MULTIGRID ALGORITHMS FOR SYMMETRIC DISCONTINUOUS GALERKIN METHODS ON GRADED MESHES

S.C. BRENNER, J. CUI, T. GUDI, AND L.-Y. SUNG

ABSTRACT. We study a class of symmetric discontinuous Galerkin methods on graded meshes. Optimal order error estimates are derived in both the energy norm and the L_2 norm, and we establish the uniform convergence of V-cycle, F-cycle and W-cycle multigrid algorithms for the resulting discrete problems. Numerical results that confirm the theoretical results are also presented.

1. INTRODUCTION

In a recent paper [15], the symmetric interior penalty Galerkin (SIPG) method [29, 39, 3] on graded meshes was analyzed using weighted Sobolev spaces. Optimal error estimates in both the L_2 norm and the energy norm were derived and the uniform convergence of the W-cycle algorithm for the resulting discrete problem was also established.

In this paper we extend the results in [15] to a class of symmetric, stable and consistent discontinuous Galerkin (DG) methods, and also to the uniform convergence of V-cycle and F-cycle algorithms. For simplicity we will focus on the following model problem:
Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall\, v \in H_0^1(\Omega), \tag{1.1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$, and f belongs to $L_2(\Omega)$ (or the space $L_{2,\mu}(\Omega)$ defined in (2.3) below).

Let $\mathcal{T}_h$ be a triangulation of Ω,

$$V_h = \{v \in L_2(\Omega) : \, v_T = v\big|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h\}$$

be the discontinuous P_1 finite element space associated with $\mathcal{T}_h$. In order to define the class of DG methods studied in this paper, we first recall the concepts of jump and mean over the edges of $\mathcal{T}_h$.

Let $H^\theta(\Omega, \mathcal{T}_h)$ $(\theta \geq 1)$ be the space of piecewise Sobolev functions defined by

$$H^\theta(\Omega, \mathcal{T}_h) = \{v \in L_2(T) : \, v_T = v\big|_T \in H^\theta(T) \quad \forall\, T \in \mathcal{T}_h\}. \tag{1.2}$$

1991 *Mathematics Subject Classification.* 65N30, 65N15, 65N55.

Key words and phrases. discontinuous Galerkin methods, graded meshes, multigrid algorithms.

The work of the first author was supported in part by the National Science Foundation under Grant No. DMS 07-38028 and Grant No. DMS-07-13835. The work of the second and fourth authors was supported in part by the National Science Foundation under Grant No. DMS-07-13835.

Let e be an interior edge of $\mathcal{T}_h$ shared by two triangles $T_\pm \in \mathcal{T}_h$, and $\boldsymbol{n}_\pm$ be the unit normals of e pointing towards the outside of $T_\pm$. We define on e

$$[\![v]\!] = v_+\boldsymbol{n}_+ + v_-\boldsymbol{n}_- \qquad \forall\, v \in H^1(\Omega, \mathcal{T}_h),$$

where $v_\pm = v\big|_{T_\pm}$, and

$$\{\!\{\nabla v\}\!\} = \frac{1}{2}(\nabla v_+ + \nabla v_-) \qquad \forall\, v \in H^\theta(\Omega, \mathcal{T}_h),\ \theta > 3/2,$$
$$\{\!\{\boldsymbol{w}\}\!\} = \frac{1}{2}(\boldsymbol{w}_+ + \boldsymbol{w}_-) \qquad \forall\, \boldsymbol{w} \in H^1(\Omega, \mathcal{T}_h) \times H^1(\Omega, \mathcal{T}_h),$$

where $\boldsymbol{w}_\pm = \boldsymbol{w}\big|_{T_\pm}$.

Let e be a boundary edge of $\mathcal{T}_h$. Then $e \subset \partial T$ for some $T \in \mathcal{T}_h$. We define on e

$$[\![v]\!] = v_{T}\boldsymbol{n} \qquad \forall\, v \in H^1(\Omega, \mathcal{T}_h),$$
$$\{\!\{\boldsymbol{w}\}\!\} = \boldsymbol{w}_{T} \qquad \forall\, \boldsymbol{w} \in H^1(\Omega, \mathcal{T}_h) \times H^1(\Omega, \mathcal{T}_h),$$

where $\boldsymbol{n}$ is the unit normal of e pointing towards the outside of Ω.

Next we define for any edge e of $\mathcal{T}_h$ the lifting operator $r_e : L_2(e) \times L_2(e) \longrightarrow V_h \times V_h$ by

$$\int_\Omega r_e(\boldsymbol{v}) \cdot \boldsymbol{w}\, dx = -\int_e \boldsymbol{v} \cdot \{\!\{\boldsymbol{w}\}\!\}\, ds \qquad \forall\, \boldsymbol{w} \in V_h \times V_h. \tag{1.3}$$

Let $\mathcal{E}_h$ be the set of the edges of $\mathcal{T}_h$. The global lifting $r_h : L_2(\mathcal{E}_h) \times L_2(\mathcal{E}_h) \longrightarrow V_h \times V_h$ is defined by

$$r_h(\boldsymbol{v}) = \sum_{e \in \mathcal{E}_h} r_e(\boldsymbol{v}). \tag{1.4}$$

We can now introduce the DG methods to be studied in this paper:
Find $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_h, \tag{1.5}$$

where

$$\begin{aligned} a_h(w, v) = \sum_{T \in \mathcal{T}_h} \int_T \nabla w \cdot \nabla v\, dx - \sum_{e \in \mathcal{E}_h} \int_e \{\!\{\nabla w\}\!\} \cdot [\![v]\!]\, ds - \sum_{e \in \mathcal{E}_h} \int_e \{\!\{\nabla v\}\!\} \cdot [\![w]\!]\, ds \\ + \delta \int_\Omega r_h\big([\![w]\!]\big) \cdot r_h\big([\![v]\!]\big)\, dx + J_h(w, v) \qquad \forall\, v, w \in V_h, \end{aligned} \tag{1.6}$$

$\delta = 1$ or 0, and $J_h = J^j$ or J^r. The jump terms J^j and J^r are defined by

$$J^j(w, v) = \eta \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \int_e [\![w]\!] \cdot [\![v]\!]\, ds \qquad \forall\, v, w \in V_h, \tag{1.7}$$

$$J^r(w, v) = \eta \sum_{e \in \mathcal{E}_h} \int_\Omega r_e\big([\![w]\!]\big) \cdot r_e\big([\![v]\!]\big)\, ds \qquad \forall\, v, w \in V_h, \tag{1.8}$$

where $|e|$ is the length of the edge e and $\eta > 0$ is a penalty parameter.

The different choices for δ and J_h lead to four different DG methods (cf. Table 1.1). They are symmetric, consistent and stable under the condition on η given in Table 1.1 (cf. the references cited in Table 1.1 and also [4]), where η_* is a sufficiently large positive number that depends only on the shape regularity of $\mathcal{T}_h$. We assume from here on that these conditions on η are satisfied for the respective methods.

Table 1.1. DG Methods

Method [Ref.]	δ	J_h	η
Brezzi et al. [23]	1	J^r	$\eta > 0$
LDG [26, 25]	1	J^j	$\eta > 0$
Bassi et al. [8]	0	J^r	$\eta > 3$
SIPG [29, 39, 3]	0	J^j	$\eta > \eta^*$

The rest of the paper is organized as follows. We discuss weighted Sobolev spaces, elliptic regularity and graded meshes in Section 2. The error estimates for the DG methods are derived in Section 3. Section 4 contains the descriptions of the multigrid algorithms. We establish the convergence of W-cycle algorithms in Section 5 and the convergence of V-cycle and F-cycle algorithms in Section 6. Numerical results are reported in Section 7, and we end with some concluding remarks in Section 8.

2. Weighted Sobolev spaces, elliptic regularity and graded meshes

It is well-known [31, 27, 36] that the solution u of (1.1) in general does not belong to $H^2(\Omega)$ if Ω is nonconvex and $f \in L_2(\Omega)$. However the second order weak derivatives of u do belong to a weighted L_2 space, even for f belonging to a larger space.

Let $\omega_1, \ldots, \omega_L$ be the interior angles at the corners $c_1, \ldots, c_L$ of Ω. Let the parameters $\mu_1, \ldots, \mu_L$ be chosen according to

$$\begin{cases} \mu_\ell = 1 & \omega_\ell < \pi \\ \dfrac{1}{2} < \mu_\ell < \dfrac{\pi}{\omega_\ell} & \omega_\ell > \pi \end{cases} \tag{2.1}$$

and the weight function ϕ_μ be defined by

$$\phi_\mu(x) = \prod_{\ell=1}^{L} |x - c_\ell|^{1-\mu_\ell}. \tag{2.2}$$

The space $L_{2,\mu}(\Omega)$ is defined by

$$L_{2,\mu}(\Omega) = \{f \in L_{2,\mathrm{loc}}(\Omega) :\ \|f\|^2_{L_{2,\mu}(\Omega)} = \int_\Omega \phi^2_\mu(x) f^2(x)\, dx < \infty\}. \tag{2.3}$$

Note that $L_2(\Omega)$ is embedded in $L_{2,\mu}(\Omega)$.

Since ϕ_μ^{-1} belongs to $L_4(\Omega)$, it follows from the (generalized) Hölder inequality and the Sobolev inequality [1] that

$$\int_\Omega |fv|\,dx = \int_\Omega |\phi_\mu^{-1}(\phi_\mu f)v|\,dx \le C_\Omega \|f\|_{L_{2,\mu}(\Omega)} \|v\|_{H^1(\Omega)} \qquad \forall\, v \in H^1(\Omega). \tag{2.4}$$

Hence the model problem (1.1) has a unique solution u for any $f \in L_{2,\mu}(\Omega)$. In fact u belongs to the weighted Sobolev space $H^2_\mu(\Omega)$, i.e.,

$$\phi^{|\alpha|-2}(\partial^\alpha u/\partial x^\alpha) \in L_{2,\mu}(\Omega) \qquad \text{for} \quad |\alpha| \le 2,$$

where $\phi(x) = \prod_{\ell=1}^{L} |x - c_\ell|$, and the following regularity estimate holds:

$$\|u\|_{H^2_\mu(\Omega)} = \Big(\sum_{|\alpha|\le 2} \|\phi^{|\alpha|-2}(\partial^\alpha u/\partial x^\alpha)\|^2_{L_{2,\mu}(\Omega)}\Big)^{1/2} \le C_\Omega \|f\|_{L_{2,\mu}(\Omega)}. \tag{2.5}$$

The weighted Sobolev space $H^2_\mu(\Omega)$ is embedded in the Sobolev space $H^s(\Omega)$, where

$$s = \min_{\omega_\ell > \pi} (1 + \mu_\ell) > 3/2,$$

and hence $H^2_\mu(\Omega)$ is also embedded in $C(\bar{\Omega})$ by the Sobolev inequality. More precisely, let $\delta > 0$ be small enough so that the neighborhoods

$$\Omega_{\ell,\delta} = \{x \in \Omega : \ |x - c_\ell| < \delta\}$$

around the corners c_ℓ for $1 \le \ell \le L$ are disjoint. Then at a reentrant corner c_ℓ where $\omega_\ell > \pi$ we have $v \in H^{1+\mu_\ell}(\Omega_{\ell,\delta})$ for all $v \in H^2_\mu(\Omega)$ and

$$\|v\|_{H^{1+\mu_\ell}(\Omega_{\ell,\delta})} \le C_\Omega \|v\|_{H^2_\mu(\Omega)}, \tag{2.6}$$

which together with (2.5) implies

$$\|v\|_{H^s(\Omega)} \le C_\Omega \|v\|_{H^2_\mu(\Omega)} \qquad \forall\, v \in H^2_\mu(\Omega).$$

Details of the elliptic regularity theory in weighted Sobolev spaces can be found for example in [34, 27, 36]. The proof of the embedding result for $H^2_\mu(\Omega)$ is also outlined in [15, p.483].

