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Research Article

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A Nonconforming Finite Element Method for an Acoustic Fluid-Structure Interaction Problem

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Abstract: We study a nonconforming finite element approximation of the vibration modes of an acoustic fluid-structure interaction. Displacement variables are used for both the fluid and the solid. The numerical scheme is based on an irrotational fluid displacement formulation and hence it is free of spurious eigenmodes. The method uses weakly continuous P_1 vector fields for the fluid and classical piecewise linear elements for the solid, and it has $O(h^2)$ convergence for the eigenvalues on properly graded meshes. The theoretical results are confirmed by numerical experiments.

Keywords: Nonconforming Finite Element Method, Fluid-Structure Interaction, Acoustic Fluid

MSC 2010: 65N25, 65N30, 74F10

1 Introduction

We consider the problem of determining the vibration modes of a linear elastic structure containing an acoustic (barotropic, inviscid and compressible) fluid.

Let Ω_F and Ω_S be the bounded polygonal domains in $\mathbb{R}^2$ occupied by the fluid and the solid, respectively, as in Figure 1. We assume Ω_F to be simply connected but not necessarily convex. Let Γ_I denote the interface between the solid and the fluid and let $\boldsymbol{n}$ be the unit normal vector pointing toward Ω_S. The exterior boundary of the solid is the union of Γ_D and Γ_N: the structure is fixed along Γ_D and free of stress along Γ_N. Let $\boldsymbol{\eta}$ denote the unit outward normal vector along Γ_N.

We denote

$$H^{s,\gamma}(\mathrm{div};\Omega_F) = \{\boldsymbol{u} \in [H^s(\Omega_F)]^2 : \nabla\cdot\boldsymbol{u} \in H^\gamma(\Omega_F)\}$$

and

$$\|\boldsymbol{u}\|^2_{H^{s,\gamma}(\mathrm{div};\Omega_F)} = \|\boldsymbol{u}\|^2_{H^s(\Omega_F)} + \|\nabla\cdot\boldsymbol{u}\|^2_{H^\gamma(\Omega_F)}.$$

The classical acoustic model for small-amplitude motions yields the following eigenvalue problem for the free vibration modes of the coupled system (see [8, 21]).

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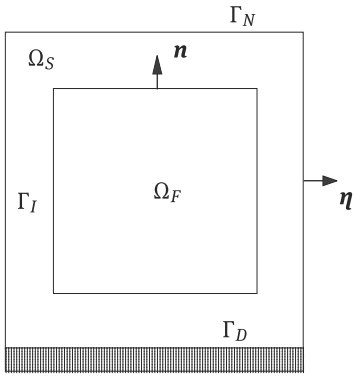

Figure 1: Domains of fluid and solid.

Problem. Find $\lambda \geq 0$ and $(\boldsymbol{u}, \boldsymbol{w}) \in H^{0,1}(\mathrm{div}; \Omega_F) \times [H^1(\Omega_S)]^2$, $(\boldsymbol{u}, \boldsymbol{w}) \neq (\boldsymbol{0}, \boldsymbol{0})$, such that

$$c^2\nabla(\nabla \cdot \boldsymbol{u}) + \lambda \boldsymbol{u} = \boldsymbol{0} \quad \text{in } \Omega_F, \tag{1.1a}$$

$$\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{w}) + \lambda \rho_S \boldsymbol{w} = \boldsymbol{0} \quad \text{in } \Omega_S, \tag{1.1b}$$

$$\boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{n} - (c^2\rho_F \nabla \cdot \boldsymbol{u})\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma_I, \tag{1.1c}$$

$$\boldsymbol{w} \cdot \boldsymbol{n} - \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_I, \tag{1.1d}$$

$$\boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{\eta} = \boldsymbol{0} \quad \text{on } \Gamma_N, \tag{1.1e}$$

$$\boldsymbol{w} = \boldsymbol{0} \quad \text{on } \Gamma_D. \tag{1.1f}$$

The physical meanings of the terms in equation (1.1) are as follows:

- $\boldsymbol{u}$ (respectively, $\boldsymbol{w}$) is the displacement of the fluid (respectively, solid) particle.
- $\sqrt{\lambda} = \omega$ is the frequency of the eigenmode.
- ρ_F (respectively, ρ_S) is the density of fluid (respectively, solid).
- $\boldsymbol{\epsilon}(\boldsymbol{w}) = \frac{1}{2}(\nabla \boldsymbol{w} + (\nabla \boldsymbol{w})^T)$ is the strain tensor for the displacement $\boldsymbol{w}$ of the solid.
- $\boldsymbol{\sigma}(\boldsymbol{w}) = 2\mu_S \boldsymbol{\epsilon}(\boldsymbol{w}) + \lambda_S(\nabla \cdot \boldsymbol{w})\boldsymbol{\delta}$ is the stress tensor, where $\boldsymbol{\delta}$ is the 2×2 identity matrix. Here $\mu_S = \frac{M}{2(1+\nu)}$ and $\lambda_S = \frac{M\nu}{(1+\nu)(1-2\nu)}$ are the Lamé constants, where M is the Young's modulus and ν is the Poisson ratio of the solid.

Remark 1.1. Equations (1.1a) and (1.1b) must be understood in the sense of distributions. Equations (1.1c) and (1.1d) hold in the sense of $H^{-1/2}(\Gamma_I)$. But since $\nabla \cdot \boldsymbol{u} \in H^1(\Omega_F)$ and $\boldsymbol{w} \cdot \boldsymbol{n} \in H^{1/2}(\Gamma_I)$, both can be considered as equalities in $L_2(\Gamma_I)$.

Problem (1.1) is the displacement formulation for the acoustic fluid-structure interaction. An advantage of this formulation is that it is easy to maintain compatibility when displacement variables are used for both the fluid and the solid [23, 28]. On the other hand this is a non-elliptic formulation and hence the solution operator associated with the source problem is not a compact operator. Indeed it has an infinite-dimensional kernel (cf. Section 2.1). The functions in the infinite-dimensional eigenspace for $\lambda = 0$ are pure rotations in the fluid that do not produce vibrations in the solid. Hence they are not physically relevant. But a naive discretization of the continuous problem would generate spurious eigenmodes that approximate these nonphysical eigenfunctions (cf. [21]). Such positive spurious eigenvalues pollute the approximation of the physical positive eigenvalues of (1.1).

Several approaches have been proposed to circumvent this drawback. A penalty method was introduced in [21]. It penalizes the curl-free condition so that the spurious eigenmodes are pushed towards higher frequencies and hence can be separated from the physical eigenvalues. An alternative approach [2, 6] uses standard piecewise linear elements for solid and the lowest order Raviart–Thomas elements for fluid. In this approach the discrete nonphysical 0 eigenvalue is isolated and all the positive discrete eigenvalues are spectrally correct. Optimal-order convergence on quasi-uniform meshes was established in [29]. This method has been adapted to deal with incompressible fluids in [3, 29], curved interfaces in [30] and to three dimensions

in [5]. Related work can also be found in [4, 7, 18, 25, 26]. The convergence analysis in these papers requires sophisticated techniques due to the non-elliptic nature of the formulation.

In this paper, we introduce and analyze a nonconforming finite element method for (1.1) that is based on an elliptic formulation using only irrotational fluid displacement. Since all the pure rotational motions that are not physically relevant are excluded from the variational problem, the spurious eigenvalues of the discrete problem disappear. The method uses weakly continuous P_1 vector fields for the fluid and classical piecewise linear elements for the solid. Furthermore, the convergence for the source problem is $O(h)$ and the convergence of the eigenvalues is $O(h^2)$ (a well-known doubling phenomenon) on general domains, provided that three consistency terms involving the jumps of the vector fields across element boundaries in the fluid and on the fluid-solid interface are included in the discretization and properly graded meshes are used. The analysis of the numerical scheme is facilitated by its connections to the nonconforming finite element methods studied in [9–11] for the Maxwell's equations. Furthermore, the compactness of the underlying operator greatly simplifies the analysis of the method as an eigensolver, which is similar to that in [12] for Maxwell's eigenvalues.

The rest of the paper is organized as follows. In Section 2, we introduce a nonconforming finite element method for the source problem associated with (1.1). Optimal order convergence of the method is established in both the energy norm and the L_2-norm. In Section 3, we present the convergence analysis for the nonconforming method as an eigensolver for (1.1). Results of a series of numerical experiments are reported in Section 4. We end the paper with a few concluding remarks in Section 5.

2 A Nonconforming Finite Element Method for the Source Problem

In this section, we introduce and analyze a nonconforming method on graded meshes for the source problem corresponding to (1.1).

2.1 The Source Problem

Let $H^1_{\Gamma_D}(\Omega_S)$ be the subspace of $H^1(\Omega_S)$ whose members vanish on Γ_D and

$$\mathcal{V} = \{(\boldsymbol{v}, \boldsymbol{w}) : \boldsymbol{v} \in H^{0,1}(\mathrm{div}; \Omega_F),\ \boldsymbol{w} \in [H^1_{\Gamma_D}(\Omega_S)]^2,\ \boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{w} \cdot \boldsymbol{n} \text{ on } \Gamma_I\}. \tag{2.1}$$

Note that $\mathcal{V}$ is a Hilbert space under the inner product defined by

$$((\boldsymbol{v}_1, \boldsymbol{w}_1), (\boldsymbol{v}_2, \boldsymbol{w}_2))_{\mathcal{V}} = \int_\Omega [\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + (\nabla \cdot \boldsymbol{v}_1)(\nabla \cdot \boldsymbol{v}_2) + \boldsymbol{w}_1 \cdot \boldsymbol{w}_2 + \nabla \boldsymbol{w}_1 : \nabla \boldsymbol{w}_2]\, dx.$$

The corresponding norm is denoted by $\|(\boldsymbol{v}, \boldsymbol{w})\|$. We also use $|(\boldsymbol{v}, \boldsymbol{w})|$ to denote the standard norm on $[L_2(\Omega_F)]^2 \times [L_2(\Omega_S)]^2$.

Given $(\boldsymbol{f}, \boldsymbol{g}) \in [L_2(\Omega_F)]^2 \times [L_2(\Omega_S)]^2$, the weak form of the source problem corresponding to (1.1) is to find $(\boldsymbol{u}, \boldsymbol{w}) \in \mathcal{V}$ such that

$$a((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{v}, \boldsymbol{z})) = b((\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{v}, \boldsymbol{z})) \quad \text{for all } (\boldsymbol{v}, \boldsymbol{z}) \in \mathcal{V},$$

where

$$a((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{v}, \boldsymbol{z})) = \int_{\Omega_F} (\rho_F c^2)(\nabla \cdot \boldsymbol{u})(\nabla \cdot \boldsymbol{v})\, dx + \int_{\Omega_S} (\boldsymbol{\sigma}(\boldsymbol{w}) : \boldsymbol{\epsilon}(\boldsymbol{z}))\, dx, \tag{2.2}$$

$$b((\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{v}, \boldsymbol{z})) = \int_{\Omega_F} \rho_F \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \int_{\Omega_S} \rho_S \boldsymbol{g} \cdot \boldsymbol{z}\, dx. \tag{2.3}$$

Note that

$$a((\boldsymbol{v}_1, \boldsymbol{w}_1), (\boldsymbol{v}_2, \boldsymbol{w}_2)) = 0 \quad \text{for all } (\boldsymbol{v}_1, \boldsymbol{w}_1) \in E_0 \text{ and } (\boldsymbol{v}_2, \boldsymbol{w}_2) \in \mathcal{V},$$

where E_0 is the infinite-dimensional subspace of $\mathcal{V}$ given by

$$E_0 = [\nabla \times H_0^1(\Omega_F)] \times \{\mathbf{0}\}.$$

Therefore this is a non-elliptic problem.

