise averages of p_h.

Proof. Let us begin by noting that, if $(\boldsymbol{w}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h, \delta_h) \in \boldsymbol{W}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h \times M_h^\partial$ solves the equations (2.7), then $(\boldsymbol{w}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)$ solves equations (2.4).

To do that, let us note first that $(\boldsymbol{w}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) \in \boldsymbol{W}_h \times \widetilde{\boldsymbol{V}}_h(\boldsymbol{g}) \times \boldsymbol{M}_h$. Indeed, $\boldsymbol{u}_h \in \boldsymbol{V}_h$ and, by equations (2.7c), (2.7g), and (2.7e) with $\boldsymbol{\mu} := \eta \boldsymbol{n}$ for $\eta \in M_h^\partial$, we readily see that $\boldsymbol{u}_h$ also belongs to $\widetilde{\boldsymbol{V}}_h(\boldsymbol{g})$.

Note that the equation (2.7a) is the same as (2.4a). The fact that $(\boldsymbol{w}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)$ satisfies (2.4b) follows from equations (2.7b), the fact that $\boldsymbol{v} \in \widetilde{\boldsymbol{V}}_h(\boldsymbol{0}) \subset \boldsymbol{V}_h$, and by equation (2.7f). The equation (2.4d) is a direct consequence of (2.7f) with vectors $\boldsymbol{\mu}$ with only tangential components. Finally, the equation (2.4e) holds in view of (2.7e) and (2.7g).

Now, let us prove that the solution (2.7) exists and is unique. Since this is a square system, we only have to prove that, when $(\boldsymbol{f}, \boldsymbol{g}) := (\boldsymbol{0}, \boldsymbol{0})$, the only solution is the trivial one.

Since $(\boldsymbol{w}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)$ solves equations (2.4), we get, by Proposition 2.1 that $(\boldsymbol{w}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$. It remains to show that $(p_h, \delta_h) = (0, 0)$.

To do that, we begin by noting that equations (2.7b) and (2.7f) are now reduced to

$$(\nabla p_h, \boldsymbol{v})_{\mathcal{T}_h} + \langle \delta_h \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial \mathcal{T}_h} = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}_h, \tag{2.8a}$$

$$\langle p_h + \delta_h, \boldsymbol{\mu} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h \setminus \partial\Omega} = 0 \qquad \forall\, \boldsymbol{\mu} \in \boldsymbol{M}_h. \tag{2.8b}$$

By [6, Lemma 4.5], we have that

$$\mathcal{R}_k(\partial T) = \{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial T} :\, \boldsymbol{v} \in \boldsymbol{P}_k(T)^{\perp}\} \oplus \{q|_{\partial T} :\, q \in P_k(T)^{\perp}\}$$

is an orthogonal decomposition in $L_2(\partial T)$. As a consequence, there exists $\boldsymbol{v} \in \boldsymbol{V}_h$ such that $\boldsymbol{v}|_{\partial T} \in \boldsymbol{P}_k(T)^{\perp}$ and $\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial T} = \delta_h|_{\partial T} \quad \forall\, T \in \mathcal{T}_h$. Using this $\boldsymbol{v}$ as a test function in (2.8a), we readily get $\delta_h = 0$. We now take $\boldsymbol{v} := \nabla p_h$ and conclude that p_h is a constant on each element $T \in \mathcal{T}_h$.

Since the equation (2.8b) implies that $p_h \in H^1(\Omega)$, we conclude that p_h is a constant on Ω and by equation (2.7h), that such a constant is actually zero. This completes the proof. $\square$

2.4 HDG methods with completely discontinuous velocities

Here, we are going to establish a relation between the HDG method just considered and the HDG method introduced and studied in [2]. Such HDG method uses spaces for the velocity which are completely discontinuous and uses a stabilization function τ_n in order to control the interelement jumps of their normal component.