It follows from the estimate (2.5) that u is less regular near the reentrant corners. Hence the approximation of u by piecewise linear polynomials on a quasi-uniform triangulation becomes less accurate on the triangles near the reentrant corners and the overall approximation of u will no longer be quasi-optimal. This situation can be remedied by using graded meshes [5], where the triangles near the reentrant corners become progressively smaller.

We therefore assume that the triangulation $\mathcal{T}_h$ of Ω has the following property: There exist positive constants C_1 and C_2 such that

$$C_1 h_T \le \Phi_\mu(T) h \le C_2 h_T \qquad \forall\, T \in \mathcal{T}_h, \tag{2.7}$$

where $h_T = \operatorname{diam} T$, $h = \max_{T \in \mathcal{T}_h} h_T$ is the mesh parameter, and $\Phi_\mu(T)$ is defined by

$$\Phi_\mu(T) = \prod_{\ell=1}^{L} |c_\ell - c_T|^{1-\mu_\ell}. \tag{2.8}$$

Here $c_1, \ldots, c_L$ are the corners of Ω, c_T is the center of T, and the grading parameters are chosen according to (2.1). From here on we use C (with or without subscript) to denote a generic positive constant independent of the mesh parameter that can take different values at different occurrences.

The construction of graded meshes that satisfy (2.7) can be found for example in [2, 13, 10]. Note that for a given set of grading parameters the graded meshes satisfy the minimum angle condition.

It was shown in [6] that quasi-optimal error estimates for conforming finite element methods on nonconvex domains are recovered if graded meshes are used. We will prove in the next section that this is also true for the DG methods introduced in Section 1.

3. Error Estimates for the DG Methods

Since $H^2_\mu(\Omega) \subset H^s(\Omega)$ for $s > 3/2$, the bilinear form $a_h(\cdot,\cdot)$ (cf. (1.6)) is defined on $H^2_\mu(\Omega) + V_h$ by the trace theorem [1]. The consistency of the DG methods means that the solution u of (1.1) satisfies

$$a_h(u, v) = \int_\Omega f v \, dx \qquad \forall\, v \in V_h. \tag{3.1}$$

Let the mesh-dependent energy norm $\|\cdot\|_h$ on $H^2_\mu(\Omega) + V_h$ be defined by

$$\|v\|_h^2 = \sum_{T \in \mathcal{T}_h} \|\nabla v\|^2_{L_2(T)} + \eta^{-1} \sum_{e \in \mathcal{E}_h} |e|\, \|\{\!\{\nabla v\}\!\}\|^2_{L_2(e)} + \eta \sum_{e \in \mathcal{E}_h} |e|^{-1} \|[\![v]\!]\|^2_{L_2(e)}. \tag{3.2}$$

The key to the error analysis of the DG methods is the boundedness (on $H^2_\mu(\Omega) + V_h$) and coercivity (on V_h) of the bilinear form $a_h(\cdot,\cdot)$ with respect to the norm $\|\cdot\|_h$.

Lemma 3.1. *The bilinear form $a_h(\cdot,\cdot)$ for all four DG methods is bounded by the $\|\cdot\|_h$ norm:*

$$a_h(w, v) \le C_b \|w\|_h \|v\|_h \qquad \forall\, v, w \in H^2_\mu(\Omega) + V_h, \tag{3.3}$$

where the positive constant C_b is independent of the penalty parameter η as long as η is bounded away from 0 *and satisfies the restrictions given in Table* 1.1.

Proof. It follows from the Cauchy-Schwarz inequality that

$$\Big| \sum_{e \in \mathcal{E}_h} \int_e \{\!\{\nabla w\}\!\} \cdot [\![v]\!] \, ds \Big| + \Big| \sum_{e \in \mathcal{E}_h} \int_e \{\!\{\nabla v\}\!\} \cdot [\![w]\!] \, ds \Big|$$

$$\le \Big(\eta^{-1} \sum_{e \in \mathcal{E}_h} |e| \|\{\!\{\nabla w\}\!\}\|^2_{L_2(e)} + \eta \sum_{e \in \mathcal{E}_h} |e|^{-1} \|[\![w]\!]\|^2_{L_2(e)} \Big)^{1/2} \tag{3.4}$$

$$\times\Big(\eta^{-1}\sum_{e\in\mathcal{E}_h}|e|\|\{\!\{\nabla v\}\!\}\|_{L_2(e)}^2+\eta\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![v]\!]\|_{L_2(e)}^2\Big)^{1/2}$$

for all $v,w\in H^2_\mu(\Omega)+V_h$, which immediately implies (3.3) for the SIPG method.

The boundedness estimate for the other three DG methods follow from (3.4) and the two estimates below (cf. [23, 4]):

$$\|r_e([\![v]\!])\|_{L_2(\Omega)}^2\leq C|e|^{-1}\|[\![v]\!]\|_{L_2(e)}^2 \qquad \forall\, v\in H^2_\mu(\Omega)+V_h, \tag{3.5}$$

$$\|r_h([\![v]\!])\|_{L_2(\Omega)}^2\leq C\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![v]\!]\|_{L_2(e)}^2 \qquad \forall\, v\in H^2_\mu(\Omega)+V_h, \tag{3.6}$$

where the positive constant C depends only on the shape regularity of $\mathcal{T}_h$. □

Lemma 3.2. *The bilinear form $a_h(\cdot,\cdot)$ is coercive for all four DG methods*:

$$a_h(v,v)\geq C_c\|v\|_h^2 \qquad \forall\, v\in V_h, \tag{3.7}$$

where the positive constant C_c is independent of the penalty parameter η as long as η is bounded away from 0*.*

Proof. Let $|\!|\!|\cdot|\!|\!|_h$ be defined by

$$|\!|\!|v|\!|\!|_h^2=\sum_{T\in\mathcal{T}_h}\|\nabla v\|_{L_2(T)}^2+\eta\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![v]\!]\|_{L_2(e)}^2.$$

Since the two norms $\|\cdot\|_h$ and $|\!|\!|\cdot|\!|\!|_h$ are equivalent on V_h (cf. for example [20, Section 10.5]), it suffices to establish the coercivity of $a_h(\cdot,\cdot)$ with respect to $|\!|\!|\cdot|\!|\!|_h$.

From [4] we have the estimate

$$a_h(v,v)\geq C\Big(\sum_{T\in\mathcal{T}_h}\|\nabla v\|_{L_2(T)}^2+\eta\sum_{e\in\mathcal{E}_h}\|r_e([\![v]\!]\|_{L_2(e)}^2\Big) \qquad \forall\, v\in V_h, \tag{3.8}$$

for all four DG methods under the restrictions on η as stated in Table 1.1, where the positive constant C is independent of η as long as it is bounded away from 0. The coercivity with respect to $|\!|\!|\cdot|\!|\!|_h$ then follows from the estimate [23, 4]

$$|e|^{-1}\|[\![v]\!]\|_{L_2(e)}^2\leq C\|r_e([\![v]\!])\|_{L_2(e)}^2 \qquad \forall\, v\in V_h.$$

□

Combining (3.1), (3.3) and (3.7), we have a quasi-optimal error estimate (cf. for example [20, Section 10.5]) for the four DG methods:

$$\|u-u_h\|_h\leq C\inf_{v\in V_h}\|u-v\|_h, \tag{3.9}$$

where the positive constant C is independent of the penalty parameter η as long as η is bounded away from 0.

Remark 3.3. One can also obtain *a priori* error estimates for the DG methods without relying on (3.1) or the fact that $u\in H^s(\Omega)$ for some $s>3/2$. We refer the readers to [32] for details.

Let $\Pi_h : C(\bar{\Omega}) \longrightarrow V_h$ be the nodal interpolation operator for the conforming P_1 finite element, i.e., $\Pi_h\zeta \in V_h$ agrees with ζ at the vertices of $\mathcal{T}_h$. The following result, which is based on the property (2.7) of graded meshes and the elliptic regularity estimates (2.5) and (2.6), is proved in [15, Lemma 2.2].

Lemma 3.4. *Let $f \in L_{2,\mu}(\Omega)$ and $u \in H_0^1(\Omega)$ satisfy* (1.1)*. Then we have*

$$\|u - \Pi_h u\|_h \leq Ch\|f\|_{L_{2,\mu}(\Omega)}, \tag{3.10}$$

where the positive constant C depends only on the shape regularity of $\mathcal{T}_h$ and the constants in (2.5)–(2.6)*.*

Using (3.9) and (3.10), we can immediately establish the energy norm error estimate for the four DG methods.

Theorem 3.5. *Let $f \in L_{2,\mu}(\Omega)$, u be the solution of* (1.1)*, and u_h be the solution of one of the four DG methods associated with a triangulation $\mathcal{T}_h$ that satisfies* (2.7)*. We have the following error estimate*:

$$\|u - u_h\|_h \leq Ch\|f\|_{L_{2,\mu}(\Omega)}, \tag{3.11}$$

where the positive constant C is independent of the penalty parameter η as long as η is bounded away from 0 *and satisfies the restrictions given in Table* 1.1*.*

We can also establish an error estimate for the DG methods in the norm

$$\|\varrho\|^2_{L_{2,-\mu}(\Omega)} = \int_\Omega \phi_\mu^{-2}(x)\varrho^2(x)\,dx, \tag{3.12}$$

which is the norm for $L_{2,-\mu}(\Omega)$, the dual space of $L_{2,\mu}(\Omega)$.