It is known (cf. [2, Lemma 2.3]) that the orthogonal complement of E_0 in $\mathcal{V}$ is the space $\mathring{\mathcal{V}}$ defined by

$$\mathring{\mathcal{V}} = \{(\boldsymbol{v}, \boldsymbol{w}) \in \mathcal{V} : \boldsymbol{v} \in H(\mathrm{curl}^0; \Omega_F)\},$$

where

$$H(\mathrm{curl}^0; \Omega_F) = \left\{ \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in [L_2(\Omega_F)]^2 : \nabla \times \boldsymbol{u} = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} = 0 \right\}.$$

We can avoid the complications of the non-elliptic weak problem by switching to the following source problem:

Problem. Find $(\boldsymbol{u}, \boldsymbol{w}) \in \mathring{\mathcal{V}}$ such that

$$a((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{v}, \boldsymbol{z})) = b((\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{v}, \boldsymbol{z})) \quad \text{for all } (\boldsymbol{v}, \boldsymbol{z}) \in \mathring{\mathcal{V}}. \tag{2.4}$$

The well-posedness of (2.4) follows from Korn's inequality, the compatibility condition (1.1d) and the following Friedrichs inequality [27, Section 4.4]:

$$\|\boldsymbol{v}\|_{L_2(\Omega_F)} \le C_{\Omega_F}(\|\nabla \cdot \boldsymbol{v}\|_{L_2(\Omega_F)} + \|\boldsymbol{n} \cdot \boldsymbol{v}\|_{L_2(\Gamma_I)}) \quad \text{for all } \boldsymbol{v} \in H(\mathrm{div}; \Omega_F) \cap H(\mathrm{curl}^0; \Omega_F).$$

Next we will show that the strong form of the reduced problem (2.4) is given by

$$-c^2 \nabla(\nabla \cdot \boldsymbol{u}) = Q\boldsymbol{f} \quad \text{in } \Omega_F, \tag{2.5a}$$

$$-\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{w}) = \rho_S \boldsymbol{g} \quad \text{in } \Omega_S, \tag{2.5b}$$

$$\boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{n} - (\rho_F c^2 \nabla \cdot \boldsymbol{u})\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_I, \tag{2.5c}$$

$$\boldsymbol{w} \cdot \boldsymbol{n} - \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_I, \tag{2.5d}$$

$$\boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{\eta} = \mathbf{0} \quad \text{on } \Gamma_N, \tag{2.5e}$$

$$\boldsymbol{w} = \mathbf{0} \quad \text{on } \Gamma_D, \tag{2.5f}$$

where Q is the orthogonal projection from $[L_2(\Omega_F)]^2$ onto $H(\mathrm{curl}^0; \Omega_F)$.

Indeed, as $(\boldsymbol{u}, \boldsymbol{w}) \in \mathring{\mathcal{V}}$, conditions (2.5d) and (2.5f) are automatically satisfied. Let $(\mathbf{0}, \boldsymbol{z})$ with $\boldsymbol{z} \in [D(\Omega_S)]^2$ be a test function. Then

$$\int_{\Omega_S} (\boldsymbol{\sigma}(\boldsymbol{w}) : \boldsymbol{\epsilon}(\boldsymbol{z}))\, dx = \int_{\Omega_S} \rho_S \boldsymbol{g} \cdot \boldsymbol{z}\, dx,$$

which implies (2.5b)

Let $\boldsymbol{v} \in [D(\Omega_F)]^2$ be a test function. Then $\boldsymbol{v} - Q\boldsymbol{v} \in \nabla \times H_0^1(\Omega_F)$. Hence $(\boldsymbol{v} - Q\boldsymbol{v}, \mathbf{0}) \in \mathcal{V}$, which implies $(Q\boldsymbol{v}, \mathbf{0}) \in \mathring{\mathcal{V}}$. It follows from (2.4) that

$$\begin{aligned}
\int_{\Omega_F} (\rho_F c^2)(\nabla \cdot \boldsymbol{u})(\nabla \cdot \boldsymbol{v})\, dx &= \int_{\Omega_F} (\rho_F c^2)(\nabla \cdot \boldsymbol{u})(\nabla \cdot [Q\boldsymbol{v} + (\boldsymbol{v} - Q\boldsymbol{v})])\, dx \\
&= \int_{\Omega_F} (\rho_F c^2)(\nabla \cdot \boldsymbol{u})(\nabla \cdot Q\boldsymbol{v})\, dx \\
&= \int_{\Omega_F} \rho_F \boldsymbol{f} \cdot (Q\boldsymbol{v})\, dx = \int_{\Omega_F} \rho_F (Q\boldsymbol{f}) \cdot \boldsymbol{v}\, dx,
\end{aligned}$$

which implies (2.5a). Now, for any $(\boldsymbol{v}, \boldsymbol{z}) \in \mathring{\mathcal{V}}$, integration by parts in (2.4), together with (2.5a), (2.5b), (2.5d) and (2.5f), gives

$$\int_{\Gamma_I} [(\rho_F c^2 \nabla \cdot \boldsymbol{u})\boldsymbol{n} - \boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{n}] \cdot \boldsymbol{z}\, ds + \int_{\Gamma_N} [\boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{\eta}] \cdot \boldsymbol{z}\, ds = 0, \tag{2.6}$$

which implies (2.5c) and (2.5e) in the sense of $H^{-1/2}(\Gamma_I)$ and $H^{-1/2}(\Gamma_N)$, respectively. In fact, we can take any $\boldsymbol{z} \in [H^1_{\Gamma_D}(\Omega_S)]^2$ as a test function in (2.6) since there exists $\boldsymbol{v} \in H^{0,1}(\mathrm{div}; \Omega_F)$ such that $(\boldsymbol{v}, \boldsymbol{z}) \in \mathring{\mathcal{V}}$. For instance, we can take $\boldsymbol{v} = \nabla q$, where $q \in H^1(\Omega_F)$ is a function satisfying $\frac{\partial q}{\partial \boldsymbol{n}} = \boldsymbol{z} \cdot \boldsymbol{n}$ on Γ_I.

2.2 Regularity of the Source Problem

The main difficulty arises at the interface where on one side we have the solid and on the other side we have the fluid. The regularity of the coupled problem is closely related to the regularity of the elasticity problem and because of the curl-free condition, the regularity of the Laplacian. Away from the interface, we can think of the problems separately and use the well-known regularity results [19, 20] for the solid and the fluid. However, on the interface the solutions from both sides are coupled through the interface conditions. In the following we focus on the singularities at a corner on the fluid-solid interface Γ_I.

First consider the homogeneous source problem defined as follows:

$$\begin{aligned} -c^2\nabla(\nabla\cdot\boldsymbol{u}) &= 0 && \text{in } \Omega_F,\\ -\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{w}) &= \boldsymbol{0} && \text{in } \Omega_S,\\ \boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{n} - (\rho_F c^2\nabla\cdot\boldsymbol{u})\boldsymbol{n} &= \boldsymbol{0} && \text{on } \Gamma_I,\\ \boldsymbol{w}\cdot\boldsymbol{n} - \boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{on } \Gamma_I,\\ \boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{\eta} &= \boldsymbol{0} && \text{on } \Gamma_N,\\ \boldsymbol{w} &= \boldsymbol{0} && \text{on } \Gamma_D. \end{aligned}$$

For a given corner c of Γ_I, we switch to the polar coordinates (r,θ) at c as the origin. We represent a vector function $\boldsymbol{v}(r,\theta)$ in polar coordinates as $\boldsymbol{v}(r,\theta) = v_r\hat{\boldsymbol{r}} + v_\theta\hat{\boldsymbol{\theta}}$, where

$$\hat{\boldsymbol{r}} = \begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} \quad\text{and}\quad \hat{\boldsymbol{\theta}} = \begin{pmatrix}-\sin(\theta)\\ \cos(\theta)\end{pmatrix}.$$

In other words, with respect to the basis $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}$,

$$\boldsymbol{v}(r,\theta) = \begin{pmatrix}v_r\\ v_\theta\end{pmatrix}.$$

Assume that c is at the origin and the edges of Γ_I emanating from c are defined by $\theta = -\omega_S$ and $\theta = \omega_S$ (cf. Figure 2). We apply separation of variables on the whole coupled problem. For the elasticity part we obtain

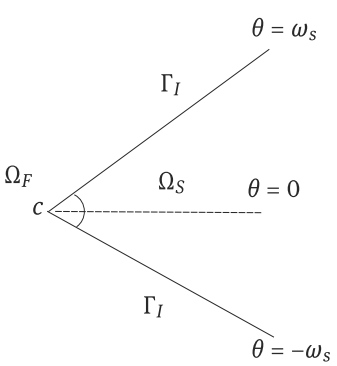

Figure 2: Corner c with angle $2\omega_s$.

the following general solution (cf. [24, Section 3.1.3]):

$$\boldsymbol{w}(r,\theta) = r^\gamma\begin{pmatrix} A\cos((\gamma+1)\theta) + B\sin((\gamma+1)\theta) + C\cos((\gamma-1)\theta) + D\sin((\gamma-1)\theta)\\ B\cos((\gamma+1)\theta) - A\sin((\gamma+1)\theta) + \Theta D\cos((\gamma-1)\theta) - \Theta C\sin((\gamma-1)\theta)\end{pmatrix},$$

where

$$\Theta := \frac{\frac{\mu_S}{\lambda_S+2\mu_S}(\gamma-1) - (\gamma+1)}{\frac{\mu_S}{\lambda_S+2\mu_S}(\gamma+1) - (\gamma-1)}$$

and the constants A, B, C, D are arbitrary. The curl-free condition implies that the problem in the fluid part is equivalent to the Laplacian problem which has the general solution of the following form:

$$\boldsymbol{u}(r,\theta) = r^{\gamma}\begin{pmatrix} E\cos((\gamma+1)\theta) + F\sin((\gamma+1)\theta) \\ F\cos((\gamma+1)\theta) - E\sin((\gamma+1)\theta) \end{pmatrix},$$

where E, F are arbitrary constants. When $\gamma \neq 1$ (a value that is not important for our purposes), the equation $\nabla(\nabla\cdot\boldsymbol{u}) = 0$ for this type of $\boldsymbol{u}$ implies that $\nabla\cdot\boldsymbol{u} = 0$. Therefore the first interface condition is the same as the traction boundary condition for the elasticity problem, i.e.,

$$\boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma_I.$$

This is equivalent to the following set of equations:

$$\begin{aligned}
\cos((\gamma+1)\omega_S)B + \frac{(\gamma-1)(\mu_S+\lambda_S)}{(\mu_S+\lambda_S)\gamma-(3\mu_S+\lambda_S)}\cos((\gamma-1)\omega_S)D &= 0,\\
\sin((\gamma+1)\omega_S)A + \frac{(\gamma-1)(\mu_S+\lambda_S)}{(\mu_S+\lambda_S)\gamma-(3\mu_S+\lambda_S)}\sin((\gamma-1)\omega_S)C &= 0,\\
\cos((\gamma+1)\omega_S)A + \frac{(\gamma+1)(\mu_S+\lambda_S)}{(\mu_S+\lambda_S)\gamma-(3\mu_S+\lambda_S)}\cos((\gamma-1)\omega_S)C &= 0,\\
\sin((\gamma+1)\omega_S)B + \frac{(\gamma+1)(\mu_S+\lambda_S)}{(\mu_S+\lambda_S)\gamma-(3\mu_S+\lambda_S)}\sin((\gamma-1)\omega_S)D &= 0.
\end{aligned}$$

Therefore, the coupled problem has a nontrivial solution when γ satisfies

$$\gamma^2\sin^2(2\omega_S) - \sin^2(2\gamma\omega_S) = 0, \tag{2.7}$$

where $\omega_S \in (0,\pi)$.

The second interface condition yields

$$\begin{aligned}
B\cos((\gamma+1)\omega_S) + \Theta D\cos((\gamma-1)\omega_S) + F\cos((\gamma+1)\omega_F) &= 0,\\
A\sin((\gamma+1)\omega_S) + \Theta C\sin((\gamma-1)\omega_S) + E\sin((\gamma+1)\omega_F) &= 0,
\end{aligned}$$

where $\omega_F = \pi - \omega_S$.

Assume $B = D = F = 0$, $\sin((\gamma+1)\omega_F) \neq 0$ and solve for A and E in terms of C. This choice corresponds to $\gamma\sin(2\omega_S) + \sin(2\gamma\omega_S) = 0$, one of the two factors of equation (2.7). We have

$$\begin{aligned}
A &= -\left(\frac{(\gamma-1)(\mu_S+\lambda_S)}{(\mu_S+\lambda_S)\gamma-(3\mu_S+\lambda_S)}\right)\frac{\sin((\gamma-1)\omega_S)}{\sin((\gamma+1)\omega_S)}C,\\
E &= -\frac{2(2\mu_S+\lambda_S)}{(\mu_S+\lambda_S)\gamma-(3\mu_S+\lambda_S)}\frac{\sin((\gamma-1)\omega_S)}{\sin((\gamma+1)\omega_F)}C.
\end{aligned}$$

Therefore, letting $C = 1$, we have

$$\boldsymbol{w}(r,\theta) = r^{\gamma}\begin{pmatrix} -\left(\frac{(\gamma-1)(\mu_S+\lambda_S)}{(\mu_S+\lambda_S)\gamma-(3\mu_S+\lambda_S)}\right)\frac{\sin((\gamma-1)\omega_S)}{\sin((\gamma+1)\omega_S)}\cos((\gamma+1)\theta) + \cos((\gamma-1)\theta) \\ \left(\frac{(\gamma-1)(\mu_S+\lambda_S)}{(\mu_S+\lambda_S)\gamma-(3\mu_S+\lambda_S)}\right)\frac{\sin((\gamma-1)\omega_S)}{\sin((\gamma+1)\omega_S)}\sin((\gamma+1)\theta) - \Theta\sin((\gamma-1)\theta) \end{pmatrix} \tag{2.8}$$

and

$$\boldsymbol{u}(r,\theta) = -r^{\gamma}\begin{pmatrix} \frac{2(2\mu_S+\lambda_S)}{(\mu_S+\lambda_S)\gamma-(3\mu_S+\lambda_S)}\frac{\sin((\gamma-1)\omega_S)}{\sin((\gamma+1)\omega_F)}\cos((\gamma+1)\theta) \\ -\frac{2(2\mu_S+\lambda_S)}{(\mu_S+\lambda_S)\gamma-(3\mu_S+\lambda_S)}\frac{\sin((\gamma-1)\omega_S)}{\sin((\gamma+1)\omega_F)}\sin((\gamma+1)\theta) \end{pmatrix}. \tag{2.9}$$

A similar solution can be obtained if we assume $A = C = E = 0$, $\cos((\gamma+1)\omega_F) \neq 0$ and solve for B, F in terms of D. This choice corresponds to $\gamma\sin(2\omega_S) - \sin(2\gamma\omega_S) = 0$, the other factor of equation (2.7).