The method is defined as follows. We look for $(\boldsymbol{w}^{\tau_h}, \boldsymbol{u}^{\tau_h}, p_h^{\tau_h}, \widehat{\boldsymbol{u}}_h^{\tau_h}) \in \boldsymbol{W}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h$ such that

$$(\boldsymbol{w}_h^{\tau_h}, \boldsymbol{\tau})_{\mathcal{T}_h} - (\boldsymbol{u}_h^{\tau_h}, \nabla \times \boldsymbol{\tau})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h^{\tau_h}, \boldsymbol{\tau} \times \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = 0, \tag{2.9a}$$

$$(\nabla \times \boldsymbol{w}_h^{\tau_h} + \nabla p_h^{\tau_h}, \boldsymbol{v})_{\mathcal{T}_h} + \langle \boldsymbol{S}_\tau(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \boldsymbol{v}\rangle_{\partial\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}, \tag{2.9b}$$

$$-(\boldsymbol{u}_h^{\tau_h}, \nabla q)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{u}}_h^{\tau_h}, q\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.9c}$$

$$\langle \boldsymbol{n} \times \boldsymbol{w}_h^{\tau_h} + p_h^{\tau_h}\boldsymbol{n} + \boldsymbol{S}_\tau(\boldsymbol{u}_h^{\tau_h} - \widehat{\boldsymbol{u}}_h^{\tau_h}), \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} = 0, \tag{2.9d}$$

$$\langle \widehat{\boldsymbol{u}}_h^{\tau_h}, \boldsymbol{\mu}\rangle_{\partial\Omega} = \langle \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega}, \tag{2.9e}$$

$$(p_h^{\tau_h}, 1)_{\mathcal{T}_h} = 0. \tag{2.9f}$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \boldsymbol{W}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h$. Here

$$\boldsymbol{S}_\tau \boldsymbol{\mu} := \tau_t \boldsymbol{n} \times \boldsymbol{\mu} \times \boldsymbol{n} + \tau_n(\boldsymbol{\mu}\cdot\boldsymbol{n})\,\boldsymbol{n},$$

and the stabilization function τ_n is taken to be constant on each face on $\partial\mathcal{T}_h$. We make here explicit dependence of τ_n in the notation since we want to study the behavior of the approximations as τ_n tends to infinity. We want to rewrite this method in the same manner we wrote the HDG method (2.7). To do that, we introduce the quantity

$$\delta_h^{\tau_n} = \tau_n(\boldsymbol{u}_h^{\tau_n} - \widehat{\boldsymbol{u}}_h^{\tau_n})\cdot\boldsymbol{n} \in M_h^\partial$$

as an unknown, and then split (2.9c) into two equations, one corresponding to $q \in P_{k-1}(T)$ and the other one corresponding to $q \in P_k^\perp(T)$.

We can rewrite the above method as follows. We look for the approximation $(\boldsymbol{w}^{\tau_n}, \boldsymbol{u}^{\tau_n}, p_h^{\tau_n}, \widehat{\boldsymbol{u}}_h^{\tau_n}, \delta_h^{\tau_n}) \in \boldsymbol{W}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h \times M_h^\partial$ defined by

$$(\boldsymbol{w}_h^{\tau_n}, \boldsymbol{\tau})_{\mathcal{T}_h} - (\boldsymbol{u}_h^{\tau_n}, \nabla\times\boldsymbol{\tau})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h^{\tau_n}, \boldsymbol{\tau}\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.10a}$$

$$(\nabla \times \boldsymbol{w}_h^{\tau_n} + \nabla p_h^{\tau_n}, \boldsymbol{v})_{\mathcal{T}_h} + \langle \tau_t(\boldsymbol{u}_h^{\tau_n} - \widehat{\boldsymbol{u}}_h^{\tau_n})_t + \delta_h^{\tau_n}\boldsymbol{n}, \boldsymbol{v}\rangle_{\partial\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}, \tag{2.10b}$$

$$(\nabla\cdot\boldsymbol{u}_h^{\tau_n}, q)_{\mathcal{T}_h} - \langle \tau_n^{-1}\delta_h^{\tau_n}, q\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.10c}$$

$$\langle \delta_h^{\tau_n}, q^\perp\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.10d}$$

$$\langle \widehat{\boldsymbol{u}}_h^{\tau_n}, \boldsymbol{\mu}\rangle_{\partial\Omega} = \langle \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega}, \tag{2.10e}$$

$$\langle \boldsymbol{n}\times\boldsymbol{w}_h^{\tau_n} + p_h^{\tau_n}\boldsymbol{n} + \tau_t(\boldsymbol{u}_h^{\tau_n} - \widehat{\boldsymbol{u}}_h^{\tau_n})_t + \delta_h^{\tau_n}\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} = 0, \tag{2.10f}$$

$$\langle \tau_n^{-1}\delta_h^{\tau_n} - (\boldsymbol{u}_h^{\tau_n} - \widehat{\boldsymbol{u}}_h^{\tau_n})\cdot \boldsymbol{n}, \eta\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.10g}$$

$$(p_h^{\tau_n}, 1)_{\mathcal{T}_h} = 0. \tag{2.10h}$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}, q, q^\perp, \boldsymbol{\mu}, \eta) \in \boldsymbol{W}_h \times \boldsymbol{V}_h \times P_h^{k-1} \times P_h^\perp \times \boldsymbol{M}_h \times M_h^\partial$. We can now see that the HDG method (2.7) appears when we formally set $\tau_n^{-1} = 0$ in (2.10c) and (2.10g). Next we explore an important consequence of this fact.