Theorem 3.6. *Under the assumptions of Theorem* 3.5*, we have*

$$\|u - u_h\|_{L_{2,-\mu}(\Omega)} \leq Ch^2\|f\|_{L_{2,\mu}(\Omega)}, \tag{3.13}$$

where the positive constant C is independent of the penalty parameter η as long as η is bounded away from 0 *and satisfies the restrictions given in Table* 1.1*.*

Proof. The error estimate is established by a duality argument.

Observe first that (1.5) and (3.1) imply the following Galerkin orthogonality:

$$a_h(u - u_h, v) = 0 \qquad \forall\, v \in V_h. \tag{3.14}$$

Let $\chi = \phi_\mu^{-2}(u - u_h)$. Then $\chi \in L_{2,\mu}(\Omega)$ because $\phi_\mu^{-1} \in L_2(\Omega)$ and $u - u_h \in L_\infty(\Omega)$, and

$$\|\chi\|_{L_{2,\mu}(\Omega)} = \|u - u_h\|_{L_{2,-\mu}(\Omega)}. \tag{3.15}$$

Let $\zeta \in H_0^1(\Omega)$ satisfy

$$\int_\Omega \nabla v \cdot \nabla\zeta\,dx = \int_\Omega v\chi\,dx \qquad \forall\, v \in H_0^1(\Omega). \tag{3.16}$$

It follows from (3.15) and Lemma 3.4 (applied to ζ) that

$$\|\zeta - \Pi_h\zeta\|_h \leq Ch\|u - u_h\|_{L_{2,-\mu}(\Omega)}. \tag{3.17}$$

Note that we can rewrite (3.16) as

$$a_h(v,\zeta) = \int_\Omega v\chi\,dx \qquad \forall\, v \in H_0^1(\Omega),$$

and that the consistency of the DG method implies

$$a_h(v,\zeta) = \int_\Omega v\chi\,dx \qquad \forall\, v \in V_h.$$

Hence we have, by (3.3), (3.11), (3.14) and (3.17),

$$\begin{aligned}
\|u-u_h\|^2_{L_{2,-\mu}(\Omega)} &= \int_\Omega (u-u_h)\chi\,dx \\
&= a_h(u-u_h,\zeta) \\
&= a_h(u-u_h,\zeta-\Pi_h\zeta) \\
&\le \|u-u_h\|_h\|\zeta-\Pi_h\zeta\|_h \le Ch^2\|f\|_{L_{2,\mu}(\Omega)}\|u-u_h\|_{L_{2,-\mu}(\Omega)},
\end{aligned}$$

which implies (3.13). □

The following corollary is immediate.

Corollary 3.7. *Under the assumptions of Theorem* 3.5*, we have*

$$\|u-u_h\|_{L_2(\Omega)} \le Ch^2\|f\|_{L_{2,\mu}(\Omega)}.$$

4. Multigrid Algorithms

The discrete problem (1.5) can be solved by multigrid algorithms. We start with an initial triangulation $\mathcal{T}_0$ and then obtain the triangulations $\mathcal{T}_k$ for $k \ge 1$ by the following refinement procedure, which is identical to the one in [10]. We assume that any triangle in $\mathcal{T}_0$ can have at most one vertex that is a reentrant corner.

- If none of the reentrant corners is a vertex of $T \in \mathcal{T}_k$, then we divide T uniformly by connecting the midpoints of the edges of T.
- If a reentrant corner c_ℓ is a vertex of $T \in \mathcal{T}_k$ and the other two vertices of T are denoted by p_1 and p_2, then we divide T by connecting the points m, q_1 and q_2 (cf. Figure 4.1). Here m is the midpoint of the edge p_1p_2 and q_1 (resp. q_2) is the point on the edge $c_\ell p_1$ (resp. $c_\ell p_2$) such that

  $$\frac{|c_\ell - q_i|}{|c_\ell - p_i|} = 2^{-(1/\mu_\ell)} \qquad \text{for} \quad i = 1,2,$$

  where μ_ℓ is the grading factor chosen according to (2.1).

The resulting family of triangulations $\{\mathcal{T}_k\}_{k\ge 0}$ satisfies the mesh condition (2.7) (cf. the Appendix of [15]). Without loss of generality we may also assume that

$$h_k = \frac{1}{2}h_{k-1} \qquad \text{for } k \ge 1. \tag{4.1}$$

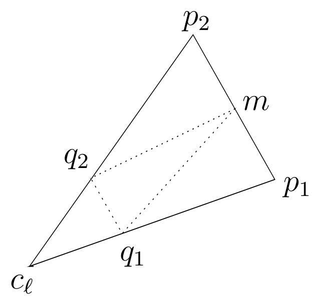

FIGURE 4.1. Refinement of a triangle at a reentrant corner

Let V_k be the discontinuous P_1 finite element space associated with $\mathcal{T}_k$ and $a_k(\cdot,\cdot)$ be the analog of $a_h(\cdot,\cdot)$. The k-th level DG method for (1.1) is:
Find $u_k \in V_k$ such that

$$a_k(u_k, v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_k. \tag{4.2}$$

Note that Lemma 3.1 and Lemma 3.2 imply

$$\|v\|_{a_k}^2 = a_k(v,v) \approx \|v\|_k^2 \qquad \forall\, v \in V_k, \tag{4.3}$$

where $\|\cdot\|_k$ is the analog of $\|\cdot\|_h$, i.e.,

$$\|v\|_k^2 = \sum_{T\in\mathcal{T}_k} |v|_{H^1(T)}^2 + \eta^{-1} \sum_{e\in\mathcal{E}_k} |e|\,\|\{\!\{\nabla v\}\!\}\|_{L_2(e)}^2 + \eta \sum_{e\in\mathcal{E}_k} |e|^{-1}\|[\![v]\!]\|_{L_2(e)}^2. \tag{4.4}$$

Moreover, the norms $\|\cdot\|_{k-1}$ and $\|\cdot\|_k$ on two consecutive levels are equivalent, i.e.,

$$\|w\|_k \approx \|w\|_{k-1} \qquad \forall\, w \in V_{k-1}. \tag{4.5}$$

The k-th level discrete problem can be written as

$$A_k u_k = f_k, \tag{4.6}$$

where $A_k : V_k \longrightarrow V_k'$ and $f_k \in V_k'$ are defined by

$$\langle A_k w, v\rangle = a_k(w,v) \qquad \forall\, v, w \in V_k, \tag{4.7a}$$

$$\langle f_k, v\rangle = \int_\Omega f v\, dx \qquad \forall\, v \in V_k. \tag{4.7b}$$

Here $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $V_k' \times V_k$.

In order to define multigrid algorithms [33, 35, 9, 37, 20] for equations of the form (4.6), we need intergrid transfer operators that move functions between grids and a good smoother to damp out the highly oscillatory part of the error. Since the finite element spaces are nested, we can take the coarse-to-fine intergrid transfer operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$ to be the natural injection and define the fine-to-coarse intergrid transfer operator $I_k^{k-1} : V_k' \longrightarrow V_{k-1}'$ to be the transpose of I_{k-1}^k with respect to the canonical bilinear forms, i.e.,

$$\langle I_k^{k-1}\alpha, v\rangle = \langle \alpha, I_{k-1}^k v\rangle \qquad \forall\, \alpha \in V_k',\ v \in V_{k-1}. \tag{4.8}$$

For simplicity we will use Richardson relaxation as our smoother. Let the operator $B_k : V_k \longrightarrow V_k'$ defined by

$$\langle B_k w, v\rangle = h_k^2 \sum_{T\in\mathcal{T}_k} \sum_{m\in\mathcal{M}_T} w(m)v(m) \qquad \forall\, v, w \in V_k, \tag{4.9}$$

where $\mathcal{M}_T$ is the set of the midpoints of the three edges of T. It is easy to see from (1.6), (4.7a) and (4.9) that the spectral radius of $B_k^{-1}A_k$ satisfies

$$\rho(B_k^{-1}A_k) \le Ch_k^{-2} \qquad \text{for} \quad k \ge 0. \tag{4.10}$$

Hence we can choose a (constant) damping factor λ so that the spectral radius $\rho(\lambda h_k^2 B_k^{-1} A_k)$ satisfies

$$\rho(\lambda h_k^2 B_k^{-1} A_k) < 1 \qquad \text{for} \quad k \ge 0. \tag{4.11}$$

Given any $g \in V_k'$, the Richardson relaxation scheme for the equation

$$A_k z = g \tag{4.12}$$

is given by

$$z_{\text{new}} = z_{\text{old}} + (\lambda h_k^2) B_k^{-1}(g - A_k z_{\text{old}}). \tag{4.13}$$

We are now ready to state the multigrid algorithms for (4.12).

Algorithm 4.1. Let $g \in V_k'$ and $z_0 \in V_k$ be an initial guess. The multigrid V-cycle algorithm for (4.12) with m_1 (resp. m_2) pre-smoothing (resp. post-smoothing) steps produces an approximate solution $MG_V(k, g, z_0, m_1, m_2)$. For $k = 0$, $MG_V(k, g, z_0, m_1, m_2) = A_0^{-1}g$. For $k \ge 1$, $MG_V(k, g, z_0, m_1, m_2)$ is computed recursively as follows.

Pre-smoothing

Apply m_1 steps of (4.13) starting with z_0 to obtain z_{m_1}.

Coarse Grid Correction

Let $r_{k-1} = I_k^{k-1}(g - A_k z_{m_1}) \in V_{k-1}'$ be the coarse grid residual. Apply the $(k-1)$-st level algorithm to the coarse grid residual equation

$$A_{k-1}e_{k-1} = r_{k-1}$$

with initial guess 0 to obtain the correction $q = MG_V(k-1, r_{k-1}, 0, m_1, m_2)$ and define

$$z_{m_1+1} = z_{m_1} + I_{k-1}^k q.$$

Post-smoothing

Apply m_2 steps of (4.13) starting with z_{m_1+1} to obtain $z_{m_1+m_2+1}$.