Remark 2.1. Equation (2.7) is precisely the transcendental equation that determines the singularity of the elasticity problem with pure traction boundary condition (cf. [20, Section 4.2] and [24, Section 4.2]). Note that it is independent of the Lamé constants.

Consider the transcendental equation

$$\gamma^2 \sin^2(\omega) - \sin^2(\gamma\omega) = 0, \quad \omega \in (0, 2\pi). \tag{2.10}$$

Define the angle $\omega_0 \in (\pi, \frac{3\pi}{2})$ by

$$\omega_0 = \tan(\omega_0).$$

The following lemma defines the singularities from the solid side.

Lemma 2.2 ([20, Lemma 3.3.1, Lemma 3.3.2]). *In the strip* $0 < \mathrm{Re}(z) < 1$, *equation* (2.10) *has no root when* $\omega < \pi$, *has only one single real root* γ_1 *when* $\pi < \omega < \omega_0$ *and has two distinct simple real roots* $\gamma_1 < \gamma_2$ *when* $\omega_0 < \omega < 2\pi$. *Moreover, when* $\omega > \pi$,

$$\frac{1}{2} < \gamma_1 < \frac{\pi}{\omega} < \gamma_2.$$

Remark 2.3. One interesting observation is that the singularities of (2.4) are completely determined by the solid side and they are independent of the Lamé constants of the solid (cf. Remark 2.1). Note that for the fluid-structure interaction problem we take $\omega = 2\omega_S$, $\omega_S \in (0, \pi)$. For example, when $\omega_S = \frac{3\pi}{4}$, the singularity index corresponding to equation $\gamma \sin(\frac{3\pi}{2}) + \sin(\gamma\frac{3\pi}{2}) = 0$ is $\gamma = 0.544483661651611$.

We conclude this subsection by the following lemma. The proof, which is based on the a priori estimates for the Poisson problem and the linear elasticity problem, is similar to the proof of [2, Theorem 2.5].

Lemma 2.4. *The solution* $(\boldsymbol{u}, \boldsymbol{w})$ *of* (2.4) *belongs to* $H^{1+\gamma_1,1}(\mathrm{div}; \Omega_F) \times [H^{1+\gamma_1}(\Omega_S)]^2$ *and*

$$\{\|\boldsymbol{u}\|_{H^{1+\gamma_1,1}(\mathrm{div};\Omega_F)} + \|\boldsymbol{w}\|_{[H^{1+\gamma_1}(\Omega_S)]^2}\} \le C|(\boldsymbol{f}, \boldsymbol{g})|, \tag{2.11}$$

where γ_1 *is as in Lemma 2.2.*

2.3 A Nonconforming Finite Element Method

Let $\mathcal{T}_h$ be a family of triangulations of $\Omega_F \cup \Omega_S$ such that every triangle is completely contained either in Ω_F or Ω_S. We denote by $h = \max_{T\in\mathcal{T}_h} h_T$ the mesh parameter of $\mathcal{T}_h$, where h_T is the diameter of the triangle T. The triangulation $\mathcal{T}_h$ is graded around the corners $c_1, \dots, c_L$ on Γ_I with property that

$$C_1 h_T \le h\Phi_\mu(T) \le C_2 h_T, \tag{2.12}$$

where

$$\Phi_\mu(T) = \prod_{\ell=1}^{L} |c_\ell - c_T|^{1-\mu_\ell}. \tag{2.13}$$

Here c_T is the center of T and the positive constants C_1 and C_2 are independent of h. The vector μ contains the grading parameters $\mu_1, \dots, \mu_L$ chosen according to

$$\begin{aligned} \mu_\ell &= 1 && \text{if } \omega_\ell \le \pi, \\ \frac{1}{2} < \mu_\ell &< \gamma_1 (< \frac{\pi}{\omega_\ell}) && \text{if } \omega_\ell > \pi, \end{aligned} \tag{2.14}$$

where $\omega_1, \dots, \omega_L$ are the interior angles at the corners $c_1, \dots, c_L$ of Ω_S. (An example of such graded meshes for the domain in Figure 1 is given in Figure 4.)

Note that

$$h_T \lesssim h \qquad \text{for all } T \in \mathcal{T}_h, \tag{2.15}$$

$$h_T \approx h^{1/\mu_\ell} \qquad \text{if the corner } c_\ell \text{ is a vertex of } T \in \mathcal{T}_h. \tag{2.16}$$

Let $\mathcal{E}_{h,\Omega_F}$ be the set of the edges in $\mathcal{T}_h \cap \Omega_F$. Let $\mathcal{E}^i_{h,\Omega_F}$ denote set of the interior edges in $\mathcal{T}_h \cap \Omega_F$ and let $\mathcal{E}^{\Gamma_I}_h$ (respectively, $\mathcal{E}^{\Gamma_N}_h$, $\mathcal{E}^{\Gamma_D}_h$) denote the set of the edges on Γ_I (respectively, Γ_N, Γ_D). We define

$$N_h(\Omega_F) = \{\boldsymbol{v} \in [L_2(\Omega_F)]^2 : \boldsymbol{v}_T = \boldsymbol{v}|_T \in [P_1(T)]^2,\ \boldsymbol{v} \text{ is continuous at the midpoint of any } e \in \mathcal{E}^i_{h,\Omega_F}\}$$

and

$$L_h(\Omega_S) = \{\boldsymbol{z} \in [H^1(\Omega_S)]^2 : \boldsymbol{z}_T = \boldsymbol{z}|_T \in [P_1(T)]^2 \text{ for all } T \in \mathcal{T}_h,\ T \in \Omega_S, \text{ and } \boldsymbol{z} = 0 \text{ on any } e \in \mathcal{E}_h^{\Gamma_D}\}.$$

Then we define

$$V_h = \left\{(\boldsymbol{v}, \boldsymbol{z}) \in N_h(\Omega_F) \times L_h(\Omega_S) : \int_e (\boldsymbol{v} - \boldsymbol{z}) \cdot \boldsymbol{n}\, ds = 0 \text{ on any } e \in \mathcal{E}_h^{\Gamma_I}\right\}.$$

Let $\nabla_h \cdot$ and ∇_h be the piecewise div and grad operators defined by

$$(\nabla_h \cdot \boldsymbol{v})_T = \nabla \cdot (\boldsymbol{v}_T) \quad \text{for all } T \in \mathcal{T}_h,$$
$$(\nabla_h \boldsymbol{v})_T = \nabla(\boldsymbol{v}_T) \quad \text{for all } T \in \mathcal{T}_h.$$

Let $e \in \mathcal{E}_h^i$ be shared by the two triangles $T_{e,1}, T_{e,2} \in \mathcal{T}_h$ and let $\boldsymbol{n}_1$ (respectively, $\boldsymbol{n}_2$) be the unit normal of e pointing towards the outside of $T_{e,1}$ (respectively, $T_{e,2}$). We define, on e,

$$[\![\boldsymbol{n} \times \boldsymbol{v}]\!] = \boldsymbol{n}_1 \times (\boldsymbol{v}_{T_{e,1}}|_e) + \boldsymbol{n}_2 \times (\boldsymbol{v}_{T_{e,2}}|_e), \tag{2.17a}$$
$$[\![\boldsymbol{n} \cdot \boldsymbol{v}]\!] = \boldsymbol{n}_1 \cdot (\boldsymbol{v}_{T_{e,1}}|_e) + \boldsymbol{n}_2 \cdot (\boldsymbol{v}_{T_{e,2}}|_e). \tag{2.17b}$$

The nonconforming finite element method for (2.4) is:

Problem. Find $(\boldsymbol{u}_h, \boldsymbol{w}_h) \in V_h$ such that

$$a_h((\boldsymbol{u}_h, \boldsymbol{w}_h), (\boldsymbol{v}, \boldsymbol{z})) = b((\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{v}, \boldsymbol{z})) \quad \text{for all } (\boldsymbol{v}, \boldsymbol{z}) \in V_h, \tag{2.18}$$

where

$$\begin{aligned} a_h((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{v}, \boldsymbol{z})) = {} & \int_{\Omega_F} (\rho_F c^2)(\nabla_h \cdot \boldsymbol{u})(\nabla_h \cdot \boldsymbol{v})\, dx + h^{-2} \int_{\Omega_F} (\rho_F c^2)(\nabla_h \times \boldsymbol{u})(\nabla_h \times \boldsymbol{v})\, dx \\ & + \int_{\Omega_S} (\boldsymbol{\sigma}(\boldsymbol{w}) : \boldsymbol{\epsilon}(\boldsymbol{z}))\, dx + \sum_{e \in \mathcal{E}_{h,\Omega_F}^i} \frac{1}{|e|} \int_e (\rho_F c^2)[\![\boldsymbol{n} \cdot \boldsymbol{u}]\!][\![\boldsymbol{n} \cdot \boldsymbol{v}]\!]\, ds \\ & + \sum_{e \in \mathcal{E}_{h,\Omega_F}^i} \frac{1}{|e|} \int_e (\rho_F c^2)[\![\boldsymbol{n} \times \boldsymbol{u}]\!]\, [\![\boldsymbol{n} \times \boldsymbol{v}]\!]\, ds \\ & + \sum_{e \in \mathcal{E}_h^{\Gamma_I}} \frac{1}{|e|} \int_e (\rho_F c^2)(\boldsymbol{n} \cdot (\boldsymbol{u} - \boldsymbol{w}))(\boldsymbol{n} \cdot (\boldsymbol{v} - \boldsymbol{z}))\, ds. \end{aligned} \tag{2.19}$$

Here $\boldsymbol{\epsilon}_h(\boldsymbol{z}) = \frac{1}{2}(\nabla_h \boldsymbol{z} + (\nabla_h \boldsymbol{z})^T)$ and $\boldsymbol{\sigma}_h(\boldsymbol{\psi}) = 2\mu_S \boldsymbol{\epsilon}_h(\boldsymbol{w}) + \lambda_S(\nabla_h \cdot \boldsymbol{w})\boldsymbol{\delta}$ are the discrete versions of the strain and stress tensors defined in Section 1. From now on we will no longer keep track of the dependence on the constants ρ_F, ρ_S, c, μ_S and λ_S.

For any $s > \frac{1}{2}$, we define a weak interpolation operator $\Pi_T : [H^s(T)]^2 \to [P_1(T)]^2$ as follows:

$$(\Pi_T \boldsymbol{\zeta})(m_{e_j}) = \frac{1}{|e_j|} \int_{e_j} \boldsymbol{\zeta}\, ds \quad \text{for } 1 \le j \le 3, \tag{2.20}$$

where e_1, e_2 and e_3 are the edges of T, and m_e and $|e|$ denote the midpoint and the length of the edge e. It follows immediately from (2.20), the midpoint rule and Green's theorem that

$$\int_T \nabla \times (\Pi_T \boldsymbol{\zeta})\, dx = \int_T \nabla \times \boldsymbol{\zeta}\, dx, \tag{2.21}$$
$$\int_T \nabla \cdot (\Pi_T \boldsymbol{\zeta})\, dx = \int_T \nabla \cdot \boldsymbol{\zeta}\, dx. \tag{2.22}$$

Furthermore, the operator Π_T satisfies a standard error estimate [16]:

$$\|\boldsymbol{\zeta} - \Pi_T \boldsymbol{\zeta}\|_{L_2(T)} + h_T^{\min(s,1)} |\boldsymbol{\zeta} - \Pi_T \boldsymbol{\zeta}|_{H^{\min(s,1)}(T)} \le C_T h_T^s |\boldsymbol{\zeta}|_{H^s(T)} \tag{2.23}$$

for all $\boldsymbol{\zeta} \in [H^s(T)]^2$ and $s \in (\frac{1}{2}, 2]$, where the positive constant C_T depends on the minimum angle of T.