2.5 The divergence-free HDG method as a limit

We now show that the divergence-free HDG method (2.7) is the limit of the HDG method (2.10) as the normal stabilization term τ_n goes to infinity. We follow [6].

Proposition 2.3 Let $(\boldsymbol{w}_h^{\tau_n}, \boldsymbol{u}_h^{\tau_n}, p_h^{\tau_n}, \widehat{\boldsymbol{u}}_h^{\tau_n}, \delta_h^{\tau_n})$ be the solution of (2.10) and $(\boldsymbol{w}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h, \delta_h)$ solve (2.7). Then

$$(\boldsymbol{w}_h^{\tau_n}, \boldsymbol{u}_h^{\tau_n}, p_h^{\tau_n}, \widehat{\boldsymbol{u}}_h^{\tau_n}, \delta_h^{\tau_n}) \to (\boldsymbol{w}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h, \delta_h),$$

in the space $\boldsymbol{W}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h \times M_h^\partial = \mathcal{P}_h(\mathcal{T}_h)$ as $\tau_n \to \infty$.

Proof. Set $s = \tau_n^{-1}$ and denote $\boldsymbol{X}(s) := (\boldsymbol{w}_h^{\tau_n}, \boldsymbol{u}_h^{\tau_n}, p_h^{\tau_n}, \widehat{\boldsymbol{u}}_h^{\tau_n}, \delta_h^{\tau_n}) \in \mathcal{P}_h(\mathcal{T}_h)$. Then (2.10) can be written as

$$\boldsymbol{A}(s)\boldsymbol{X}(s) = \boldsymbol{B},$$

where $\boldsymbol{A}(s)$ is a linear operator from $\mathcal{P}_h(\mathcal{T}_h)$ to its dual space $\mathcal{P}_h(\mathcal{T}_h)'$ and $\boldsymbol{B} \in \mathcal{P}_h(\mathcal{T}_h)'$.

Note that $\boldsymbol{A}(s)$ is an affine function of s and $\boldsymbol{A}(s)$ is invertible for all $s > 0$, see [3, 2], and also for $s = 0$, see Proposition 2.2. Therefore,

$$\lim_{s\to 0} \boldsymbol{A}(s)^{-1} = \boldsymbol{A}^{-1}(0),$$

and

$$\lim_{s\to 0}\boldsymbol{X}(s)=\boldsymbol{A}^{-1}(0)\boldsymbol{B}=(\boldsymbol{w}_h,\boldsymbol{u}_h,p_h,\widehat{\boldsymbol{u}}_h,\delta_h).$$

This completes the proof. □

3 Convergence estimates

Here, we state and prove our error estimates. We begin by introducing a projection $(\boldsymbol{\Pi w},\boldsymbol{\mathit{\Pi} u},\Pi p)$. We then obtain upper bounds for the projection of the errors, namely, $\boldsymbol{\epsilon}^{\boldsymbol{w}}:=\boldsymbol{\Pi w}-\boldsymbol{w}_h$, $\boldsymbol{\epsilon}^{\boldsymbol{u}}:=\boldsymbol{\mathit{\Pi} u}-\boldsymbol{u}_h$, $\epsilon^p:=\Pi p-p_h$ and also $\boldsymbol{\epsilon}^{\widehat{\boldsymbol{u}}}:=\boldsymbol{P}_\partial\boldsymbol{u}-\widehat{\boldsymbol{u}}_h$, where $(\boldsymbol{w}_h,\boldsymbol{u}_h,p_h,\widehat{\boldsymbol{u}}_h)$ is the approximate solution of (2.7) and $\boldsymbol{P}_\partial$ is the L_2 projection into $\boldsymbol{M}_h$. Finally, we obtain the wanted error estimates. The remainder of this section is devoted to proving the estimates.