Final Output

$$MG_V(k, g, z_0, m_1, m_2) = z_{m_1+m_2+1}$$

Algorithm 4.2. Let $g \in V_k'$ and $z_0 \in V_k$ be an initial guess. The W-cycle algorithm computes an approximate solution $MG_W(k, g, z_0, m_1, m_2)$ of (4.12). It differs from Algorithm 4.1 in the coarse grid correction step, where the coarse grid algorithm is applied twice. More precisely, the correction $q \in V_{k-1}$ is computed by

$$\begin{aligned} q' &= MG_W(k-1, r_{k-1}, 0, m_1, m_2), \\ q &= MG_W(k-1, r_{k-1}, q', m_1, m_2). \end{aligned}$$

Algorithm 4.3. Let $g \in V_k'$ and $z_0 \in V_k$ be an initial guess. The F-cycle algorithm computes an approximate solution $MG_F(k, g, z_0, m_1, m_2)$ of (4.12). It differs from Algorithm 4.1 and Algorithm 4.2 in the coarse grid correction step, where the coarse grid algorithm is applied once followed by a V-cycle algorithm. More precisely, the correction $q \in V_{k-1}$ is computed by

$$\begin{aligned} q' &= MG_F(k-1, r_{k-1}, 0, m_1, m_2), \\ q &= MG_V(k-1, r_{k-1}, q', m_1, m_2). \end{aligned}$$

5. Convergence of the W-Cycle Algorithm

Let $E_k : V_k \longrightarrow V_k$ be the error propagation operator for the k-th level W-cycle algorithm. We have the following well-known recursive relation [33, 20]:

$$E_k = R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k E_{k-1}^2 P_k^{k-1}) R_k^{m_1}, \tag{5.1}$$

where Id_k is the identity operator on V_k, the operator $R_k : V_k \longrightarrow V_k$ which measures the effect of one smoothing step is defined by

$$R_k = Id_k - (\lambda h_k^2) B_k^{-1} A_k, \tag{5.2}$$

and the operator $P_k^{k-1} : V_k \longrightarrow V_{k-1}$ is the transpose of I_{k-1}^k with respect to the variational forms, i.e.,

$$a_{k-1}(P_k^{k-1} w, v) = a_k(w, I_{k-1}^k v) = a_k(w, v) \qquad \forall\, v \in V_{k-1},\ w \in V_k. \tag{5.3}$$

The keys to the convergence analysis of the W-cycle algorithm [7, 41] are the estimates for the operators R_k^m (smoothing property) and $Id_k - I_{k-1}^k P_k^{k-1}$ (approximation property) in terms of mesh-dependent norms.

For $j = 0, 1, 2$ and $k \geq 0$, let the mesh-dependent norms $|\!|\!| v |\!|\!|_{j,k}$ be defined by

$$|\!|\!| v |\!|\!|_{j,k} = \sqrt{\langle B_k (B_k^{-1} A_k)^j v, v \rangle} \qquad \forall\, v \in V_k,\ k \geq 0. \tag{5.4}$$

In particular, we have

$$|\!|\!| v |\!|\!|_{0,k}^2 = \langle B_k v, v \rangle \qquad \forall\, v \in V_k, \tag{5.5}$$

$$|\!|\!| v |\!|\!|_{1,k}^2 = \langle A_k v, v \rangle = a_k(v, v) = \|v\|_{a_k}^2 \qquad \forall\, v \in V_k. \tag{5.6}$$

Also the Cauchy-Schwarz inequality implies that

$$|\!|\!| v |\!|\!|_{2,k} = \max_{w \in V_k \setminus \{0\}} \frac{\langle A_k v, w \rangle}{|\!|\!| w |\!|\!|_{0,k}} \qquad \forall\, v \in V_k. \tag{5.7}$$

It follows from (5.7) and (4.10) that

$$\|\!|v|\!\|_{2,k} \leq Ch_k^{-1}\|\!|v|\!\|_{1,k} \qquad \forall\, v \in V_k. \tag{5.8}$$

There is an important connection between the mesh-dependent norm $\|\!|\cdot|\!\|_{0,k}$ and the norm $\|\cdot\|_{L_{2,-\mu}(\Omega)}$ defined in (3.12). From (2.2), (2.7), (2.8), (4.9) and (5.5), we have

$$\|\!|v|\!\|_{0,k}^2 = h_k^2 \sum_{T\in\mathcal{T}_k} \sum_{m\in\mathcal{M}_T} [v(m)]^2 \approx \|v\|_{L_{2,-\mu}(\Omega)}^2 \qquad \forall\, v \in V_k, \tag{5.9}$$

where the positive constants in the equivalence depend only on the shape regularity of $\mathcal{T}_h$.

The smoothing properties in the following lemma are based on (4.11) and (5.2). Their proofs are standard [33, 20].

Lemma 5.1. *There exists a positive constant C independent of k such that*

$$\begin{aligned} \|\!|R_k v|\!\|_{j,k} &\leq \|\!|v|\!\|_{j,k} && \forall\, v \in V_k,\ k\geq 1,\ j=0,1,2,\\ \|\!|R_k^m v|\!\|_{j+1,k} &\leq Ch_k^{-1}(1+m)^{-1/2}\|\!|v|\!\|_{j,k} && \forall\, v \in V_k,\ k\geq 1,\ j=0,1. \end{aligned}$$

The proof of the approximation property in [15] uses the consistency of the DG method and the fact that

$$a_k\big(I_{k-1}^k z, (Id_k - I_{k-1}^k P_k^{k-1})v\big) = 0 \qquad \forall\, z \in V_{k-1}\cap H_0^1(\Omega),\ v\in V_k. \tag{5.10}$$

Since (5.10) is a consequence of the relation

$$a_{k-1}(I_{k-1}^k z, I_{k-1}^k v) = a_{k-1}(z,v) \qquad \forall\, z \in V_{k-1}\cap H_0^1(\Omega) \text{ and } v \in V_k$$

which is valid for all four DG methods, the following lemma can be proved using the same duality argument as in the proof of [15, Lemma 4.2], where a different scaling for the mesh-dependent norm $\|\!|\cdot|\!\|_{0,k}$ was used.

Lemma 5.2. *There exists a positive constant C independent of k such that*

$$\|\!|(Id_k - I_{k-1}^k P_k^{k-1})v|\!\|_{0,k} \leq Ch_k \|\!|(Id_k - I_{k-1}^k P_k^{k-1})v|\!\|_{1,k} \qquad \forall\, v\in V_k,\ k\geq 1. \tag{5.11}$$

The approximation property for the convergence analysis of the multigrid algorithms is provided by the next lemma.

Lemma 5.3. *There exists a positive constant C independent of k such that*

$$\|\!|(Id_k - I_{k-1}^k P_k^{k-1})v|\!\|_{0,k} \leq Ch_k^2 \|\!|v|\!\|_{2,k} \qquad \forall\, v\in V_k,\ k\geq 1. \tag{5.12}$$

Proof. Since $a_k(\cdot,\cdot)$ is an inner product on V_k, we have by (5.6) and duality,

$$\|\!|(Id_k - I_{k-1}^k P_k^{k-1})v|\!\|_{1,k} = \sup_{w\in V_k\setminus\{0\}} \frac{a_k((Id_k - I_{k-1}^k P_k^{k-1})v, w)}{\|\!|w|\!\|_{1,k}}. \tag{5.13}$$

Using (5.3), (5.7) and (5.11), the numerator on the right-hand side of (5.13) can be estimated as follows:

$$\begin{aligned} a_k((Id_k - I_{k-1}^k P_k^{k-1})v, w) &= a_k(v, (Id_k - I_{k-1}^k P_k^{k-1})w)\\ &\leq \|\!|v|\!\|_{2,k}\|\!|(Id_k - I_{k-1}^k P_k^{k-1})w|\!\|_{0,k} \leq Ch_k\|\!|v|\!\|_{2,k}\|\!|w|\!\|_{1,k}, \end{aligned}$$

which together with (5.13) implies

$$\|\!|(Id_k - I_{k-1}^k P_k^{k-1})v\|\!|_{1,k} \le Ch_k \|\!|v\|\!|_{2,k} \qquad \forall\, v \in V_k,\ k \ge 1. \tag{5.14}$$

The estimate (5.12) follows from (5.11) and (5.14). □

As in [15, Lemma 4.5], we can derive the operator bounds in the following lemma from (4.3), (4.5), (5.3), (5.6) and duality.

Lemma 5.4. *There exists a positive constant C independent of k such that*

$$\|\!|I_{k-1}^k v\|\!|_{1,k} \le C \|\!|v\|\!|_{1,k-1} \qquad \forall\, v \in V_{k-1}, \tag{5.15}$$

$$\|\!|P_k^{k-1} v\|\!|_{1,k-1} \le C \|\!|v\|\!|_{1,k} \qquad \forall\, v \in V_k. \tag{5.16}$$

Combining (5.1), Lemmas 5.1, 5.3 and 5.4, we have a convergence theorem for the W-cycle algorithm for all four DG methods. Details can be found in [15, Theorems 4.4 and 4.6].

Theorem 5.5. *There exist a positive constant C and a positive integer m_*, both independent of k, such that the output $MG_W(k, g, z_0, m_1, m_2)$ of the W-cycle algorithm (Algorithm 4.2) applied to (4.12) satisfies the estimate*

$$\|z - MG_W(k, g, z_0, m_1, m_2)\|_{a_k} \le \frac{C}{[(1+m_1)(1+m_2)]^{1/2}} \|z - z_0\|_{a_k},$$

provided $m_1 + m_2 \ge m_*$.

In particular, the W-cycle algorithm is a contraction with contraction number independent of the grid level if the number of smoothing steps is sufficiently large.

6. Convergence of the V-Cycle and F-Cycle Algorithms

The convergence analysis of V-cycle and F-cycle algorithms for the DG methods requires the additive multigrid theory developed in [11, 12]. The starting point of the additive theory is an additive expression for the error propagation operator E_k for the V-cycle algorithm with m pre-smoothing and m post-smoothing steps. By iterating the well-known recursive relation [33, 20]

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k E_{k-1} P_k^{k-1}) R_k^m$$

and taking into account that $E_0 = 0$, we have

$$\begin{aligned} E_k &= R_k^m [(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m \\ &\qquad + R_k^m I_{k-1}^k R_{k-1}^m [(Id_{k-1} - I_{k-2}^{k-1} P_{k-1}^{k-2}) + I_{k-2}^{k-1} E_{k-2} P_{k-1}^{k-2}] R_{k-1}^m P_k^{k-1} R_k^m \\ &= \sum_{j=2}^{k} T_{k,j} R_j^m (Id_j - I_{j-1}^j P_j^{j-1}) R_j^m T_{j,k}, \end{aligned} \tag{6.1}$$

where (for $j < k$) $T_{k,j}$ is the multilevel operator $R_k^m I_{k-1}^k \cdots R_{j+1}^m I_j^{j+1} : V_j \longrightarrow V_k$, $T_{j,k} = P_{j+1}^j R_j^m \cdots P_k^{k-1} R_k^m : V_k \longrightarrow V_j$ is the transpose of $T_{k,j}$ with respect to the bilinear forms $a_k(\cdot,\cdot)$ and $a_j(\cdot,\cdot)$, and $T_{k,k} = Id_k$.