We can define a global interpolation operator $\Pi_{h,\Omega_F} : H^{s,1}(\mathrm{div};\Omega_F) \to N_h$ by piecing together the local interpolation operators:

$$(\Pi_{h,\Omega_F}\boldsymbol{v})_T = \Pi_T \boldsymbol{v}_T \quad \text{for all } T \in \mathcal{T}_{h,\Omega_F}.$$

A suitable V_h-interpolation operator $\mathbf{I}_h : \{H^{1+\alpha,1}(\mathrm{div};\Omega_F) \times [H^{1+\alpha}(\Omega_S)]^2\} \cap \mathcal{V} \to V_h$ $(\alpha > \frac{1}{2})$ is defined in the following way:

$$\mathbf{I}_h(\boldsymbol{u},\boldsymbol{w})|_T = \begin{cases} (L_{h,\Omega_S}\boldsymbol{w})|_T & \text{if } T \subset \Omega_S, \\ (\Pi_{h,\Omega_F}\boldsymbol{u})|_T & \text{if } T \subset \Omega_F \text{ and } \partial T \cap \Gamma_I = \emptyset, \\ (\widehat{\Pi_{h,\Omega_F}}\boldsymbol{u})|_T & \text{if } T \subset \Omega_F \text{ and } \partial T \cap \Gamma_I \neq \emptyset, \end{cases} \tag{2.24}$$

where $L_{h,\Omega_S}\boldsymbol{w}$ is the Lagrange interpolant of $\boldsymbol{w}$ in $L_h(\Omega_S)$, and $(\widehat{\Pi_{h,\Omega_F}}\boldsymbol{u})|_T$ is the function in $N_h(\Omega_F)$ such that

$$((\widehat{\Pi_{h,\Omega_F}}\boldsymbol{u}) \times \boldsymbol{n})|_e = ((\Pi_{h,\Omega_F}\boldsymbol{u}) \times \boldsymbol{n})|_e \quad \text{for all } e \in \mathcal{E}_{h,\Omega_F} \tag{2.25}$$

and

$$(\widehat{\Pi_{h,\Omega_F}}\boldsymbol{u} \cdot \boldsymbol{n})(m_e) = \begin{cases} ((\Pi_{h,\Omega_F}\boldsymbol{u}) \cdot \boldsymbol{n})(m_e) & \text{if } e \not\subset \Gamma_I, \\ \frac{1}{|e|}\int_e ((L_{h,\Omega_S}\boldsymbol{w}) \cdot \boldsymbol{n})|_{T_e}\, ds & \text{if } e \subset \Gamma_I, \end{cases} \tag{2.26}$$

with T_e the triangle contained in Ω_S such that $\partial T \cap \partial T_e = e$, and m_e is the midpoint of e.

2.4 Preliminary Error Estimates

We will measure the discretization error in the mesh-dependent energy norm $\|(\cdot,\cdot)\|_h$ defined by

$$\begin{aligned} \|(\boldsymbol{v},\boldsymbol{z})\|_h^2 &= \|\nabla_h \cdot \boldsymbol{v}\|_{L_2(\Omega_F)}^2 + h^{-2}\|\nabla_h \times \boldsymbol{v}\|_{L_2(\Omega_F)}^2 + \|\nabla_h \boldsymbol{z}\|_{L_2(\Omega_S)}^2 + \sum_{e \in \mathcal{E}_{h,\Omega_F}^i} \frac{1}{|e|}\|[\![\boldsymbol{n} \cdot \boldsymbol{v}]\!]\|_{L_2(e)}^2 \\ &\quad + \sum_{e \in \mathcal{E}_{h,\Omega_F}^i} \frac{1}{|e|}\|[\![\boldsymbol{n} \times \boldsymbol{v}]\!]\|_{L_2(e)}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma_I}} \frac{1}{|e|}\|\boldsymbol{n} \cdot (\boldsymbol{v} - \boldsymbol{z})\|_{L_2(e)}^2. \end{aligned} \tag{2.27}$$

Observe that $a_h((\cdot,\cdot),(\cdot,\cdot))$ is bounded by the energy norm, i.e.,

$$|a_h((\boldsymbol{\phi},\boldsymbol{\psi}),(\boldsymbol{v},\boldsymbol{z}))| \le C\|(\boldsymbol{\phi},\boldsymbol{\psi})\|_h \|(\boldsymbol{v},\boldsymbol{z})\|_h \tag{2.28}$$

for all $(\boldsymbol{\phi},\boldsymbol{\psi}), (\boldsymbol{v},\boldsymbol{z}) \in \{H^{1+\alpha,1}(\mathrm{div};\Omega_F) \times [H^{1+\alpha}(\Omega_S)]^2\} \cap \mathring{\mathcal{V}} + V_h$.

Due to Korn's inequality, $a_h((\cdot,\cdot),(\cdot,\cdot))$ is also coercive with respect to $\|(\cdot,\cdot)\|_h$, i.e.,

$$a_h((\boldsymbol{v},\boldsymbol{z}),(\boldsymbol{v},\boldsymbol{z})) \ge \gamma\|(\boldsymbol{v},\boldsymbol{z})\|_h^2 \tag{2.29}$$

for all $(\boldsymbol{v},\boldsymbol{z}) \in \{H^{1+\alpha,1}(\mathrm{div};\Omega_F) \times [H^{1+\alpha}(\Omega_S)]^2\} \cap \mathring{\mathcal{V}} + V_h$.

Lemma 2.5. *Let $(\boldsymbol{u},\boldsymbol{w}) \in \mathring{\mathcal{V}}$ be the solution of* (2.4), *and let $(\boldsymbol{u}_h,\boldsymbol{w}_h)$ satisfy the discrete problem* (2.18). *It holds that*

$$\|(\boldsymbol{u},\boldsymbol{w}) - (\boldsymbol{u}_h,\boldsymbol{w}_h)\|_h \le \left(1 + \frac{C}{\gamma}\right) \inf_{(\boldsymbol{v},\boldsymbol{z}) \in V_h} \|(\boldsymbol{u},\boldsymbol{w}) - (\boldsymbol{v},\boldsymbol{z})\|_h + \frac{1}{\gamma} \sup_{(\boldsymbol{\phi},\boldsymbol{\psi}) \in V_h \setminus \{(\mathbf{0},\mathbf{0})\}} \frac{a_h((\boldsymbol{u},\boldsymbol{w}) - (\boldsymbol{u}_h,\boldsymbol{w}_h),(\boldsymbol{\phi},\boldsymbol{\psi}))}{\|(\boldsymbol{\phi},\boldsymbol{\psi})\|_h}.$$

Proof. Let $(\boldsymbol{v},\boldsymbol{z}) \in V_h$ be arbitrary. It follows from (2.28) and (2.29) that

$$\begin{aligned} \|(\boldsymbol{u},\boldsymbol{w}) - (\boldsymbol{u}_h,\boldsymbol{w}_h)\|_h &\le \|(\boldsymbol{u},\boldsymbol{w}) - (\boldsymbol{v},\boldsymbol{z})\|_h + \|(\boldsymbol{v},\boldsymbol{z}) - (\boldsymbol{u}_h,\boldsymbol{w}_h)\|_h \\ &\le \|(\boldsymbol{u},\boldsymbol{w}) - (\boldsymbol{v},\boldsymbol{z})\|_h + \frac{1}{\gamma} \sup_{(\boldsymbol{\phi},\boldsymbol{\psi}) \in V_h \setminus \{(\mathbf{0},\mathbf{0})\}} \frac{a_h((\boldsymbol{v},\boldsymbol{z}) - (\boldsymbol{u}_h,\boldsymbol{w}_h),(\boldsymbol{\phi},\boldsymbol{\psi}))}{\|(\boldsymbol{\phi},\boldsymbol{\psi})\|_h} \\ &\le \left(1 + \frac{C}{\gamma}\right)\|(\boldsymbol{u},\boldsymbol{w}) - (\boldsymbol{v},\boldsymbol{z})\|_h + \frac{1}{\gamma} \sup_{(\boldsymbol{\phi},\boldsymbol{\psi}) \in V_h \setminus \{(\mathbf{0},\mathbf{0})\}} \frac{a_h((\boldsymbol{u},\boldsymbol{w}) - (\boldsymbol{u}_h,\boldsymbol{w}_h),(\boldsymbol{\phi},\boldsymbol{\psi}))}{\|(\boldsymbol{\phi},\boldsymbol{\psi})\|_h}, \end{aligned}$$

as desired. □

The following lemma is useful for the error analysis.

Lemma 2.6. *For any* $(\boldsymbol{u}, \boldsymbol{w}) \in \{H^{1+\alpha,1}(\mathrm{div};\Omega_F) \times [H^{1+\alpha}(\Omega_S)]^2\} \cap \mathcal{V}$, $T \in \Omega_F$ *such that* $\partial T \cap \Gamma_I \neq \emptyset$ *and* $T_e \in \Omega_S$ *with* $\partial T \cap \partial T_e = e$, *there exists a positive constant* C *(depending on the minimum angle of* T *and* T_e*) such that*

$$\sum_{\substack{T\in\Omega_F\\ \partial T\cap\Gamma_I\neq\emptyset}} \|\Pi_T\boldsymbol{u} - (\widehat{\Pi_{h,\Omega_F}\boldsymbol{u}})|_T\|^2_{H(\mathrm{div};T)} \le Ch^2|(\boldsymbol{f},\boldsymbol{g})|^2. \tag{2.30}$$

Proof. Let $\boldsymbol{\phi}_{e,n}$ be the basis vectors of $\mathcal{N}_h(\Omega_F)$ corresponding to the normal vectors on e. It follows from (2.1), (2.20) and (2.26) that

$$(\Pi_{h,\Omega_F}\boldsymbol{u} - \widehat{\Pi_{h,\Omega_F}\boldsymbol{u}})|_T = \left|\frac{1}{|e|}\int_e (\boldsymbol{u} - L_{h,\Omega_S}\boldsymbol{w})\cdot\boldsymbol{n}\,ds\right|\boldsymbol{\phi}_{e,n} = \left|\frac{1}{|e|}\int_e (\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w})\cdot\boldsymbol{n}\,ds\right|\boldsymbol{\phi}_{e,n}. \tag{2.31}$$

Therefore,

$$\|\Pi_T\boldsymbol{u} - (\widehat{\Pi_{h,\Omega_F}\boldsymbol{u}})|_T\|^2_{H(\mathrm{div};T)} = \left|\frac{1}{|e|}\int_e (\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w})\cdot\boldsymbol{n}\,ds\right|^2 \|\boldsymbol{\phi}_{e,n}\|^2_{H(\mathrm{div};T)}. \tag{2.32}$$

Note that

$$\int_T |\nabla\cdot\boldsymbol{\phi}_{e,n}|^2\,dx = \frac{1}{|T|}\left|\int_T \nabla\cdot\boldsymbol{\phi}_{e,n}\,dx\right|^2 = \frac{1}{|T|}\left|\int_{\partial T}\boldsymbol{\phi}_{e,n}\cdot\boldsymbol{n}_T\,ds\right|^2 = \frac{|e|^2}{|T|}.$$

Hence $\|\boldsymbol{\phi}_{e,n}\|^2_{H(\mathrm{div};T)} \le C$ for a constant C that depends on the minimum angle of T.