3.1 The projection

Given a function $(\boldsymbol{w},\boldsymbol{u},p)$ in $\boldsymbol{H}^1(\mathcal{T}_h)\times\boldsymbol{H}^1(\mathcal{T}_h)\times H^1(\mathcal{T}_h)$, we define its projection $(\boldsymbol{\Pi w},\boldsymbol{\mathit{\Pi} u},\Pi p,\Pi^\partial\delta)\in\boldsymbol{W}_h\times\boldsymbol{V}_h\times P_h\times\boldsymbol{M}_h^\partial$ as follows. On an arbitrary element T of the triangulation $\mathcal{T}_h$, we require that

$$(\boldsymbol{\Pi w}-\boldsymbol{w},\boldsymbol{\tau})_T=0\qquad\forall\,\boldsymbol{\tau}\in\boldsymbol{P}_k(T),\tag{3.1a}$$

$$(\boldsymbol{\mathit{\Pi} u}-\boldsymbol{u},\boldsymbol{v})_T=0\qquad\forall\,\boldsymbol{v}\in\boldsymbol{P}_{k-1}(T),\tag{3.1b}$$

$$(\Pi p-p,q)_T=0\qquad\forall\,q\in P_{k-1}(T),\tag{3.1c}$$

$$\langle(\Pi p-p)+(\Pi^\partial\delta),\mu\rangle_F=0\qquad\forall\,\mu\in P_k(F),\tag{3.1d}$$

$$\langle\Pi^\partial\delta,q\rangle_{\partial T}=0\qquad\forall\,q\in P_k(T)^\perp,\tag{3.1e}$$

$$\langle(\boldsymbol{\mathit{\Pi} u}\cdot\boldsymbol{n}-\boldsymbol{u}\cdot\boldsymbol{n}),\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial T}=0\qquad\forall\,\boldsymbol{v}\in\boldsymbol{P}_k(T)^\perp.\tag{3.1f}$$

for all faces F of the tetrahedron T.

The following result states that the above projection is actually well defined and that is has optimal convergence properties.

Theorem 3.1 The system (3.1) is uniquely solvable for $(\mathbf{\Pi w}, \boldsymbol{\Pi u}, \Pi p, \Pi^{\partial}\delta)$. Moreover, if $\nabla\cdot\boldsymbol{u}=0$, there is a constant C independent of T such that

$$\|\boldsymbol{\Pi w}-\boldsymbol{w}\|_T \le C\, h_K^{\ell_{\boldsymbol{w}}+1}\, |\boldsymbol{w}|_{\boldsymbol{H}^{\ell_{\boldsymbol{u}}+1}(T)},$$
$$\|\boldsymbol{\Pi u}-\boldsymbol{u}\|_T \le C\, h_K^{\ell_{\boldsymbol{u}}+1}\, |\boldsymbol{u}|_{\boldsymbol{H}^{\ell_{\boldsymbol{u}}+1}(T)},$$
$$\|\Pi p - p\|_T \le C\, h_K^{\ell_p+1}\, |p|_{H^{\ell_p+1}(T)},$$

for $\ell_p, \ell_{\boldsymbol{w}}, \ell_{\boldsymbol{u}}$ in $[0,k]$.

3.2 Main results

To state our error estimates, we need introduce the following dual problem. For any given $\boldsymbol{\theta}\in\boldsymbol{L}_2(\Omega)$, let $(\boldsymbol{\psi},\boldsymbol{\phi},\phi)$ be the solution of

$$\boldsymbol{\psi}+\nabla\times\boldsymbol{\phi}=0 \quad \text{in } \Omega, \tag{3.2a}$$
$$-\nabla\times\boldsymbol{\psi}-\nabla\phi=\boldsymbol{\theta} \quad \text{in } \Omega, \tag{3.2b}$$
$$-\nabla\cdot\boldsymbol{\phi}=0 \quad \text{in } \Omega, \tag{3.2c}$$
$$\boldsymbol{\phi}=\mathbf{0} \quad \text{on } \partial\Omega, \tag{3.2d}$$
$$\int_\Omega \phi = 0. \tag{3.2e}$$

We assume

$$\|\boldsymbol{\psi}\|_{H^{s+1}(\Omega)}+\|\boldsymbol{\phi}\|_{H^{s+2}(\Omega)}+\|\phi\|_{H^{s+1}(\Omega)} \le C\|\boldsymbol{\theta}\|_{H^s(\Omega)}, \tag{3.3}$$

for some real number s. In three dimensional case, $s\le 0$ if the domain is convex (cf. [7]).