The convergence theory based on (6.1) has been applied successfully to classical nonconforming finite elements and DG methods on quasi-uniform meshes [12, 42, 22, 21]. Here we extend the theory to DG methods on graded meshes.

By using weighted Sobolev spaces and graded meshes, we can treat the convergence of V-cycle and F-cycle algorithms within the framework of problems with full elliptic regularity. In other words we can apply the theory in [12] for the case $\alpha = 1$ (α = index of elliptic regularity), which means that we need to establish the following estimates besides the estimates in Section 5:

$$\|I_{k-1}^k v\|_{1,k}^2 \leq (1+\theta^2)\|v\|_{1,k-1}^2 + C\theta^{-2}h_k^2\|v\|_{2,k-1}^2 \quad \forall\, v \in V_{k-1},\ \theta \in (0,1), \tag{6.2}$$

$$\|I_{k-1}^k v\|_{0,k}^2 \leq (1+\theta^2)\|v\|_{0,k-1}^2 + C\theta^{-2}h_k^2\|v\|_{1,k-1}^2 \quad \forall\, v \in V_{k-1},\ \theta \in (0,1), \tag{6.3}$$

$$\|P_k^{k-1} v\|_{0,k-1}^2 \leq (1+\theta^2)\|v\|_{0,k}^2 + C\theta^{-2}h_k^2\|v\|_{1,k}^2 \quad \forall\, v \in V_k\,,\ \theta \in (0,1)\,, \tag{6.4}$$

and

$$\|\big(Id_{k-1} - P_k^{k-1}I_{k-1}^k\big)v\|_{0,k-1} \leq Ch_k\|v\|_{1,k-1} \quad \forall\, v \in V_{k-1}. \tag{6.5}$$

The following result is useful for this purpose.

Lemma 6.1. *Given any $w \in V_k$, there exists $\psi \in L_{2,\mu}(\Omega)$ such that*

$$a_k(w,v) = \int_\Omega \psi v\,dx \qquad \forall\, v \in V_k \tag{6.6}$$

and

$$\|\psi\|_{L_{2,\mu}(\Omega)} \leq C\|w\|_{2,k}. \tag{6.7}$$

Proof. In view of (5.7) and (5.9), the linear functional $L(v) = a_k(w,v)$ defined on V_k satisfies the estimate

$$|L(v)| \leq \|w\|_{2,k}\|v\|_{0,k} \leq C\|w\|_{2,k}\|v\|_{L_{2,-\mu}(\Omega)} \qquad \forall\, v \in V_k.$$

By the Hahn-Banach Theorem [40], we can extend L to a bounded linear functional on $L_{2,-\mu}(\Omega)$ with the same bound, i.e., there exists $\psi \in L_{2,\mu}(\Omega)$ that satisfies (6.6) and (6.7). □

We begin with the estimates (6.2)–(6.4) that connect the mesh-dependent norms on two consecutive levels.

Lemma 6.2. *There exists a positive constant C such that*

$$\|\zeta_{k-1}\|_{1,k}^2 \leq \|\zeta_{k-1}\|_{1,k-1}^2 + Ch_k^2\|\zeta_{k-1}\|_{2,k-1}^2 \qquad \forall\, \zeta_{k-1} \in V_{k-1},\ k \geq 1.$$

In particular the estimate (6.2) *is valid.*

Proof. Let $\zeta_{k-1} \in V_{k-1}$ be arbitrary. By Lemma 6.1, there exists $\psi \in L_{2,\mu}(\Omega)$ such that

$$a_{k-1}(\zeta_{k-1},v) = \int_\Omega \psi v\,dx \quad \forall\, v \in V_{k-1} \quad \text{and} \quad \|\psi\|_{L_{2,\mu}(\Omega)} \leq C\|\zeta_{k-1}\|_{2,k-1}. \tag{6.8}$$

Let $\zeta \in H_0^1(\Omega) \cap H_\mu^2(\Omega)$ satisfy

$$\int_\Omega \nabla\zeta \cdot \nabla v\, dx = \int_\Omega \psi v\, dx \qquad \forall\, v \in H_0^1(\Omega). \tag{6.9}$$

In view of (6.8) and (6.9), ζ_{k-1} is the approximation of ζ by the DG method on the $(k-1)$-st level. From (1.6), (3.5), (3.6), (4.4), (5.6) and Theorem 3.5, we have

$$\begin{aligned}
|\!|\!|\zeta_{k-1}|\!|\!|_{1,k}^2 &\le |\!|\!|\zeta_{k-1}|\!|\!|_{1,k-1}^2 + C \sum_{e\in\mathcal{E}_{k-1}} \frac{1}{|e|}\|[\![\zeta_{k-1}]\!]\|_{L_2(e)}^2 \\
&= |\!|\!|\zeta_{k-1}|\!|\!|_{1,k-1}^2 + C \sum_{e\in\mathcal{E}_{k-1}} \frac{1}{|e|}\|[\![\zeta-\zeta_{k-1}]\!]\|_{L_2(e)}^2 \\
&\le |\!|\!|\zeta_{k-1}|\!|\!|_{1,k-1}^2 + C\|\zeta-\zeta_{k-1}\|_k^2 \\
&\le |\!|\!|\zeta_{k-1}|\!|\!|_{1,k-1}^2 + Ch_k^2\|\psi\|_{L_{2,\mu}(\Omega)}^2 \\
&\le |\!|\!|\zeta_{k-1}|\!|\!|_{1,k-1}^2 + Ch_k^2|\!|\!|\zeta_{k-1}|\!|\!|_{2,k-1}^2.
\end{aligned}$$

□

Lemma 6.3. *There exists a positive constant C such that*

$$\Big||\!|\!|v|\!|\!|_{0,k-1}^2 - |\!|\!|v|\!|\!|_{0,k}^2\Big| \le Ch_k\Big[\sum_{T\in\mathcal{T}_{k-1}}\|\nabla v\|_{L_2(T)}^2\Big]^{1/2}|\!|\!|v|\!|\!|_{0,k-1} \qquad \forall\, v\in V_{k-1}. \tag{6.10}$$

Proof. Let $v \in V_{k-1}$ be arbitrary. From (4.1), (4.9) and (5.5), we have

$$\begin{aligned}
|\!|\!|v|\!|\!|_{0,k}^2 &= h_k^2 \sum_{T'\in\mathcal{T}_k}\sum_{m'\in\mathcal{M}_{T'}} v_{T'}^2(m') \\
&= h_k^2 \sum_{T\in\mathcal{T}_{k-1}}\sum_{\substack{T'\subset T\\ T'\in\mathcal{T}_k}}\sum_{m'\in\mathcal{M}_{T'}} v_{T'}^2(m') \\
&= h_{k-1}^2 \sum_{T\in\mathcal{T}_{k-1}}\sum_{m\in\mathcal{M}_T} v_T^2(m) \\
&\quad + h_k^2 \sum_{T\in\mathcal{T}_{k-1}}\Big[\sum_{\substack{T'\subset T\\ T'\in\mathcal{T}_k}}\sum_{m'\in\mathcal{M}_{T'}} v_{T'}^2(m') - 4\sum_{m\in\mathcal{M}_T} v_T^2(m)\Big] \\
&= |\!|\!|v|\!|\!|_{0,k-1}^2 + R.
\end{aligned} \tag{6.11}$$

The term R in (6.11) can be estimated as follows. Let $T \in \mathcal{T}_{k-1}$, $m \in \mathcal{M}_T$, $T' \in \mathcal{T}_k$, $T' \subset T$, and $m' \in \mathcal{M}_{T'}$. It follows from the mean value theorem that

$$\begin{aligned}
\big|v_T^2(m) - v_{T'}^2(m')\big| &= \big|v_T(m) - v_{T'}(m')\big|\big|v_T(m) + v_{T'}(m')\big| \\
&\le C\|\nabla v_T\|_{L_2(T)}\|v_T\|_{L_\infty(T)},
\end{aligned}$$

and hence

$$\Big|\sum_{\substack{T'\subset T\\ T'\in\mathcal{T}_k}}\sum_{m'\in\mathcal{M}_{T'}} v_{T'}^2(m') - 4\sum_{m\in\mathcal{M}_T} v_T^2(m)\Big| \le C\|\nabla v\|_{L_2(T)}\Big[\sum_{m\in\mathcal{M}_T} v_T^2(m)\Big]^{1/2}. \tag{6.12}$$

The estimate (6.10) follows from (4.9), (5.5), (6.11), (6.12) and the Cauchy-Schwarz inequality. □

Lemma 6.4. *The estimate* (6.3) *is valid.*

Proof. Let $v \in V_{k-1}$ be arbitrary. In view of (4.3), (4.4) and (5.6), we have

$$\Big[\sum_{T\in\mathcal{T}_{k-1}}\|\nabla v\|_{L_2(T)}^2\Big]^{1/2} \le \|v\|_{k-1} \le C|\!|\!|v|\!|\!|_{1,k-1}. \tag{6.13}$$

The estimate (6.3) follows from (6.10), (6.13) and the elementary inequality

$$ab \le \theta^2 a^2 + \frac{b^2}{4\theta^2} \qquad \forall\, a, b, \theta \neq 0 \in \mathbb{R}. \tag{6.14}$$

□

Lemma 6.5. *The estimate* (6.4) *is valid.*

Proof. Let $\zeta_k \in V_k$ be arbitrary. By Lemma 6.1, there exists $\psi \in L_{2,\mu}(\Omega)$ such that

$$a_k(\zeta_k, v) = \int_\Omega \psi v\, dx \quad \forall\, v \in V_k \quad \text{and} \quad \|\psi\|_{L_{2,\mu}(\Omega)} \le C|\!|\!|\zeta_k|\!|\!|_{2,k}. \tag{6.15}$$

Let $\zeta \in H_0^1(\Omega)\cap H_\mu^2(\Omega)$ satisfy (6.9) and $\zeta_{k-1} = P_k^{k-1}\zeta_k$. Then (6.15) implies that ζ_k is the DG approximation of ζ on the k-th level, and (5.3) implies that

$$a_{k-1}(\zeta_{k-1}, v) = a_k(\zeta_k, I_{k-1}^k v) = \int_\Omega \psi v\, dx \qquad \forall\, v \in V_{k-1},$$

i.e., ζ_{k-1} is the DG approximation of ζ on the $(k-1)$-st level.