If T_e is away from the reentrant corners on Γ_I, by using (2.11), the trace theorem (with scaling) and standard interpolation results [13, 15], we have

$$\begin{aligned}\left|\frac{1}{|e|}\int_e (\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w})\cdot\boldsymbol{n}\,ds\right|^2 &\le \frac{1}{|e|}\|\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w}\|^2_{[L_2(e)]^2}\\ &\le C\Big\{h_T^{-2}\|\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w}\|^2_{[L_2(T_e)]^2} + |\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w}|^2_{[H^1(T_e)]^2}\Big\}\\ &\le Ch_T^2|\boldsymbol{w}|^2_{[H^2(T_e)]^2}.\end{aligned} \tag{2.33}$$

For triangles T_e inside the neighborhood of a reentrant corner but not touching the corner, it follows from (2.12) that

$$\begin{aligned}\left|\frac{1}{|e|}\int_e [(\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w})\cdot\boldsymbol{n}\,ds\right|^2 &\le \frac{1}{|e|}\|\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w}\|^2_{[L_2(e)]^2}\\ &\le C\Big\{h_T^{-2}\|\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w}\|^2_{[L_2(T_e)]^2} + |\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w}|^2_{[H^1(T_e)]^2}\Big\}\\ &\le Ch_T^2|\boldsymbol{w}|^2_{[H^2(T_e)]^2}\\ &\le Ch^2[\Phi_\mu(T)]^2|\boldsymbol{w}|^2_{[H^2(T_e)]^2}\\ &\le Ch^2|\boldsymbol{w}|^2_{[H^2(T_e)]^2},\end{aligned} \tag{2.34}$$

where we have applied the fact that

$$\int_0^1 r^{2(1-\mu_\ell)}r^{2(\gamma_1-2)}r\,dr < \infty \quad \text{if } \mu_\ell < \gamma_1. \tag{2.35}$$

For triangles T_e touching a reentrant corner, we can apply an interpolation error estimate for the fractional order Sobolev spaces [17] together with (2.11), (2.16), (2.14) and the trace theorem with scaling to obtain

$$\begin{aligned}\left|\frac{1}{|e|}\int_e [(\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w})\cdot\boldsymbol{n}\,ds\right|^2 &\le \frac{1}{|e|}\|\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w}\|^2_{[L_2(e)]^2}\\ &\le C\Big\{h_T^{-2}\|\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w}\|^2_{[L_2(T_e)]^2} + |\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w}|^2_{[H^1(T_e)]^2}\Big\}\\ &\le Ch_T^{2\mu_\ell}|\boldsymbol{w}|^2_{[H^{1+\mu_\ell}(T_e)]^2}.\end{aligned} \tag{2.36}$$

Estimate (2.30) then follows from the regularity result (2.11) and the summation of (2.32)–(2.36) over $T \in \Omega_F$ with $\partial T \cap \Gamma_I \neq \emptyset$. □

Lemma 2.7. *Let* $(\boldsymbol{u}, \boldsymbol{w}) \in \mathring{\mathcal{V}}$ *be the solution of* (2.4). *We have*

$$\sum_{\substack{T\in\Omega_F\\ \partial T\cap\Gamma_I=\emptyset}} \|\nabla\cdot(\boldsymbol{u}-\Pi_T\boldsymbol{u})\|^2_{L_2(T)} + \sum_{\substack{T\in\Omega_F\\ \partial T\cap\Gamma_I\neq\emptyset}} \|\nabla\cdot(\boldsymbol{u}-(\widehat{\Pi_{h,\Omega_F}}\boldsymbol{u})|_T)\|^2_{L_2(T)} \le Ch^2|(\boldsymbol{f},\boldsymbol{g})|. \tag{2.37}$$

Proof. Observe that (2.22) implies

$$\nabla\cdot(\Pi_T\boldsymbol{u}) = \Pi^0_T(\nabla\cdot\boldsymbol{u}),$$

where Π^0_T is the orthogonal projection from $L_2(\Omega)$ onto the space of piecewise constant functions with respect to $T \in \mathcal{T}_{h,\Omega_F}$. Hence, by using a standard interpolation error estimate [13, 15], we have

$$\sum_{\substack{T\in\Omega_F\\ \partial T\cap\Gamma_I=\emptyset}} \|\nabla\cdot(\boldsymbol{u}-\Pi_T\boldsymbol{u})\|^2_{L_2(T)} = \sum_{\substack{T\in\Omega_F\\ \partial T\cap\Gamma_I=\emptyset}} \|\nabla\cdot\boldsymbol{u}-\Pi^0_h(\nabla\cdot\boldsymbol{u})\|^2_{L_2(T)} \le Ch^2|\nabla\cdot\boldsymbol{u}|^2_{H^1(\Omega_F)}. \tag{2.38}$$

It then follows from (2.38), Lemma 2.6 and the regularity result (2.11) that

$$\begin{aligned}\sum_{\substack{T\in\Omega_F\\ \partial T\cap\Gamma_I\neq\emptyset}} \|\nabla\cdot(\boldsymbol{u}-(\widehat{\Pi_{h,\Omega_F}}\boldsymbol{u})|_T)\|^2_{L_2(T)} &\le \sum_{\substack{T\in\Omega_F\\ \partial T\cap\Gamma_I\neq\emptyset}} \|\nabla\cdot(\boldsymbol{u}-\Pi_T\boldsymbol{u})\|^2_{L_2(T)} + \sum_{\substack{T\in\Omega_F\\ \partial T\cap\Gamma_I\neq\emptyset}} \|\nabla\cdot(\Pi_T\boldsymbol{u}-(\widehat{\Pi_{h,\Omega_F}}\boldsymbol{u})|_T)\|^2_{L_2(T)}\\ &\le C\{h^2|\nabla\cdot\boldsymbol{u}|^2_{H^1(\Omega_F)} + h^2|(\boldsymbol{f},\boldsymbol{g})|^2\}.\end{aligned} \tag{2.39}$$

We conclude the proof of (2.37) by combining (2.11), (2.38) and (2.39). □

Note that the interpolation operator Π_{h,Ω_F} defined on $\mathcal{E}^i_{h,\Omega_F}$ is identical with the one employed in [9]. The following result can be proved similarly as in [9, Lemma 5.2]:

Lemma 2.8. *Let* $(\boldsymbol{u}, \boldsymbol{w}) \in \mathring{\mathcal{V}}$ *be the solution of* (2.4). *We have*

$$\sum_{e\in\mathcal{E}^i_{h,\Omega_F}} \frac{1}{|e|}\|[\![\boldsymbol{u}-\Pi_{h,\Omega_F}\boldsymbol{u}]\!]\|^2_{L_2(e)} \le Ch^2|(\boldsymbol{f},\boldsymbol{g})|^2. \tag{2.40}$$

Proof. Let $e \in \mathcal{E}^i_{h,\Omega_F}$ and let $\mathcal{T}_e$ be the set of the triangles in $\mathcal{T}_h$ having e as an edge. We have

$$\frac{1}{|e|}\|[\![\boldsymbol{u}-\Pi_{h,\Omega_F}\boldsymbol{u}]\!]\|^2_{L_2(e)} \lesssim \sum_{T\in\mathcal{T}_e} |e|^{-1}\|\boldsymbol{u}-\Pi_T\boldsymbol{u}\|^2_{L^2(e)}. \tag{2.41}$$

If $T \in \mathcal{T}_e$ is away from the reentrant corners on Γ_I, then we have, by the trace theorem (with scaling), (2.15) and (2.23) (with $s = 2$),

$$|e|^{-1}\|\boldsymbol{u}-\Pi_T\boldsymbol{u}\|^2_{L_2(e)} \lesssim h_T^{-2}\|\boldsymbol{u}-\Pi_T\boldsymbol{u}\|^2_{L_2(T)} + |\boldsymbol{u}-\Pi_T\boldsymbol{u}|^2_{H^1(T)} \le Ch_T^2|\boldsymbol{u}|^2_{H^2(T)}. \tag{2.42}$$

Note that if $T \in \mathcal{T}_e$ is inside the neighborhood of a reentrant corner but not touching the corner, estimate (2.42) also holds in view of (2.35). On the other hand, if $T \in \mathcal{T}_e$ has a reentrant corner c_ℓ as one of its vertices, we can use (2.16) and (2.23) (with $s = 1 + \mu_\ell$) to obtain

$$|e|^{-1}\|\boldsymbol{u}-\Pi_T\boldsymbol{u}\|^2_{L_2(e)} \lesssim h_T^{-2}\|\boldsymbol{u}-\Pi_T\boldsymbol{u}\|^2_{L_2(T)} + |\boldsymbol{u}-\Pi_T\boldsymbol{u}|^2_{H^1(T)} \lesssim h_T^{2\mu_\ell}|\boldsymbol{u}|^2_{H^{1+\mu_\ell}(T)}. \tag{2.43}$$

Estimate (2.40) follows from the regularity result (2.11) and the summation of (2.41)–(2.43) over $e \in \mathcal{E}^i_{h,\Omega_F}$. This completes the proof of the lemma. □

The following lemma, which is identical with [9, Lemma 5.3], is useful for estimating terms involving the jumps of the weakly continuous P_1 vector fields across the edges. The proof is based on the trace theorem (with scaling) and a standard interpolation error estimate [13, 15].

Lemma 2.9. *It holds that*

$$\sum_{e\in\mathcal{E}_{h,\Omega_F}} |e|\|\eta-\bar{\eta}_{T_e}\|^2_{L_2(e)} \le Ch^2|\eta|^2_{H^1(\Omega_F)} \quad \textit{for all } \eta \in H^1(\Omega_F),$$

where

$$\bar{\eta}_{T_e} = \frac{1}{|T_e|}\int_{T_e} \eta\, dx \tag{2.44}$$

is the mean of η *over* T_e, *one of the triangles in* $\mathcal{T}_h$ *that has* e *as an edge.*

Recall that Q is the L_2-orthogonal projection onto $H(\mathrm{curl}^0;\Omega_F)$. The following result is useful in addressing the consistency error caused by Q in (2.5a).

Lemma 2.10. *The following estimate holds:*

$$\|\boldsymbol{v}-Q\boldsymbol{v}\|_{[L_2(\Omega_F)]^2}\le Ch\|(\boldsymbol{v},\boldsymbol{w})\|_h \tag{2.45}$$

for all $(\boldsymbol{v},\boldsymbol{w})\in[H^{1+\alpha,1}(\mathrm{div};\Omega_F)\times[H^{1+\alpha}(\Omega_S)]^2]+V_h$.

Proof. For any $\boldsymbol{v}\in H^{1+\alpha,1}(\mathrm{div};\Omega_F)$, we have $(\boldsymbol{v}-Q\boldsymbol{v})\in\nabla\times H_0^1(\Omega_F)$, the orthogonal complement of $H(\mathrm{curl}^0;\Omega_F)$. Hence by duality,

$$\|\boldsymbol{v}-Q\boldsymbol{v}\|_{[L_2(\Omega_F)]^2}=\sup_{\eta\in H_0^1(\Omega_F)\setminus\{0\}}\frac{(\boldsymbol{v}-Q\boldsymbol{v},\nabla\times\eta)}{\|\nabla\times\eta\|_{L_2(\Omega_F)}}=\sup_{\eta\in H_0^1(\Omega_F)\setminus\{0\}}\frac{(\boldsymbol{v},\nabla\times\eta)}{\|\nabla\times\eta\|_{L_2(\Omega_F)}}. \tag{2.46}$$

It follows from integration by parts that

$$(\boldsymbol{v},\nabla\times\eta)=(\nabla_h\times\boldsymbol{v},\eta)+\sum_{e\in\mathcal{E}^i_{h,\Omega_F}}\int_e\eta[\![\boldsymbol{n}\times\boldsymbol{v}]\!]\,ds.$$

In view of (2.27) and the Poincaré–Friedrichs inequality, we have

$$(\nabla_h\times\boldsymbol{v},\eta)\le\|\nabla_h\times\boldsymbol{v}\|_{L_2(\Omega_F)}\|\eta\|_{L_2(\Omega_F)}\le Ch\|(\boldsymbol{v},\boldsymbol{w})\|_h\|\nabla\times\eta\|_{L_2(\Omega_F)}. \tag{2.47}$$

Since $[\![\boldsymbol{n}\times\boldsymbol{v}]\!]$ vanishes at the midpoints of the interior edges, using the midpoint rule we can write

$$\begin{aligned}\sum_{e\in\mathcal{E}^i_{h,\Omega_F}}\int_e\eta[\![\boldsymbol{n}\times\boldsymbol{v}]\!]\,ds&=\sum_{e\in\mathcal{E}^i_{h,\Omega_F}}\int_e(\eta-\bar\eta_{T_e})[\![\boldsymbol{n}\times\boldsymbol{v}]\!]\,ds\\&\le C\Big\{\sum_{e\in\mathcal{E}^i_{h,\Omega_F}}|e|\|\eta-\bar\eta_{T_e}\|^2_{L_2(e)}\Big\}^{1/2}\Big\{\sum_{e\in\mathcal{E}^i_{h,\Omega_F}}\frac{1}{|e|}\|[\![\boldsymbol{n}\times\boldsymbol{v}]\!]\|^2_{L_2(e)}\Big\}^{1/2}\\&\le Ch|\eta|_{H^1(\Omega_F)}\|(\boldsymbol{v},\boldsymbol{w})\|_h,\end{aligned} \tag{2.48}$$

where $\bar\eta_{T_e}$ is defined by (2.44) and we used Lemma 2.9 for the last inequality.