We are now ready to state our main results. They are going to be stated in terms for the following quantity:

$$|||(\mathbf{\Pi w}-\boldsymbol{w}, \boldsymbol{\Pi u}-\boldsymbol{u})|||_{\tau_t,\partial\mathcal{T}_h} := \|\tau_t^{-1/2}\boldsymbol{n}\times(\mathbf{\Pi w}-\boldsymbol{w})+\tau_t^{1/2}(\boldsymbol{\Pi u}-\boldsymbol{u})_t\|_{\partial\mathcal{T}_h},$$

which can be estimated by the following result (c.f. [2, Proposition 2.2]).

Proposition 3.2 ([2]) For all $(\boldsymbol{\zeta}, \boldsymbol{\eta}) \in \boldsymbol{H}^1(\mathcal{T}_h) \times \boldsymbol{H}^1(\mathcal{T}_h)$, we have

$$\begin{aligned}\|\!|(\boldsymbol{\Pi}\boldsymbol{\zeta} - \boldsymbol{\zeta}, \boldsymbol{\Pi}\boldsymbol{\eta} - \boldsymbol{\eta})\|\!|_{\tau_t, \partial\mathcal{T}_h} \leq C \max_{T \in \mathcal{T}_h} h_T^{1/2} \, \|\tau_t^{-1}\|_{L^\infty(\partial T)}^{1/2} \, E(\boldsymbol{\Pi}\boldsymbol{\zeta}, \boldsymbol{\zeta}) \\ + C \max_{T \in \mathcal{T}_h} h_T^{1/2} \, \|\tau_t\|_{L^\infty(\partial T)}^{1/2} \, E(\boldsymbol{\Pi}\boldsymbol{\eta}, \boldsymbol{\eta}),\end{aligned}$$

where

$$E^2(\boldsymbol{\Theta}, \boldsymbol{\theta}) := \inf_{\boldsymbol{S} \in \boldsymbol{W}_h} \sum_{T \in \mathcal{T}_h} \left(h_T^{-2} (\|\boldsymbol{\Theta} - \boldsymbol{\theta}\|_T^2 + \|\boldsymbol{S} - \boldsymbol{\theta}\|_T^2) + \|\nabla(\boldsymbol{S} - \boldsymbol{\theta})\|_T^2 \right),$$

and C is a constant depending on the shape-regularity constant of the elements and on the polynomial degree k.

Theorem 3.3 Assume Ω is simply connected and that $\tau_t > 0$ on $\partial\mathcal{T}_h$. Then

$$\begin{aligned}\|\boldsymbol{\epsilon}^{\boldsymbol{w}}\|_\Omega &\leq \|\!|(\boldsymbol{\Pi}\boldsymbol{w} - \boldsymbol{w}, \boldsymbol{\Pi}\boldsymbol{u} - \boldsymbol{P}_\partial \boldsymbol{u})\|\!|_{\tau_t, \partial\mathcal{T}_h}, \\ \|\epsilon^p\|_\Omega &\leq |\overline{(\Pi p - p)}||\Omega|^{1/2} + C_{\tau_t} \, \|\!|(\boldsymbol{\Pi}\boldsymbol{w} - \boldsymbol{w}, \boldsymbol{\Pi}\boldsymbol{u} - \boldsymbol{P}_\partial \boldsymbol{u})\|\!|_{\tau_t, \partial\mathcal{T}_h},\end{aligned}$$

where

$$C_{\tau_t} := 1 + \big(\max_{T \in \mathcal{T}_h} h_T \, \|\tau_t\|_{L^\infty(\partial T)} \big)^{1/2}.$$

Moreover, if the elliptic regularity inequality (3.3) holds with $s = 0$, we have

$$\|\boldsymbol{\epsilon}^{\boldsymbol{u}}\|_\Omega \leq H_{\tau_t} \, \|\!|(\boldsymbol{\Pi}\boldsymbol{w} - \boldsymbol{w}, \boldsymbol{\Pi}\boldsymbol{u} - \boldsymbol{P}_\partial \boldsymbol{u})\|\!|_{\tau_t, \partial\mathcal{T}_h},$$

where

$$H_{\tau_t} := C_{\tau_t}^2 \max_{T \in \mathcal{T}_h} h_T^{1/2} \|\tau_t^{-1}\|_{L^\infty(\partial T)}^{1/2}.$$

It follows from Proposition 3.2 and Theorem 3.3 that, when τ_t and τ_t^{-1} are of order one, the L^2 norm of the projection of the errors in the approximate vorticity and pressure converge to zero with order $k+1/2$ and with order $k+1$ in the approximate velocity for $k \geq 0$.