Let $\theta \in (0,1)$ be arbitrary. From Lemma 6.3, (6.13) and (6.14), we have

$$|\!|\!|\zeta_{k-1}|\!|\!|_{0,k-1}^2 \le |\!|\!|\zeta_{k-1}|\!|\!|_{0,k}^2 + \frac{\theta^2}{2}|\!|\!|\zeta_{k-1}|\!|\!|_{0,k-1}^2 + C\theta^{-2}h_k^2|\!|\!|\zeta_{k-1}|\!|\!|_{1,k-1}^2,$$

and hence

$$\begin{aligned}|\!|\!|\zeta_{k-1}|\!|\!|_{0,k-1}^2 &\le \frac{1}{1-(\theta^2/2)}|\!|\!|\zeta_{k-1}|\!|\!|_{0,k}^2 + C\theta^{-2}h_k^2|\!|\!|\zeta_{k-1}|\!|\!|_{1,k-1}^2\\ &\le (1+\theta^2)|\!|\!|\zeta_{k-1}|\!|\!|_{0,k}^2 + C\theta^{-2}h_k^2|\!|\!|\zeta_k|\!|\!|_{1,k}^2,\end{aligned} \tag{6.16}$$

where we have also used (5.16).

On the other hand, we have, by (5.9), (6.14), (6.15) and Theorem 3.6,

$$\begin{aligned}|\!|\!|\zeta_{k-1}|\!|\!|_{0,k}^2 &\le (|\!|\!|\zeta_k|\!|\!|_{0,k} + |\!|\!|\zeta_{k-1}-\zeta_k|\!|\!|_{0,k})^2\\ &\le (1+\theta^2)|\!|\!|\zeta_k|\!|\!|_{0,k}^2 + (1+\theta^{-2})|\!|\!|\zeta_{k-1}-\zeta_k|\!|\!|_{0,k}^2\end{aligned}$$

$$
\begin{aligned}
(6.17)\qquad &\leq (1+\theta^2)\|\!|\zeta_k|\!\|_{0,k}^2 + C\theta^{-2}\|\zeta_{k-1}-\zeta_k\|_{L_{2,-\mu}(\Omega)}^2 \\
&\leq (1+\theta^2)\|\!|\zeta_k|\!\|_{0,k}^2 + C\theta^{-2}\big(\|\zeta_{k-1}-\zeta\|_{L_{2,-\mu}(\Omega)} + \|\zeta-\zeta_k\|_{L_{2,-\mu}(\Omega)}\big)^2 \\
&\leq (1+\theta^2)\|\!|\zeta_k|\!\|_{0,k}^2 + C\theta^{-2}h_k^4\|\psi\|_{L_{2,\mu}(\Omega)}^2 \leq (1+\theta^2)\|\!|\zeta_k|\!\|_{0,k}^2 + C\theta^{-2}h_k^4\|\!|\zeta_k|\!\|_{2,k}^2.
\end{aligned}
$$

Combining (5.8), (6.16) and (6.17), we find

$$
\|\!|P_k^{k-1}\zeta_k|\!\|_{0,k-1}^2 \leq (1+\theta^2)^2\|\!|\zeta_k|\!\|_{0,k}^2 + C\theta^{-2}h_k^2\|\!|\zeta_k|\!\|_{1,k}^2,
$$

which implies that (6.4) holds for ζ_k because $\theta \in (0,1)$ is arbitrary. □

Finally we prove the estimate (6.5) which is a special feature of nonconforming methods where in general $Id_{k-1} \neq P_k^{k-1}I_{k-1}^k$.

Lemma 6.6. *The estimate* (6.5) *is valid.*

Proof. Let $\zeta_{k-1} \in V_{k-1}$ be arbitrary. By Lemma 6.1, there exists $\psi \in L_{2,\mu}(\Omega)$ such that

$$
a_k(\zeta_{k-1}, v) = \int_\Omega \psi v\,dx \quad \forall\, v \in V_k \quad \text{and} \quad \|\psi\|_{L_{2,\mu}(\Omega)} \leq C\|\!|\zeta_{k-1}|\!\|_{2,k}. \tag{6.18}
$$

Let $\zeta \in H_0^1(\Omega)\cap H_\mu^2(\Omega)$ satisfy (6.9). In view of (6.18), ζ_{k-1} is the DG approximation of ζ on the k-th level, and as in the proof of Lemma 6.5, $P_k^{k-1}\zeta_{k-1}$ is the DG approximation of ζ on the $(k-1)$-st level.

It follows from Theorem 3.6, (5.8), (5.15) and (6.18) that

$$
\begin{aligned}
\|\!|(Id_{k-1}-P_k^{k-1}I_{k-1}^k)\zeta_{k-1}|\!\|_{0,k-1} &= \|\!|\zeta_{k-1}-P_k^{k-1}\zeta_{k-1}|\!\|_{0,k-1} \\
&\leq C\|\zeta_{k-1}-P_k^{k-1}\zeta_{k-1}\|_{L_{2,-\mu}(\Omega)} \\
&\leq C\big[\|\zeta_{k-1}-\zeta\|_{L_{2,-\mu}(\Omega)} + \|\zeta-P_k^{k-1}\zeta_{k-1}\|_{L_{2,-\mu}(\Omega)}\big] \\
&\leq Ch_k^2\|\psi\|_{L_{2,\mu}(\Omega)} \\
&\leq Ch_k^2\|\!|\zeta_{k-1}|\!\|_{2,k} \\
&\leq Ch_k\|\!|\zeta_{k-1}|\!\|_{1,k} \leq Ch_k\|\!|\zeta_{k-1}|\!\|_{1,k-1}.
\end{aligned}
$$

□

We have verified the assumptions for the additive theory. Therefore we can apply the results in [12] to obtain the following convergence theorems for the V-cycle and F-cycle algorithms.

Theorem 6.7. *The output $MG_V(k,g,z_0,m,m)$ of the V-cycle algorithm (Algorithm 4.1) applied to* (4.12) *satisfies the following estimate:*

$$
\|z - MG_V(k,g,z_0,m,m)\|_{a_k} \leq \frac{C}{m}\|z-z_0\|_{a_k},
$$

where the positive constant C is independent of the grid level, provided that the number of smoothing steps m is greater than a positive integer m_* that is also independent of the grid level.

Theorem 6.8. *The output $MG_F(k, g, z_0, m, m)$ of the F-cycle algorithm (Algorithm 4.3) applied to (4.12) satisfies the following estimate:*

$$\|z - MG_F(k, g, z_0, m, m)\|_{a_k} \leq \frac{C}{m}\|z - z_0\|_{a_k},$$

where the positive constant C is independent of the grid level, provided that the number of smoothing steps m is greater than a positive integer m_* that is also independent of the grid level.

In particular, both the V-cycle and the F-cycle algorithm are contractions with contraction number independent of the grid level if the number of smoothing steps is sufficiently large.

Remark 6.9. Since we can take $\alpha = 1$ in the additive theory and use natural injection to connect the nested finite element spaces, the convergence analysis here is simpler than the convergence analysis in [12, 42, 22, 21].

7. Numerical Experiments

In this section we report results of several numerical experiments for the model problem (1.1) on the L-shaped domain $(-1, 1)^2 \setminus ([0, 1] \times [-1, 0])$. The triangulations $\mathcal{T}_0, \mathcal{T}_1, \ldots,$ are generated by the refinement procedure described at the beginning of Section 4, where $\mathcal{T}_0$ has four elements (created by connecting the origin to each of the three vertices $(-1, -1)$, $(-1, 1)$ and $(1, 1)$) and the grading parameter at the reentrant corner is taken to be 2/3. The mesh parameter of $\mathcal{T}_k$ is $h_k = 2^{-k}$.

We take the exact solution to be

$$u(x, y) = (1 - x^2)(1 - y^2)r^{2/3} \sin\big(2\theta/3\big),$$

where (r, θ) are the polar coordinates at the origin. We computed the energy error and L_2 error for the solution u_k of the method of Brezzi et al. (respectively the LDG method, the method of Bassi et al. and the SIPG method) with $\eta = 1$ (respectively $\eta = 1$, $\eta = 4$ and $\eta = 10$) for $0 \leq k \leq 7$. The results are plotted against the mesh size in log-log scale and presented in Figure 7.1 and Figure 7.2. The order of convergence is 1 for the energy norm and 2 for the L_2 norm in all four cases.

We also computed the contraction numbers of the W-cycle, F-cycle and V-cycle algorithms for the DG methods on the graded meshes $\mathcal{T}_1, \ldots, \mathcal{T}_7$.

We used $\eta = 1$ and $\lambda = 1/35$ for the method of Brezzi et al. and tabulated the contraction numbers in Tables 7.1–7.3. We found that the W-cycle (respectively F-cycle and V-cycle) is a contraction for $m \geq 2$ (respectively $m \geq 3$ and $m \geq 5$).

For the LDG method, we used $\eta = 1$ and $\lambda = 1/20$. The results are reported in Tables 7.4–7.6. In this case the W-cycle (respectively V-cycle and F-cycle) algorithm is a contraction for $m = 3$ (respectively $m \geq 4$ and $m \geq 5$).

We used $\eta = 4$ and $\lambda = 1/80$ for the method of Bassi et al. The contraction numbers are tabulated in Tables 7.7–7.9. We found that the W-cycle (respectively F-cycle and V-cycle) algorithm is a contraction for $m \geq 1$ (respectively $m \geq 3$ and $m \geq 4$).