Estimate (2.45) follows from (2.46), (2.47) and (2.48). □

2.5 Convergence Analysis

We begin with the approximation property of V_h:

Lemma 2.11. *Let* $(\boldsymbol{u},\boldsymbol{w})\in\mathring{\mathcal{V}}$ *be the solution of* (2.4). *We have*

$$\inf_{(\boldsymbol{v},\boldsymbol{z})\in V_h}\|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{v},\boldsymbol{z})\|_h\le\|(\boldsymbol{u},\boldsymbol{w})-\mathbf{I}_h(\boldsymbol{u},\boldsymbol{w})\|_h\le Ch|(\boldsymbol{f},\boldsymbol{g})|. \tag{2.49}$$

Proof. Since $\nabla\times\boldsymbol{u}=0$ implies $\nabla_h\times(\Pi_{h,\Omega_F}\boldsymbol{u})=0$, and because of (2.21), (2.24) and (2.25), we have

$$\begin{aligned}\|(\boldsymbol{u},\boldsymbol{w})-\mathbf{I}_h(\boldsymbol{u},\boldsymbol{w})\|_h^2=&\sum_{\substack{T\in\Omega_F\\ \partial T\cap\Gamma_I=\emptyset}}\|\nabla\cdot(\boldsymbol{u}-\Pi_T\boldsymbol{u})\|^2_{L_2(T)}+\sum_{\substack{T\in\Omega_F\\ \partial T\cap\Gamma_I\neq\emptyset}}\|\nabla\cdot(\boldsymbol{u}-(\widehat{\Pi_{h,\Omega_F}}\boldsymbol{u})|_T)\|^2_{L_2(T)}\\&+\|\nabla_h(\boldsymbol{w}-L_{h,\Omega_S}\boldsymbol{w})\|^2_{L_2(\Omega_S)}+\sum_{e\in\mathcal{E}^i_{h,\Omega_F}}\frac{1}{|e|}\|[\![\boldsymbol{n}\cdot(\boldsymbol{u}-\Pi_{h,\Omega_F}\boldsymbol{u})]\!]\|^2_{L_2(e)}\\&+\sum_{e\in\mathcal{E}^i_{h,\Omega_F}}\frac{1}{|e|}\|[\![\boldsymbol{n}\times(\boldsymbol{u}-\Pi_{h,\Omega_F}\boldsymbol{u})]\!]\|^2_{L_2(e)}\\&+\sum_{e\in\mathcal{E}^{\Gamma_I}_h}\frac{1}{|e|}\|\boldsymbol{n}\cdot((\boldsymbol{u}-\widehat{\Pi_{h,\Omega_F}}\boldsymbol{u})-(\boldsymbol{w}-L_{h,\Omega_S}\boldsymbol{w}))\|^2_{L_2(e)}.\end{aligned} \tag{2.50}$$

The first two terms on the right-hand side of (2.50) have been estimated by Lemma 2.7. The fourth and fifth terms are estimated by Lemma 2.8. The following estimate for the third term on the right-hand side of (2.50) can be obtained by a similar argument as in the proof of Lemma 2.6:

$$\|\nabla_h(\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w})\|^2_{L_2(\Omega_S)} = \sum_{T\in\Omega_S} |\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w}|^2_{[H^1(T)]} \le Ch^2|(\boldsymbol{f},\boldsymbol{g})|^2. \tag{2.51}$$

For the last term, we have

$$\begin{aligned}\sum_{e\in\mathcal{E}_h^{\Gamma_I}} &\frac{1}{|e|}\|\boldsymbol{n}\cdot((\boldsymbol{u} - \widehat{\Pi_{h,\Omega_F}\boldsymbol{u}}) - (\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w}))\|^2_{L_2(e)} \\ &\le C\sum_{e\in\mathcal{E}_h^{\Gamma_I}}\Big\{\frac{1}{|e|}\|\boldsymbol{n}\cdot(\boldsymbol{u} - \widehat{\Pi_{h,\Omega_F}\boldsymbol{u}})\|^2_{L_2(e)} + \frac{1}{|e|}\|\boldsymbol{n}\cdot(\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w})\|^2_{L_2(e)}\Big\}.\end{aligned} \tag{2.52}$$

The second term on the right-hand side of (2.52) can be bounded by $Ch^2|(\boldsymbol{f},\boldsymbol{g})|^2$, by a similar argument as in Lemma 2.6.

It only remains to estimate the first term on the right-hand side of (2.52). Note that

$$\frac{1}{|e|}\|\boldsymbol{n}\cdot(\boldsymbol{u} - \widehat{\Pi_{h,\Omega_F}\boldsymbol{u}})\|^2_{[L_2(e)]^2} \le \frac{1}{|e|}\|\boldsymbol{n}\cdot(\boldsymbol{u} - \Pi_{h,\Omega_F}\boldsymbol{u})\|^2_{[L_2(e)]^2} + \frac{1}{|e|}\|\boldsymbol{n}\cdot(\Pi_{h,\Omega_F}\boldsymbol{u} - \widehat{\Pi_{h,\Omega_F}\boldsymbol{u}})\|^2_{[L_2(e)]^2}. \tag{2.53}$$

The first term on the right-hand side of (2.53) can be estimated by a trace theorem as in Lemma 2.8. Moreover,

$$(\Pi_{h,\Omega_F}\boldsymbol{u} - \widehat{\Pi_{h,\Omega_F}\boldsymbol{u}})|_T = \Big|\frac{1}{|e|}\int_e [(\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w})\cdot\boldsymbol{n}]\,ds\Big|\boldsymbol{\phi}_{e,n},$$

where $T \in \Omega_F$ has e as an edge and $\boldsymbol{\phi}_{e,n}$ is defined as in (2.31). Therefore, the second term on the right-hand side of (2.53) becomes

$$\frac{1}{|e|}\|\boldsymbol{n}\cdot(\Pi_{h,\Omega_F}\boldsymbol{u} - \widehat{\Pi_{h,\Omega_F}\boldsymbol{u}})\|^2_{[L_2(e)]^2} = \Big|\frac{1}{|e|}\int_e [(\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w})\cdot\boldsymbol{n}]\,ds\Big|^2\Big(\frac{1}{|e|}\|\boldsymbol{n}\cdot\boldsymbol{\phi}_{e,n}\|^2_{L_2(e)}\Big). \tag{2.54}$$

Note that $\frac{1}{|e|}\|\boldsymbol{n}\cdot\boldsymbol{\phi}_{e,n}\|^2_{L_2(e)} \le C$, where C is a constant that depends on the minimum angle of T. Therefore, we can derive from (2.52), the summation of (2.53)–(2.54) over $e \in \Gamma_I$, and similar arguments as in Lemma 2.6 and Lemma 2.8 that

$$\sum_{e\in\mathcal{E}_h^{\Gamma_I}} \frac{1}{|e|}\|\boldsymbol{n}\cdot((\boldsymbol{u} - \widehat{\Pi_{h,\Omega_F}\boldsymbol{u}}) - (\boldsymbol{w} - L_{h,\Omega_S}\boldsymbol{w}))\|^2_{L_2(e)} \le Ch^2|(\boldsymbol{f},\boldsymbol{g})|^2. \tag{2.55}$$

Estimate (2.49) then follows from (2.51), (2.55) and Lemmas 2.7–2.8. □

Lemma 2.12. *Let* $(\boldsymbol{u},\boldsymbol{w}) \in \mathring{\mathcal{V}}$ *be the solution of* (2.4), *and* $(\boldsymbol{u}_h,\boldsymbol{w}_h)$ *satisfy the discrete problem* (2.18). *It holds that*

$$\sup_{(\boldsymbol{\phi},\boldsymbol{\psi})\in V_h\setminus\{(\boldsymbol{0},\boldsymbol{0})\}} \frac{a_h((\boldsymbol{u},\boldsymbol{w}) - (\boldsymbol{u}_h,\boldsymbol{w}_h),(\boldsymbol{\phi},\boldsymbol{\psi}))}{\|(\boldsymbol{\phi},\boldsymbol{\psi})\|_h} \le Ch|(\boldsymbol{f},\boldsymbol{g})|. \tag{2.56}$$

Proof. Let $(\boldsymbol{\phi},\boldsymbol{\psi}) \in V_h$ be arbitrary. Using (2.5), integration by parts, the fact that $\nabla\times\boldsymbol{u} = 0$ in Ω_F, and $\boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{n} = [(\rho_F c^2)(\nabla\cdot\boldsymbol{u})]\boldsymbol{n}$, we find

$$\begin{aligned}a_h((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{\phi},\boldsymbol{\psi})) &= \sum_{T\in\Omega_F}\int_T (\rho_F c^2)(\nabla\cdot\boldsymbol{u})(\nabla\cdot\boldsymbol{\phi})\,dx + \int_{\Omega_S}(\boldsymbol{\sigma}_h(\boldsymbol{w}) : \boldsymbol{\epsilon}_h(\boldsymbol{\psi}))\,dx \\ &= b((Q\boldsymbol{f},\boldsymbol{g}),(\boldsymbol{\phi},\boldsymbol{\psi})) + \sum_{e\in\mathcal{E}_{h,\Omega_F}}\int_e (\rho_F c^2)(\nabla\cdot\boldsymbol{u})[\![\boldsymbol{n}\cdot\boldsymbol{\phi}]\!]\,ds - \sum_{e\in\mathcal{E}_h^{\Gamma_I}}\int_e (\boldsymbol{\sigma}_h(\boldsymbol{w})\boldsymbol{n})\cdot\boldsymbol{\psi}\,ds \\ &= b((Q\boldsymbol{f},\boldsymbol{g}),(\boldsymbol{\phi},\boldsymbol{\psi})) + \sum_{e\in\mathcal{E}^i_{h,\Omega_F}}\int_e (\rho_F c^2)(\nabla\cdot\boldsymbol{u})[\![\boldsymbol{n}\cdot\boldsymbol{\phi}]\!]\,ds \\ &\quad + \sum_{e\in\mathcal{E}_h^{\Gamma_I}}\int_e (\rho_F c^2)(\nabla\cdot\boldsymbol{u})[\boldsymbol{n}\cdot(\boldsymbol{\phi} - \boldsymbol{\psi})]\,ds.\end{aligned} \tag{2.57}$$

Therefore, by (2.18),

$$\begin{aligned} a_h((\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h),(\boldsymbol{\phi},\boldsymbol{\psi})) &= b((Q\boldsymbol{f}-\boldsymbol{f},\boldsymbol{0}),(\boldsymbol{\phi},\boldsymbol{\psi})) + \sum_{e\in\mathcal{E}^i_{h,\Omega_F}} \int_e (\rho_F c^2)(\nabla\cdot\boldsymbol{u})[\![\boldsymbol{n}\cdot\boldsymbol{\phi}]\!]\,ds \\ &\quad + \sum_{e\in\mathcal{E}^{\Gamma_I}_h} \int_e (\rho_F c^2)(\nabla\cdot\boldsymbol{u})[\boldsymbol{n}\cdot(\boldsymbol{\phi}-\boldsymbol{\psi})]\,ds \\ &= \int_{\Omega_F} \rho_F \boldsymbol{f}\cdot(Q\boldsymbol{\phi}-\boldsymbol{\phi})\,dx + \sum_{e\in\mathcal{E}^i_{h,\Omega_F}} \int_e (\rho_F c^2)(\nabla\cdot\boldsymbol{u})[\![\boldsymbol{n}\cdot\boldsymbol{\phi}]\!]\,ds \\ &\quad + \sum_{e\in\mathcal{E}^{\Gamma_I}_h} \int_e (\rho_F c^2)(\nabla\cdot\boldsymbol{u})[\boldsymbol{n}\cdot(\boldsymbol{\phi}-\boldsymbol{\psi})]\,ds. \end{aligned} \tag{2.58}$$

We observe from (2.58) that there are three sources for the consistency error of scheme (2.18), namely the projection Q, the discontinuity of the vector fields V_h inside Ω_F, and on Γ_I.