Combining Theorem 3.1 and Theorem 3.3, we immediately obtain the errors in the following Corollary.

Corollary 3.4 Under the hypotheses of Theorem 3.3, and when the solution is very smooth, we have that

$$\begin{aligned}\|\boldsymbol{w}-\boldsymbol{w}_h\|_\Omega \leq & C\, h^{k+1/2},\\ \|p-p_h\|_\Omega \leq & C\, h^{k+1/2},\\ \|\boldsymbol{u}-\boldsymbol{u}_h\|_\Omega \leq & C\, h^{k+1},\end{aligned}$$

provided τ_t and τ_t^{-1} are of order one on $\partial\mathcal{T}_h$.

3.3 Proof of the approximation properties of the projection

Here, we give a detailed proof of Theorem 3.1. We proceed in several steps.

Step 1: The projection is well defined

We begin by showing the following result.

Proposition 3.5 The projection given by (3.1) is well defined.

Proof. We first note that, by (3.1a), the $\mathbf{\Pi}\boldsymbol{w}$ is the simple L^2-projection of $\boldsymbol{w}$ into $\boldsymbol{P}_k(T)$.

Next, we note that the equations defining the projection of the velocity are

$$\begin{aligned}(\boldsymbol{\Pi}\boldsymbol{u}-\boldsymbol{u},\boldsymbol{v})_T = 0 \qquad & \forall\ \boldsymbol{v}\in\boldsymbol{P}_{k-1}(T),\\ \langle(\boldsymbol{\Pi}\boldsymbol{u}\cdot\boldsymbol{n}-\boldsymbol{u}\cdot\boldsymbol{n}),\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial T} = 0 \qquad & \forall\ \boldsymbol{v}\in\boldsymbol{P}_k(T)^{\perp}.\end{aligned}$$

We can see that this is a square system and that if $\boldsymbol{u}=\mathbf{0}$, we immediately get that $\boldsymbol{\Pi}\boldsymbol{u}=\mathbf{0}$. The projection for the velocity is thus well defined.

Now, we have that Πp is given by

$$\begin{aligned}(\Pi p - p, q)_T = 0 \qquad & \forall\, q\in P_{k-1}(T),\\ \langle \Pi p - p, q\rangle_{\partial T} = 0 \qquad & \forall\, q\in P_k^{\perp}(T),\end{aligned}$$

since $\langle \Pi^{\partial}\delta, q\rangle_{\partial T} = 0$ for all $q \in P_k^{\perp}(T)$. This readily implies that Πp is well defined.

Finally, we have that $\Pi^{\partial}\delta$ satisfies

$$\langle \Pi^{\partial}\delta, \mu\rangle_{\partial T} = \langle \Pi p - p, \mu\rangle_{\partial T} \qquad \forall \mu \in \mathcal{R}_k(\partial T),$$

This also defines $\Pi^{\partial}\delta$ since the right-hand side is equal to zero whenever $\mu = q \in P_k^{\perp}(T)$. This completes the proof. □

Step 2: The projection for the HDG method in [2]

Next, we recall the projection considered in [2]. We do this because we are going to show that such projection converges to the projection just defined as the stabilization function τ_n tends for infinity.

The projection is defined as follows:

$$(\mathbf{\Pi}^{\tau_n}\boldsymbol{w}, \boldsymbol{\tau})_T = (\boldsymbol{w}, \boldsymbol{\tau})_T \qquad \forall\ \boldsymbol{\tau} \in \boldsymbol{P}_k(T), \tag{3.6a}$$

$$(\boldsymbol{\Pi}^{\tau_n}\boldsymbol{u}, \boldsymbol{v})_T = (\boldsymbol{u}, \boldsymbol{v})_T \qquad \forall\ \boldsymbol{v} \in \boldsymbol{P}_{k-1}(T), \tag{3.6b}$$

$$(\Pi^{\tau_n}p, q)_T = (p, q)_T \qquad \forall\ q \in P_{k-1}(T), \tag{3.6c}$$

$$\langle \Pi^{\tau_n}p + \tau_n(\boldsymbol{\Pi}^{\tau_n}\boldsymbol{u})\cdot\boldsymbol{n}, \mu\rangle_F = \langle p + \tau_n\boldsymbol{u}\cdot\boldsymbol{n}, \mu\rangle_F \qquad \forall\ \mu \in P_k(F), \tag{3.6d}$$

Note that the $\mathbf{\Pi}^{\tau_n}\boldsymbol{w}$ is the L^2-projection of $\boldsymbol{w}$ into $\boldsymbol{P}_k(T)$ and that $(\boldsymbol{\Pi}^{\tau_n}\boldsymbol{u}, \Pi^{\tau_n}p)$ is the same projection used in the analysis of HDG methods for diffusion problems in [5] with the stabilization parameter τ used therein replaced by $1/\tau_n$. Consequently, we have the following result.