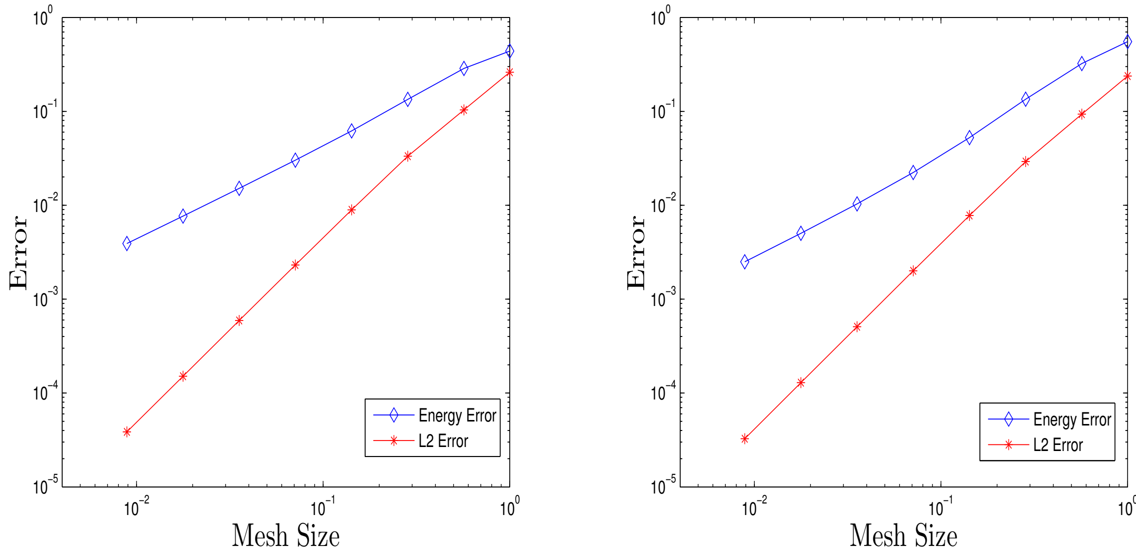

FIGURE 7.1. Energy errors and L_2 errors for the method of Brezzi et al. (left, $\eta = 1$) and for the LDG method (right, $\eta = 1$)

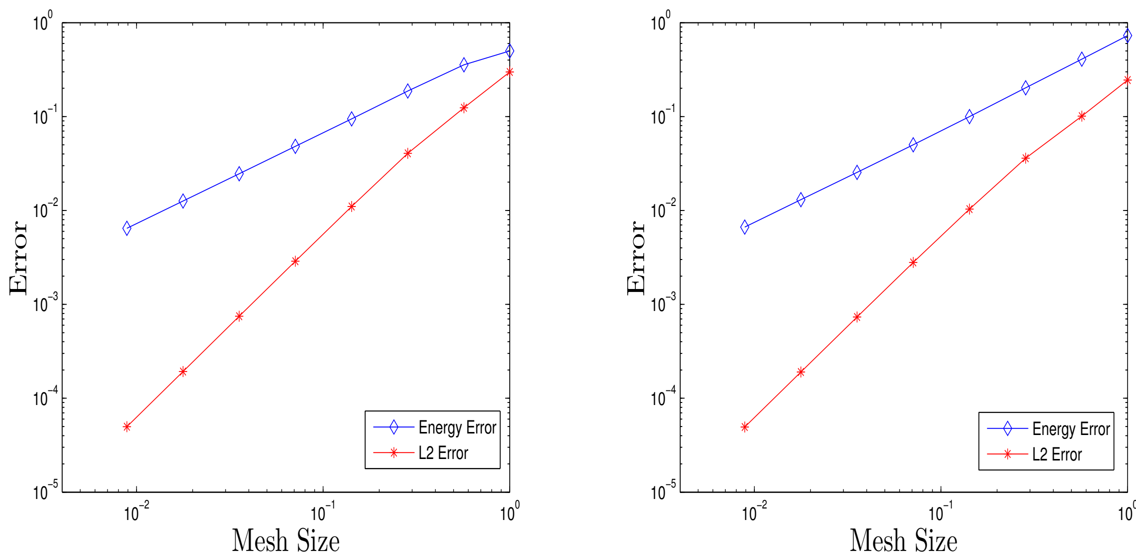

FIGURE 7.2. Energy errors and L_2 errors for the method of Bassi et al. (left, $\eta = 4$) and for the SIPG method (right, $\eta = 10$)

For the SIPG method, we used $\eta = 10$ and $\lambda = 1/40$. The W-cycle (respectively F-cycle and V-cycle) algorithm is a contraction for $m \geq 2$ (respectively $m \geq 4$ and $m \geq 6$). Since

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=2$	0.69	0.75	0.76	0.79	0.78	0.79	0.79
$m=3$	0.53	0.65	0.72	0.72	0.74	0.74	0.75
$m=4$	0.44	0.60	0.66	0.68	0.69	0.71	0.71
$m=5$	0.39	0.55	0.61	0.64	0.65	0.66	0.67
$m=6$	0.33	0.50	0.56	0.61	0.62	0.63	0.64
$m=7$	0.30	0.47	0.54	0.57	0.59	0.60	0.60
$m=8$	0.26	0.34	0.51	0.55	0.57	0.57	0.58
$m=9$	0.22	0.41	0.48	0.52	0.53	0.54	0.54

TABLE 7.1. Contraction numbers of the W-cycle algorithm on the L-shaped domain for the method of Brezzi et al. ($\eta = 1$)

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=3$	0.53	0.65	0.72	0.72	0.74	0.74	0.75
$m=4$	0.44	0.60	0.66	0.68	0.70	0.71	0.71
$m=5$	0.39	0.55	0.61	0.64	0.65	0.66	0.67
$m=6$	0.33	0.50	0.56	0.61	0.62	0.63	0.64
$m=7$	0.30	0.47	0.54	0.57	0.60	0.60	0.60
$m=8$	0.26	0.44	0.51	0.55	0.57	0.58	0.58
$m=9$	0.22	0.41	0.48	0.52	0.53	0.54	0.54
$m=10$	0.20	0.39	0.44	0.47	0.52	0.52	0.53

TABLE 7.2. Contraction numbers of the F-cycle algorithm on the L-shaped domain for the method of Brezzi et al. ($\eta = 1$)

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=5$	0.39	0.58	0.65	0.69	0.69	0.73	0.90
$m=6$	0.33	0.54	0.60	0.62	0.68	0.68	0.70
$m=7$	0.30	0.50	0.56	0.52	0.62	0.65	0.66
$m=8$	0.26	0.47	0.53	0.57	0.61	0.63	0.63
$m=9$	0.22	0.44	0.48	0.56	0.57	0.60	0.61
$m=10$	0.20	0.42	0.46	0.53	0.57	0.58	0.59
$m=11$	0.19	0.40	0.44	0.49	0.54	0.57	0.57
$m=12$	0.17	0.38	0.43	0.46	0.50	0.54	0.55

TABLE 7.3. Contraction numbers of the V-cycle algorithm on the L-shaped domain for the method of Brezzi et al. ($\eta = 1$)

the results are similar to the ones reported in [15] (where $\mathcal{T}_0$ has six elements instead of four), we do not present them here.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=3$	0.99	0.88	0.65	0.63	0.63	0.63	0.64
$m=4$	0.65	0.43	0.51	0.55	0.57	0.56	0.57
$m=5$	0.43	0.38	0.45	0.49	0.50	0.51	0.52
$m=6$	0.28	0.33	0.41	0.44	0.46	0.47	0.47
$m=7$	0.18	0.29	0.37	0.39	0.42	0.43	0.44
$m=8$	0.13	0.26	0.33	0.37	0.39	0.39	0.40
$m=9$	0.11	0.24	0.30	0.32	0.36	0.37	0.38
$m=10$	0.09	0.22	0.29	0.32	0.35	0.36	0.36

TABLE 7.4. Contraction numbers of the W-cycle algorithm on the L-shaped domain for the LDG method ($\eta = 1$)

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=4$	0.65	0.43	0.51	0.54	0.57	0.57	0.58
$m=5$	0.43	0.38	0.45	0.49	0.50	0.52	0.52
$m=6$	0.28	0.33	0.41	0.44	0.45	0.47	0.47
$m=7$	0.18	0.29	0.37	0.39	0.42	0.43	0.44
$m=8$	0.13	0.26	0.33	0.37	0.39	0.39	0.40
$m=9$	0.11	0.24	0.30	0.32	0.36	0.37	0.38
$m=10$	0.09	0.22	0.29	0.32	0.35	0.36	0.36
$m=11$	0.07	0.20	0.22	0.31	0.33	0.34	0.34

TABLE 7.5. Contraction numbers of the F-cycle algorithm on the L-shaped domain for the LDG method ($\eta = 1$)

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=5$	0.43	0.45	0.53	0.64	0.72	0.77	0.81
$m=6$	0.28	0.35	0.36	0.42	0.48	0.48	0.50
$m=7$	0.18	0.31	0.38	0.41	0.48	0.48	0.49
$m=8$	0.13	0.28	0.34	0.39	0.41	0.45	0.46
$m=9$	0.11	0.25	0.30	0.35	0.39	0.43	0.43
$m=10$	0.09	0.23	0.28	0.35	0.38	0.40	0.41
$m=11$	0.07	0.21	0.28	0.33	0.37	0.38	0.39
$m=12$	0.06	0.20	0.25	0.32	0.36	0.36	0.38

TABLE 7.6. Contraction numbers of the V-cycle algorithm on the L-shaped domain for the LDG method ($\eta = 1$)

Remark 7.1. For all four DG methods, the W-cycle algorithm and the F-cycle algorithm have similar contraction numbers when they are both contractions. The asymptotic behavior of the contraction number of the V-cycle and W-cycle algorithms for the SIPG method can

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=1$	0.81	0.89	0.91	0.91	0.91	0.91	0.91
$m=2$	0.79	0.85	0.86	0.87	0.87	0.87	0.87
$m=3$	0.72	0.80	0.83	0.83	0.84	0.83	0.83
$m=4$	0.65	0.75	0.81	0.80	0.81	0.80	0.81
$m=5$	0.61	0.71	0.76	0.78	0.78	0.79	0.79
$m=6$	0.55	0.70	0.74	0.78	0.77	0.77	0.78
$m=7$	0.51	0.67	0.72	0.75	0.76	0.77	0.77
$m=8$	0.49	0.66	0.71	0.72	0.74	0.75	0.75
$m=9$	0.45	0.64	0.68	0.71	0.72	0.74	0.74

TABLE 7.7. Contraction numbers of the W-cycle algorithm on the L-shaped domain for the method of Bassi et al. ($\eta = 4$)

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=3$	0.72	0.82	0.83	0.83	0.83	0.83	0.84
$m=4$	0.65	0.78	0.81	0.80	0.81	0.80	0.81
$m=5$	0.61	0.76	0.76	0.78	0.78	0.79	0.79
$m=6$	0.55	0.71	0.74	0.78	0.77	0.77	0.78
$m=7$	0.51	0.69	0.72	0.75	0.76	0.77	0.77
$m=8$	0.49	0.68	0.71	0.72	0.74	0.75	0.75
$m=9$	0.45	0.65	0.68	0.71	0.72	0.74	0.74
$m=10$	0.41	0.63	0.68	0.68	0.71	0.72	0.72
$m=11$	0.39	0.60	0.64	0.68	0.68	0.71	0.71

TABLE 7.8. Contraction numbers of the F-cycle algorithm on the L-shaped domain for the method of Bassi et al. ($\eta = 4$)

be found in [15, Figure 5.2], where the theoretical decay rate of $1/m$ is clearly visible. The asymptotic behaviors of the other three DG methods are similar.