In view of Lemma 2.10, the first term on the right-hand side of equation (2.58) satisfies the estimate

$$\int_{\Omega_F} \boldsymbol{f}\cdot(Q\boldsymbol{\phi}-\boldsymbol{\phi})\,dx \le C\|\boldsymbol{f}\|_{[L_2(\Omega_F)]^2}\|Q\boldsymbol{\phi}-\boldsymbol{\phi}\|_{[L_2(\Omega_F)]^2} \le Ch|(\boldsymbol{f},\boldsymbol{g})|\|(\boldsymbol{\phi},\boldsymbol{\psi})\|_h. \tag{2.59}$$

By the definition of V_h, the Cauchy–Schwarz inequality, (2.27) and Lemma 2.9, we can estimate the second term on the right-hand side of (2.58) as follows:

$$\begin{aligned} \sum_{e\in\mathcal{E}^i_{h,\Omega_F}} \int_e (\nabla\cdot\boldsymbol{u})[\![\boldsymbol{n}\cdot\boldsymbol{\phi}]\!]\,ds &= \sum_{e\in\mathcal{E}^i_{h,\Omega_F}} \int_e (\nabla\cdot\boldsymbol{u}-\overline{(\nabla\cdot\boldsymbol{u})}_{T_e})[\![\boldsymbol{n}\cdot\boldsymbol{\phi}]\!]\,ds \\ &\le \Big\{\sum_{e\in\mathcal{E}^i_{h,\Omega_F}} |e|\|\nabla\cdot\boldsymbol{u}-\overline{(\nabla\cdot\boldsymbol{u})}_{T_e}\|^2_{L_2(e)}\Big\}^{1/2} \Big\{\sum_{e\in\mathcal{E}^i_{h,\Omega_F}} \frac{1}{|e|}\|[\![\boldsymbol{n}\cdot\boldsymbol{\phi}]\!]\|^2_{L_2(e)}\Big\}^{1/2} \\ &\le Ch|\nabla\cdot\boldsymbol{u}|_{H^1(\Omega_F)}\|(\boldsymbol{\phi},\boldsymbol{\psi})\|_h \\ &\le Ch|(\boldsymbol{f},\boldsymbol{g})|\|(\boldsymbol{\phi},\boldsymbol{\psi})\|_h. \end{aligned} \tag{2.60}$$

Now we turn to estimate the third term on the right-hand side of (2.58). For any $e\in\mathcal{E}^{\Gamma_I}_h$, note that

$$\int_e (\boldsymbol{\phi}-\boldsymbol{\psi})\cdot\boldsymbol{n}\,ds = 0.$$

Let $T^F_e\in\Omega_F$ and $T^S_e\in\Omega_S$ be the triangles such that $T^F_e\cap T^S_e = e$. By the Cauchy–Schwarz inequality,

$$\begin{aligned} \Big|\int_e (\nabla\cdot\boldsymbol{u})[(\boldsymbol{\phi}-\boldsymbol{\psi})\cdot\boldsymbol{n}]\,ds\Big| &= \Big|\int_e (\nabla\cdot\boldsymbol{u}-\overline{(\nabla\cdot\boldsymbol{u})}_{T^F_e})(\boldsymbol{\phi}\cdot\boldsymbol{n}-\boldsymbol{\psi}\cdot\boldsymbol{n})\,ds\Big| \\ &\le (|e|^{1/2}\|\nabla\cdot\boldsymbol{u}-\overline{(\nabla\cdot\boldsymbol{u})}_{T^F_e}\|_{L_2(e)})(|e|^{-1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}-\boldsymbol{\psi}\cdot\boldsymbol{n}\|_{L_2(e)}). \end{aligned} \tag{2.61}$$

In view of Lemma 2.9, we have

$$\sum_{e\in\mathcal{E}^{\Gamma_I}_h} |e|\|\nabla\cdot\boldsymbol{u}-\overline{(\nabla\cdot\boldsymbol{u})}_{T^F_e}\|^2_{L_2(e)} \le Ch^2|\nabla\cdot\boldsymbol{u}|^2_{H^1(\Omega_F)} \le Ch^2|(\boldsymbol{f},\boldsymbol{g})|^2. \tag{2.62}$$

Combining (2.27), (2.61), and (2.62), we obtain that

$$\sum_{e\in\mathcal{E}^{\Gamma_I}_h} \int_e (\rho_F c^2)(\nabla\cdot\boldsymbol{u})[\boldsymbol{n}\cdot(\boldsymbol{\phi}-\boldsymbol{\psi})]\,ds \le Ch|(\boldsymbol{f},\boldsymbol{g})|\|(\boldsymbol{\phi},\boldsymbol{\psi})\|_h. \tag{2.63}$$

Finally, estimate (2.56) follows from (2.58), (2.59), (2.60) and (2.63). This completes the proof of the lemma. □

The following theorem is an immediate consequence of Lemma 2.5, Lemma 2.11 and Lemma 2.12.

Theorem 2.13. *The following discretization error estimates hold for the solution* $(\boldsymbol{u}_h, \boldsymbol{w}_h) \in V_h$ *of* (2.18):

$$\|(\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h)\|_h \le Ch|(\boldsymbol{f}, \boldsymbol{g})|.$$

In the rest of this section, we derive the error estimate in the standard L_2-norm $|(\cdot,\cdot)|$ on $[L_2(\Omega_F)]^2 \times [L_2(\Omega_S)]^2$.

Theorem 2.14. *Let* $(\boldsymbol{u}, \boldsymbol{w}) \in \mathring{\mathcal{V}}$ *be the solution of* (2.4) *and* $(\boldsymbol{u}_h, \boldsymbol{w}_h) \in V_h$ *satisfy* (2.18). *Then we have*

$$|(\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h)| \le C(h^2|(\boldsymbol{f}, \boldsymbol{g})| + h\|(\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h)\|_h). \tag{2.64}$$

Proof. Let $(\boldsymbol{p}, \boldsymbol{q}) \in \mathring{\mathcal{V}}$ satisfy

$$a((\boldsymbol{v}, \boldsymbol{z}), (\boldsymbol{p}, \boldsymbol{q})) = b((\boldsymbol{v}, \boldsymbol{z}), (\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h)) \tag{2.65}$$

for all $(\boldsymbol{v}, \boldsymbol{z}) \in \mathring{\mathcal{V}}$. Here $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by (2.2) and (2.3).

Note that the strong form of (2.65) is

$$-c^2\nabla(\nabla\cdot\boldsymbol{p}) = Q(\boldsymbol{u} - \boldsymbol{u}_h) \quad \text{in } \Omega_F, \tag{2.66}$$

$$-\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{q}) = \rho_S(\boldsymbol{w} - \boldsymbol{w}_h) \quad \text{in } \Omega_S, \tag{2.67}$$

and we have the following estimate:

$$|\nabla\cdot\boldsymbol{p}|_{H^1(\Omega_F)} \le C|(\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h)|. \tag{2.68}$$

Furthermore, we can rewrite (2.65) as

$$a_h((\boldsymbol{v}, \boldsymbol{z}), (\boldsymbol{p}, \boldsymbol{q})) = b((\boldsymbol{v}, \boldsymbol{z}), (\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h)) \tag{2.69}$$

for all $(\boldsymbol{v}, \boldsymbol{z}) \in \mathring{\mathcal{V}}$. Here $a_h(\cdot,\cdot)$ is defined by (2.19).

It follows from (2.65), (2.66), (2.67), and integration by parts that the following analog of (2.57) holds:

$$\begin{aligned}
a_h((\boldsymbol{u}_h, \boldsymbol{w}_h), (\boldsymbol{p}, \boldsymbol{q})) &= \sum_{T\in\Omega_F}\int_T (\rho_F c^2)(\nabla\cdot\boldsymbol{q})(\nabla\cdot\boldsymbol{u}_h)\,dx + \int_{\Omega_S}(\boldsymbol{\sigma}_h(\boldsymbol{q}) : \boldsymbol{\epsilon}_h(\boldsymbol{w}_h))\,dx \\
&= b((\boldsymbol{u}_h, \boldsymbol{w}_h), (Q(\boldsymbol{u} - \boldsymbol{u}_h), (\boldsymbol{w} - \boldsymbol{w}_h))) + \sum_{e\in\mathcal{E}_{h,\Omega_F}}\int_e (\rho_F c^2)(\nabla\cdot\boldsymbol{p})[\![\boldsymbol{n}\cdot\boldsymbol{u}_h]\!]\,ds \\
&\quad - \sum_{e\in\mathcal{E}_h^{\Gamma_I}}\int_e (\boldsymbol{\sigma}_h(\boldsymbol{q})\boldsymbol{n})\cdot\boldsymbol{w}_h\,ds \\
&= b((\boldsymbol{u}_h, \boldsymbol{w}_h), (Q(\boldsymbol{u} - \boldsymbol{u}_h), (\boldsymbol{w} - \boldsymbol{w}_h))) + \sum_{e\in\mathcal{E}_{h,\Omega_F}^i}\int_e (\rho_F c^2)(\nabla\cdot\boldsymbol{p})[\![\boldsymbol{n}\cdot\boldsymbol{u}_h]\!]\,ds \\
&\quad + \sum_{e\in\mathcal{E}_h^{\Gamma_I}}\int_e (\rho_F c^2)(\nabla\cdot\boldsymbol{p})[\boldsymbol{n}\cdot(\boldsymbol{u}_h - \boldsymbol{w}_h)]\,ds.
\end{aligned} \tag{2.70}$$

Combining (2.69) and (2.70), we have

$$\begin{aligned}
|(\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h)|^2 &\approx b((\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h), (\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h)) \\
&= b((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h)) - b((\boldsymbol{u}_h, \boldsymbol{w}_h), (\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h)) \\
&= a_h((\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h), (\boldsymbol{p}, \boldsymbol{q})) - b((\boldsymbol{u}_h, \boldsymbol{w}_h), ((I - Q)(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{0})) \\
&\quad + \sum_{e\in\mathcal{E}_{h,\Omega_F}^i}\int_e (\rho_F c^2)(\nabla\cdot\boldsymbol{p})[\![\boldsymbol{n}\cdot\boldsymbol{u}_h]\!]\,ds + \sum_{e\in\mathcal{E}_h^{\Gamma_I}}\int_e (\rho_F c^2)(\nabla\cdot\boldsymbol{p})[\boldsymbol{n}\cdot(\boldsymbol{u}_h - \boldsymbol{w}_h)]\,ds.
\end{aligned} \tag{2.71}$$

We will estimate the four terms on the right-hand side of (2.71) separately.

We can rewrite the first term as

$$a_h((\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h), (\boldsymbol{p}, \boldsymbol{q})) = a_h((\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h), (\boldsymbol{p}, \boldsymbol{q}) - \mathbf{I}_h(\boldsymbol{p}, \boldsymbol{q})) + a_h((\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h), \mathbf{I}_h(\boldsymbol{p}, \boldsymbol{q})). \tag{2.72}$$

Then from (2.28) and Lemma 2.11 (applied to $(\boldsymbol{p},\boldsymbol{q})$) we immediately have

$$\begin{aligned} a_h((\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h),(\boldsymbol{p},\boldsymbol{q})-\mathbf{I}_h(\boldsymbol{p},\boldsymbol{q})) &\le C\|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)\|_h\|(\boldsymbol{p},\boldsymbol{q})-\mathbf{I}_h(\boldsymbol{p},\boldsymbol{q})\|_h \\ &\le Ch\|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)\|_h|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)|. \end{aligned} \tag{2.73}$$

We can rewrite the second term on the right-hand side of (2.72) as

$$\begin{aligned} a_h((\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h),\mathbf{I}_h(\boldsymbol{p},\boldsymbol{q})) &= b((Q\boldsymbol{f}-\boldsymbol{f},\mathbf{0}),\mathbf{I}_h(\boldsymbol{p},\boldsymbol{q})) + \sum_{e\in\mathcal{E}^i_{h,\Omega_F}}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{u})[\![\boldsymbol{n}\cdot(\Pi_{h,\Omega_F}\boldsymbol{p})]\!]\,ds \\ &\quad+\sum_{e\in\mathcal{E}^{\Gamma_I}_h}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{u})[\boldsymbol{n}\cdot(\widehat{\Pi_{h,\Omega_F}}\boldsymbol{p}-L_{h,\Omega_S}\boldsymbol{q})]\,ds. \end{aligned} \tag{2.74}$$

Since $\boldsymbol{p}\in H(\mathrm{curl}^0;\Omega_F)$, we have $Q\boldsymbol{p}=\boldsymbol{p}$. By Lemma 2.10 (applied to $\boldsymbol{p}$) and Lemma 2.11 (applied to $(\boldsymbol{p},\boldsymbol{q})$), we have

$$\begin{aligned} b((Q\boldsymbol{f}-\boldsymbol{f},\mathbf{0}),\mathbf{I}_h(\boldsymbol{p},\boldsymbol{q})) &= \rho_F(Q\boldsymbol{f}-\boldsymbol{f},\overline{\Pi_{h,\Omega_F}}\boldsymbol{p})_{\Omega_F} \\ &= \rho_F(\boldsymbol{f},Q(\overline{\Pi_{h,\Omega_F}}\boldsymbol{p})-\overline{\Pi_{h,\Omega_F}}\boldsymbol{p})_{\Omega_F} \\ &= \rho_F(\boldsymbol{f},Q(\overline{\Pi_{h,\Omega_F}}\boldsymbol{p}-\boldsymbol{p})-(\overline{\Pi_{h,\Omega_F}}\boldsymbol{p}-\boldsymbol{p}))_{\Omega_F} \\ &\le Ch\|\boldsymbol{f}\|_{L_2(\Omega_F)}\|(\boldsymbol{p},\boldsymbol{q})-\mathbf{I}_h(\boldsymbol{p},\boldsymbol{q})\|_h \\ &\le Ch^2|(\boldsymbol{f},\boldsymbol{g})||(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)|. \end{aligned} \tag{2.75}$$

Here

$$\overline{\Pi_{h,\Omega_F}}\boldsymbol{p}|_T = \begin{cases} (\Pi_{h,\Omega_F}\boldsymbol{p})|_T & \text{if } T\subset\Omega_F \text{ and } \partial T\cap\Gamma_I=\emptyset, \\ (\widehat{\Pi_{h,\Omega_F}}\boldsymbol{p})|_T & \text{if } T\subset\Omega_F \text{ and } \partial T\cap\Gamma_I\neq\emptyset. \end{cases}$$