Theorem 3.6 ([5]) Assume that $\tau_n|_{\partial T}$ is nonnegative and that $(\tau_n^{\min})_T := \min \tau_n|_T > 0$. Then the projection given by (3.6) is well defined. Moreover, if $\nabla\cdot\boldsymbol{u} = 0$, there is a constant C independent of T and τ_n such that

$$\|\boldsymbol{\Pi}^{\tau_n}\boldsymbol{w} - \boldsymbol{w}\,\|_T \le C\, h_K^{\ell_{\boldsymbol{w}}+1}\,|\boldsymbol{w}|_{\boldsymbol{H}^{\ell_{\boldsymbol{u}}+1}(T)},$$

$$\|\boldsymbol{\Pi}^{\tau_n}\boldsymbol{u}-\boldsymbol{u}\|_T \le C\,h_K^{\ell_{\boldsymbol{u}}+1}\,|\boldsymbol{u}|_{\boldsymbol{H}^{\ell_{\boldsymbol{u}}+1}(T)} + C\,\frac{h_T^{\ell_p+1}}{(\tau_n)_T^*}\,|p|_{H^{\ell_p+1}(T)},$$
$$\|\Pi^{\tau_n}p-p\|_T \le C\,h_K^{\ell_p+1}\,|p|_{H^{\ell_p+1}(T)},$$

for $\ell_{\boldsymbol{w}}, \ell_p, \ell_{\boldsymbol{u}}$ in $[0,k]$. Here $(\tau_n)_T^* := \min \tau_h|_{\partial T\setminus F^*}$, where F^* is a face of T at which $\tau_n|_T$ is minimum.

Step 3: Letting τ_n go to infinity

In order to compare the two projections introduced above, we are gong to rewrite the last projection in a suitable manner.

To this effect, we set

$$\Pi_e^{\partial}\delta := \tau_n\big(\boldsymbol{\Pi}^{\tau_n}\boldsymbol{u}\cdot\boldsymbol{n} - P_{\partial T}(\boldsymbol{u}\cdot\boldsymbol{n})\big) \in \mathcal{R}_k(\partial T),$$

and treat it as an additional unknown. Here $P_{\partial T} : L^2(\partial T) \to \mathcal{R}_k(\partial T)$ is the orthogonal projection onto $\mathcal{R}_k(\partial T)$.

Then, we define $(\mathbf{\Pi}_e\boldsymbol{w}, \boldsymbol{\Pi}_e\boldsymbol{u}, \Pi_e p, \Pi_e^{\partial}\delta) \in \boldsymbol{W}_h \times \boldsymbol{V}_h \times P_h \times M_h^{\partial}$ on the element $T \in \mathcal{T}_h$ as the solution of

$$(\mathbf{\Pi}_e\boldsymbol{w},\boldsymbol{\tau})_T = (\boldsymbol{w},\boldsymbol{\tau})_T \qquad \forall\ \boldsymbol{\tau}\in\boldsymbol{P}_k(T), \tag{3.7a}$$
$$(\boldsymbol{\Pi}_e\boldsymbol{u},\boldsymbol{v})_T = (\boldsymbol{u},\boldsymbol{v})_T \qquad \forall\ \boldsymbol{v}\in\boldsymbol{P}_{k-1}(T), \tag{3.7b}$$
$$(\Pi_e p, q)_T = (p,q)_T \qquad \forall\ q\in P_{k-1}(T), \tag{3.7c}$$
$$\langle \Pi_e p + (\Pi_e^{\partial}\delta), \mu\rangle_F = \langle p,\mu\rangle_F \qquad \forall\ \mu\in P_k(F), \tag{3.7d}$$
$$\tau_n^{-1}\,\langle \Pi_e^{\partial}\delta, q\rangle_{\partial T} = -(\nabla\cdot\boldsymbol{u}, q)_T \qquad \forall\ q\in P_k(T)^{\perp}, \tag{3.7e}$$
$$\langle \tau_n^{-1}\Pi_e^{\partial}\delta - \boldsymbol{\Pi}_e\boldsymbol{u}\cdot\boldsymbol{n}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial T} = -\langle \boldsymbol{u}\cdot\boldsymbol{n}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial T} \quad \forall\ \boldsymbol{v}\in\boldsymbol{P}_k(T)^{\perp}. \tag{3.7f}$$

Note that if $\nabla\cdot\boldsymbol{u} = 0$ and we formally set $\tau_n^{-1} = 0$, we obtain equations defining the projection (3.1). In fact, we have the following result.