Remark 7.2. The method of Brezzi et al. and the LDG method have similar sparsity. We notice that the magnitudes of both the energy and the L_2 errors for the LDG method are smaller, and when the multigrid algorithms are contractions for both methods, the contraction numbers for the LDG method are also smaller. The method of Bassi et al. and the SIPG method also have similar sparsity. The magnitudes of the energy and the L_2 errors for these methods are essentially the same. However, when the multigrid algorithms are contractions for both methods, the contraction numbers for the SIPG method are smaller. We refer the reader to [24] for the comparison of other aspects (e.g., storage) of the DG methods.

8. Concluding Remarks

We have demonstrated that a class of symmetric discontinuous Galerkin methods on graded meshes can be analyzed in a unified framework, and that the additive multigrid

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=4$	0.65	0.76	0.83	0.80	0.82	0.93	0.99
$m=5$	0.61	0.73	0.77	0.80	0.81	0.81	0.82
$m=6$	0.55	0.73	0.74	0.78	0.80	0.81	0.80
$m=7$	0.51	0.70	0.73	0.76	0.79	0.80	0.80
$m=8$	0.49	0.69	0.71	0.76	0.78	0.78	0.78
$m=9$	0.45	0.67	0.69	0.73	0.75	0.77	0.77
$m=10$	0.41	0.65	0.69	0.71	0.75	0.75	0.76
$m=11$	0.39	0.64	0.67	0.69	0.70	0.73	0.74
$m=12$	0.37	0.62	0.66	0.68	0.69	0.72	0.73

TABLE 7.9. Contraction numbers of the V-cycle algorithm on the L-shaped domain for the method of Bassi et al. ($\eta = 4$)

theory can be applied to DG methods on graded meshes. The uniform convergence of W-cycle, V-cycle and F-cycle algorithm established in Theorems 5.5, 6.7 and 6.8 complements existing multigrid results for DG methods [30, 22, 38, 21, 28, 19]. The results of this paper are also relevant for other nonconforming methods where graded meshes play a crucial role [16, 18, 17, 14].

For simplicity we have only considered conforming meshes in this paper. With some modifications, the results of this paper can be extended to nonconforming meshes. Research in this direction will be carried out in the future.

References

[1] R.A. Adams and J.J.F. Fournier. *Sobolev Spaces* (*Second Edition*). Academic Press, Amsterdam, 2003.
[2] Th. Apel, A.-M. Sändig, and J.R. Whiteman. Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains. *Math. Methods Appl. Sci.*, 19:63–85, 1996.
[3] D.N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19:742–760, 1982.
[4] D.N. Arnold, F. Brezzi, B. Cockburn, and L.D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39:1749–1779, 2001/02.
[5] I. Babuška. Finite element method for domains with corners. *Computing*, 6:264–273, 1970.
[6] I. Babuška, R.B. Kellogg, and J. Pitkäranta. Direct and inverse error estimates for finite elements with mesh refinements. *Numer. Math.*, 33:447–471, 1979.
[7] R.E. Bank and T.F. Dupont. An optimal order process for solving finite element equations. *Math. Comp.*, 36:35–51, 1981.
[8] F. Bassi, S. Rebay, G. Mariotti, S. Pedinotti, and M. Savini. A high-order accurate discontinuous finite element method for inviscid and viscous turbomachinery flows. In R. Decuypere and G. Dibelius, editors, *Proceedings of 2nd European Conference on Turbomachinery, Fluid Dynamics and Thermodynamics*, pages 99–108. Technologisch Instituut, Antwerpen, Belgium, 1997.
[9] J.H. Bramble and X. Zhang. The Analysis of Multigrid Methods. In P.G. Ciarlet and J.L. Lions, editors, *Handbook of Numerical Analysis, VII*, pages 173–415. North-Holland, Amsterdam, 2000.

[10] J.J. Brannick, H. Li, and L.T. Zikatanov. Uniform convergence of the multigrid V-cycle on graded meshes for corner singularities. *Numer. Linear Algebra Appl.*, 15:291–306, 2008.

[11] S.C. Brenner. Convergence of the multigrid V-cycle algorithm for second order boundary value problems without full elliptic regularity. *Math. Comp.*, 71:507–525, 2002.

[12] S.C. Brenner. Convergence of nonconforming V-cycle and F-cycle multigrid algorithms for second order elliptic boundary value problems. *Math. Comp.*, 73:1041–1066 (electronic), 2004.

[13] S.C. Brenner and C. Carstensen. Finite Element Methods. In E. Stein, R. de Borst, and T.J.R. Hughes, editors, *Encyclopedia of Computational Mechanics*, pages 73–118. Wiley, Weinheim, 2004.

[14] S.C. Brenner, J. Cui, F. Li, and L.-Y. Sung. A nonconforming finite element method for a two-dimensional curl-curl and grad-div problem. *Numer. Math.*, 109:509–533, 2008.

[15] S.C. Brenner, J. Cui, and L.-Y. Sung. Multigrid methods for the symmetric interior penalty method on graded meshes. *Numer. Linear Algebra Appl.*, 16:481–501, 2009.

[16] S.C. Brenner, F. Li, and L.-Y. Sung. A locally divergence-free nonconforming finite element method for the time-harmonic Maxwell equations. *Math. Comp.*, 76:573–595, 2007.

[17] S.C. Brenner, F. Li, and L.-Y. Sung. A locally divergence-free interior penalty method for two dimensional curl-curl problems. *SIAM J. Numer. Anal.*, 46:1190–1211, 2008.

[18] S.C. Brenner, F. Li, and L.-Y. Sung. A nonconforming penalty method for a two dimensional curl-curl problem. *M3AS*, 19:651–668, 2009.

[19] S.C. Brenner and L. Owens. A W-cycle algorithm for a weakly over-penalized interior penalty method. *Comput. Methods Appl. Mech. Engrg.*, 196:3823–3832, 2007.

[20] S.C. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods* (*Third Edition*). Springer-Verlag, New York, 2008.

[21] S.C. Brenner and L.-Y. Sung. Multigrid algorithms for C^0 interior penalty methods. *SIAM J. Numer. Anal.*, 44:199–223, 2006.

[22] S.C. Brenner and J. Zhao. Convergence of multigrid algorithms for interior penalty methods. *Appl. Numer. Anal. Comput. Math.*, 2:3–18, 2005.

[23] F. Brezzi, G. Manzini, D. Marini, P. Pietra, and A. Russo. Discontinuous Galerkin approximations for elliptic problems. *Numer. Methods Partial Differential Equations*, 16:365–378, 2000.

[24] P. Castillo. Performance of discontinuous Galerkin methods for elliptic PDEs. *SIAM J. Sci. Comput.*, 24:524–547, 2002.

[25] P. Castillo, B. Cockburn, I. Perugia, and D. Schötzau. An a priori error analysis of the local discontinuous Galerkin method for elliptic problems. *SIAM J. Numer. Anal.*, 38:1676–1706 (electronic), 2000.

[26] B. Cockburn and C.-W. Shu. The local discontinuous Galerkin method for time-dependent convection-diffusion systems. *SIAM J. Numer. Anal.*, 35:2440–2463 (electronic), 1998.

[27] M. Dauge. *Elliptic Boundary Value Problems on Corner Domains,* Lecture Notes in Mathematics 1341. Springer-Verlag, Berlin-Heidelberg, 1988.

[28] V.A. Dobrev, R.D. Lazarov, P.S. Vassilevski, and L.T. Zikatanov. Two-level preconditioning of discontinuous Galerkin approximations of second-order elliptic equations. *Numer. Linear Algebra Appl.*, 13:753–770, 2006.

[29] J. Douglas, Jr. and T. Dupont. Interior penalty procedures for elliptic and parabolic Galerkin methods. In *Computing Methods in Applied Sciences* (*Second Internat. Sympos., Versailles, 1975*), pages 207–216. Lecture Notes in Phys., Vol. 58. Springer, Berlin, 1976.

[30] J. Gopalakrishnan and G. Kanschat. A multilevel discontinuous Galerkin method. *Numer. Math.*, 95:527–550, 2003.

[31] P. Grisvard. *Elliptic Problems in Non Smooth Domains.* Pitman, Boston, 1985.

[32] T. Gudi. A new error analysis for discontinuous finite element methods for linear elliptic problems. *Math. Comp.*, 79:2169–2189, 2010.

[33] W. Hackbusch. *Multi-grid Methods and Applications*. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1985.

[34] V. Kondratiev. Boundary value problems for elliptic equations in domains with conical or angular points. *Trans. Moscow Math. Soc.*, pages 227–313, 1967.

[35] S.F. McCormick, editor. *Multigrid Methods*, volume 3 of *Frontiers in Applied Mathematics*. SIAM, Philadelphia, 1987.

[36] S.A. Nazarov and B.A. Plamenevsky. *Elliptic Problems in Domains with Piecewise Smooth Boundaries*. de Gruyter, Berlin-New York, 1994.

[37] U. Trottenberg, C. Oosterlee, and A. Schüller. *Multigrid*. Academic Press, San Diego, 2001.

[38] M. H. van Raalte and P. W. Hemker. Two-level multigrid analysis for the convection-diffusion equation discretized by a discontinuous Galerkin method. *Numer. Linear Algebra Appl.*, 12:563–584, 2005.

[39] M.F. Wheeler. An elliptic collocation-finite-element method with interior penalties. *SIAM J. Numer. Anal.*, 15:152–161, 1978.

[40] K. Yosida. *Functional Analysis*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the sixth (1980) edition.

[41] H. Yserentant. The convergence of multilevel methods for solving finite-element equations in the presence of singularities. *Math. Comp.*, 47:399–409, 1986.

[42] J. Zhao. Convergence of nonconforming V-cycle and F-cycle methods for the biharmonic problem using the Morley element. *Electron. Trans. Numer. Anal.*, 17:112–132, 2004.

Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803

E-mail address: brenner@math.lsu.edu

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803 (Current address: Institute for Mathematics and Its Applications, University of Minnesota, Minneapolis, MN 55455)

E-mail address: cui@math.lsu.edu (Current *E-mail address*:jcui@ima.umn.edu)

Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803 (Current Address: Department of Mathematics, Indian Institute of Science, Bangalore, 560012)

E-mail address: tgudi@cct.lsu.edu (Current *E-mail address*:gudi@math.iisc.ernet.in)

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803

E-mail address: sung@math.lsu.edu