Proposition 3.7 Under the same hypotheses of Theorem 3.6, the projection given by (3.7) is well-defined. Moreover,

$$(\mathbf{\Pi}_e \boldsymbol{w}, \boldsymbol{\mathit{\Pi}}_e \boldsymbol{u}, \Pi_e p, \Pi_e^{\partial}\delta) = \big(\mathbf{\Pi}^{\tau_n} \boldsymbol{w}, \boldsymbol{\mathit{\Pi}}^{\tau_n} \boldsymbol{u}, \Pi^{\tau_n} p, \tau_n(\boldsymbol{\mathit{\Pi}}^{\tau_n}\boldsymbol{u}\cdot\boldsymbol{n} - P_{\partial T}(\boldsymbol{u}\cdot\boldsymbol{n}))\big),$$

and

$$(\mathbf{\Pi}_e \boldsymbol{w}, \boldsymbol{\mathit{\Pi}}_e \boldsymbol{u}, \Pi_e p, \Pi_e^{\partial}\delta) \to (\mathbf{\Pi} \boldsymbol{w}, \boldsymbol{\mathit{\Pi}} \boldsymbol{u}, \Pi p, \Pi^{\partial}\delta)$$

as $\tau_n \to \infty$.

Proof. The fact that the projection given by (3.7) is well-defined can be proven in a way similar as the proof that the projection given by (3.1) is well defined.

To prove the first identity, we only have to show that the solution of (3.6) with $\Pi_e^{\partial}\delta = \tau_n(\boldsymbol{\mathit{\Pi}}^{\tau_n}\boldsymbol{u}\cdot\boldsymbol{n} - P_{\partial T}(\boldsymbol{u}\cdot\boldsymbol{n}))$ solves (3.7). In other words, we only have to deal with (3.7d), (3.7e), and (3.7f).

Note that $\langle \tau_n\big(P_{\partial T}(\boldsymbol{u}\cdot\boldsymbol{n}) - \boldsymbol{u}\cdot\boldsymbol{n}\big), \mu\rangle_F = 0 \quad \forall \mu \in P_k(F),\ \forall F \in \partial T$. Hence (3.7d) follows from (3.6d) and the definition of Π_e^{∂}. Moreover, (3.7f) is a direct consequence of the definition of Π_e^{∂} and $P_{\partial T}$.

Finally,

$$\begin{aligned}
\langle \tau_n^{-1}\Pi_e^{\partial}\delta, q\rangle_{\partial T} =& \langle (\boldsymbol{\mathit{\Pi}}^{\tau_n}\boldsymbol{u} - \boldsymbol{u})\cdot\boldsymbol{n}, q\rangle_{\partial T}\\
=& \big(\nabla\cdot(\boldsymbol{\mathit{\Pi}}^{\tau_n}\boldsymbol{u} - \boldsymbol{u}), q\big)_T + (\boldsymbol{\mathit{\Pi}}^{\tau_n}\boldsymbol{u} - \boldsymbol{u}, \nabla q)_T\\
=& \big(\nabla\cdot(\boldsymbol{\mathit{\Pi}}^{\tau_n}\boldsymbol{u}), q\big)_T - (\nabla\cdot\boldsymbol{u}, q)_T\\
=& -(\nabla\cdot\boldsymbol{u}, q)_T \qquad \forall\, q \in P_k(T)^{\perp},
\end{aligned}$$

which implies (3.7e).

We now can set τ_n goes to infinity and show that the solution of (3.7) converges to the solution of (3.1). The argument is exactly the same as the one in Proposition 2.3.

This completes the proof. $\square$

3.4 Proof of the error estimates

We are now ready to prove Theorems 3.1 and 3.3.

Theorem 3.1 follows from Theorem 3.6 by letting τ_n go to infinity and by applying Proposition 2.3.

The estimates Theorem 3.3 follow from similar estimates in [2] by letting τ_n go to infinity and by applying Propositions 2.3 and 3.7.

This completes the proof of our main results.

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