We can rewrite the second term on the right-hand side of (2.74) using the notation introduced in (2.44) as

$$\begin{aligned} \sum_{e\in\mathcal{E}^i_{h,\Omega_F}}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{u})[\![\boldsymbol{n}\cdot(\Pi_{h,\Omega_F}\boldsymbol{p})]\!]\,ds &= \sum_{e\in\mathcal{E}^i_{h,\Omega_F}}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{u}-\overline{(\nabla\cdot\boldsymbol{u})_{T_e}})[\![\boldsymbol{n}\cdot(\Pi_{h,\Omega_F}\boldsymbol{p})]\!]\,ds \\ &= \sum_{e\in\mathcal{E}^i_{h,\Omega_F}}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{u}-\overline{(\nabla\cdot\boldsymbol{u})_{T_e}})[\![\boldsymbol{n}\cdot(\Pi_{h,\Omega_F}\boldsymbol{p}-\boldsymbol{p})]\!]\,ds, \end{aligned} \tag{2.76}$$

since $\boldsymbol{n}\cdot\boldsymbol{p}$ is continuous at the midpoints of any edge $e\in\mathcal{E}^i_{h,\Omega_F}$. It then follows from the Cauchy–Schwarz inequality, (2.11), Lemma 2.8 (applied to $(\boldsymbol{p},\boldsymbol{q})$) and Lemma 2.9 that

$$\begin{aligned} \sum_{e\in\mathcal{E}^i_{h,\Omega_F}}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{u})[\![\boldsymbol{n}\cdot(\Pi_{h,\Omega_F}\boldsymbol{p})]\!]\,ds &\le C\Big[\sum_{e\in\mathcal{E}^i_{h,\Omega_F}}|e|\|\nabla\cdot\boldsymbol{u}-\overline{(\nabla\cdot\boldsymbol{u})_{T_e}}\|^2_{L_2(e)}\Big]^{1/2} \\ &\quad\times\Big[\sum_{e\in\mathcal{E}^i_{h,\Omega_F}}|e|^{-1}\|[\![\boldsymbol{n}\cdot(\Pi_{h,\Omega_F}\boldsymbol{p}-\boldsymbol{p})]\!]\|^2_{L_2(e)}\Big]^{1/2} \\ &\le C(h|\nabla\cdot\boldsymbol{u}|_{H^1(\Omega_F)})(h|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)|) \\ &\le Ch^2|(\boldsymbol{f},\boldsymbol{g})||(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)|. \end{aligned} \tag{2.77}$$

Using the definition of $\widehat{\Pi_{h,\Omega_F}}$ and the fact that $\boldsymbol{n}\cdot\boldsymbol{p}=\boldsymbol{n}\cdot\boldsymbol{q}$ on Γ_I, we can rewrite the third term on the right-hand side of (2.74) as

$$\begin{aligned} &\sum_{e\in\mathcal{E}^{\Gamma_I}_h}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{u})[\boldsymbol{n}\cdot(\widehat{\Pi_{h,\Omega_F}}\boldsymbol{p}-L_{h,\Omega_S}\boldsymbol{q})]\,ds \\ &\quad= \sum_{e\in\mathcal{E}^{\Gamma_I}_h}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{u}-\overline{(\nabla\cdot\boldsymbol{u})_{T_e}})[\boldsymbol{n}\cdot(\widehat{\Pi_{h,\Omega_F}}\boldsymbol{p}-L_{h,\Omega_S}\boldsymbol{q})]\,ds \\ &\quad= \sum_{e\in\mathcal{E}^{\Gamma_I}_h}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{u}-\overline{(\nabla\cdot\boldsymbol{u})_{T_e}})[\boldsymbol{n}\cdot(\boldsymbol{p}-\widehat{\Pi_{h,\Omega_F}}\boldsymbol{p})-\boldsymbol{n}\cdot(\boldsymbol{q}-L_{h,\Omega_S}\boldsymbol{q})]\,ds. \end{aligned} \tag{2.78}$$

It follows from the Cauchy–Schwarz inequality, (2.11), Lemma 2.9 and similar arguments as in Lemma 2.11 that

$$\begin{aligned}\sum_{e\in\mathcal{E}_h^{\Gamma_I}}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{u})[\boldsymbol{n}\cdot(\widehat{\Pi_{h,\Omega_F}}\boldsymbol{p}-L_{h,\Omega_S}\boldsymbol{q})]\,ds &\le C\Big[\sum_{e\in\mathcal{E}_h^{\Gamma_I}}|e|\|\nabla\cdot\boldsymbol{u}-\overline{(\nabla\cdot\boldsymbol{u})_{T_e}}\|_{L_2(e)}^2\Big]^{1/2}\\ &\quad\times\Big[\sum_{e\in\mathcal{E}_h^{\Gamma_I}}|e|^{-1}\|\boldsymbol{n}\cdot(\boldsymbol{p}-\widehat{\Pi_{h,\Omega_F}}\boldsymbol{p})-\boldsymbol{n}\cdot(\boldsymbol{q}-L_{h,\Omega_S}\boldsymbol{q})\|_{L_2(e)}^2\Big]^{1/2}\\ &\le C(h|\nabla\cdot\boldsymbol{u}|_{H^1(\Omega_F)})(h|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)|)\\ &\le Ch^2|(\boldsymbol{f},\boldsymbol{g})||(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)|. \qquad (2.79)\end{aligned}$$

Combining (2.72)–(2.79), we obtain

$$a_h((\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h),(\boldsymbol{p},\boldsymbol{q}))\le C(h^2|(\boldsymbol{f},\boldsymbol{g})|+h\|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)\|_h)|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)|. \qquad (2.80)$$

Next we estimate the second term on the right-hand side of (2.71). By Lemma 2.10 and the fact that $(I-Q)\boldsymbol{u}=0$, we have

$$\begin{aligned}-b((\boldsymbol{u}_h,\boldsymbol{w}_h),((I-Q)(\boldsymbol{u}-\boldsymbol{u}_h),\boldsymbol{0})) &= b((\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w}-\boldsymbol{w}_h),\ ((I-Q)(\boldsymbol{u}-\boldsymbol{u}_h),\boldsymbol{0}))\\ &\le Ch|(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w}-\boldsymbol{w}_h)|\|(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w}-\boldsymbol{w}_h)\|_h. \qquad (2.81)\end{aligned}$$

We then estimate the third term on the right-hand side of (2.71). Since $\boldsymbol{n}\cdot\boldsymbol{u}_h$ is continuous at the midpoints of the interior edges and $[\![\boldsymbol{n}\cdot\boldsymbol{u}]\!]=0$, we get, using the notation introduced in (2.44),

$$\begin{aligned}\sum_{e\in\mathcal{E}_{h,\Omega_F}^i}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{p})[\![\boldsymbol{n}\cdot\boldsymbol{u}_h]\!]\,ds &= \sum_{e\in\mathcal{E}_{h,\Omega_F}^i}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{p}-\overline{(\nabla\cdot\boldsymbol{p})_{T_e}})[\![\boldsymbol{n}\cdot\boldsymbol{u}_h]\!]\,ds\\ &= \sum_{e\in\mathcal{E}_{h,\Omega_F}^i}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{p}-\overline{(\nabla\cdot\boldsymbol{p})_{T_e}})[\![\boldsymbol{n}\cdot(\boldsymbol{u}_h-\boldsymbol{u})]\!]\,ds. \qquad (2.82)\end{aligned}$$

Using the Cauchy–Schwarz inequality, (2.27), Lemma 2.9 (applied to $\boldsymbol{p}$) and (2.68), we can obtain that

$$\begin{aligned}\sum_{e\in\mathcal{E}_{h,\Omega_F}^i}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{p})[\![\boldsymbol{n}\cdot\boldsymbol{u}_h]\!]\,ds &\le C\Big[\sum_{e\in\mathcal{E}_{h,\Omega_F}^i}|e|\|\nabla\cdot\boldsymbol{p}-\overline{(\nabla\cdot\boldsymbol{p})_{T_e}}\|_{L_2(e)}^2\Big]^{1/2}\Big[\sum_{e\in\mathcal{E}_h^i}|e|^{-1}\|[\![\boldsymbol{u}_h-\boldsymbol{u}]\!]\|_{L_2(e)}^2\Big]^{1/2}\\ &\le Ch|\nabla\cdot\boldsymbol{p}|_{H^1(\Omega_F)}\|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)\|_h\\ &\le Ch|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)|\|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)\|_h. \qquad (2.83)\end{aligned}$$

Finally, we estimate the fourth term on the right-hand side of (2.71). Using the notation introduced in (2.44) and the fact that on any $e\in\Gamma_I$, $\boldsymbol{n}\cdot\boldsymbol{u}=\boldsymbol{n}\cdot\boldsymbol{w}$ and $\int_e\boldsymbol{n}\cdot(\boldsymbol{u}_h-\boldsymbol{w}_h)\,ds=0$, we have

$$\begin{aligned}\sum_{e\in\mathcal{E}_h^{\Gamma_I}}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{p})[\boldsymbol{n}\cdot(\boldsymbol{u}_h-\boldsymbol{w}_h)]\,ds &= \sum_{e\in\mathcal{E}_h^{\Gamma_I}}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{p}-\overline{(\nabla\cdot\boldsymbol{p})_{T_e}})[\boldsymbol{n}\cdot(\boldsymbol{u}_h-\boldsymbol{w}_h)]\,ds\\ &= \sum_{e\in\mathcal{E}_h^{\Gamma_I}}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{p}-\overline{(\nabla\cdot\boldsymbol{p})_{T_e}})[\boldsymbol{n}\cdot(\boldsymbol{u}_h-\boldsymbol{u})-\boldsymbol{n}\cdot(\boldsymbol{w}_h-\boldsymbol{w})]\,ds. \qquad (2.84)\end{aligned}$$

Using the Cauchy–Schwarz inequality, (2.27), Lemma 2.9 (applied to $\boldsymbol{p}$) and (2.68), we can obtain that

$$\begin{aligned}\sum_{e\in\mathcal{E}_h^{\Gamma_I}}\int_e(\rho_F c^2)(\nabla\cdot\boldsymbol{p})[\boldsymbol{n}\cdot(\boldsymbol{u}_h-\boldsymbol{w}_h)]\,ds &\le C\Big[\sum_{e\in\mathcal{E}_{h,\Omega_F}^{\Gamma_I}}|e|\|\nabla\cdot\boldsymbol{p}-\overline{(\nabla\cdot\boldsymbol{p})_{T_e}}\|_{L_2(e)}^2\Big]^{1/2}\\ &\quad\times\Big[\sum_{e\in\mathcal{E}_h^{\Gamma_I}}|e|^{-1}\|\boldsymbol{n}\cdot(\boldsymbol{u}_h-\boldsymbol{u})-\boldsymbol{n}\cdot(\boldsymbol{w}_h-\boldsymbol{w})\|_{L_2(e)}^2\Big]^{1/2}\\ &\le Ch|\nabla\cdot\boldsymbol{p}|_{H^1(\Omega_F)}\|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)\|_h\\ &\le Ch|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)|\|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)\|_h. \qquad (2.85)\end{aligned}$$

Estimate (2.64) follows from (2.71), (2.80), (2.81)–(2.85). □

Combining Theorem 2.13 and Theorem 2.14, we have the following corollary for the L_2-error estimate.

Corollary 2.15. *The following estimate holds for the solution* $(\boldsymbol{u}_h, \boldsymbol{w}_h) \in V_h$ *of* (2.18):

$$|(\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h)| \le Ch^2|(\boldsymbol{f}, \boldsymbol{g})|.$$

3 A Nonconforming Finite Element Method for the Eigenproblem

The nonconforming method studied in Section 2 can be applied to (1.1) as an eigensolver. In this section, we summarize the convergence property of the nonconforming eigensolver by using the results from Section 2.

We consider the following weak problem:

Problem. Find $\lambda \in \mathbb{R}$ and $(\boldsymbol{u}, \boldsymbol{w}) \in \mathring{\mathcal{V}}$ such that

$$a((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{v}, \boldsymbol{z})) = \lambda b((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{v}, \boldsymbol{z})) \quad \text{for all } (\boldsymbol{v}, \boldsymbol{z}) \in \mathring{\mathcal{V}}, \tag{3.1}$$

where $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are defined by (2.2) and (2.3).

Given $(\boldsymbol{f}, \boldsymbol{g}) \in [L_2(\Omega_F)]^2 \times [L_2(\Omega_S)]^2$, we define $T(\boldsymbol{f}, \boldsymbol{g}) \in \mathring{\mathcal{V}}$ by

$$a(T(\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{v}, \boldsymbol{z})) = b((\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{v}, \boldsymbol{z})) \quad \text{for all } (\boldsymbol{v}, \boldsymbol{z}) \in \mathring{\mathcal{V}}.$$

It is clear that $(\lambda, (\boldsymbol{u}, \boldsymbol{w}))$ is a solution of (3.1) if and only if $(\frac{1}{\lambda}, (\boldsymbol{u}, \boldsymbol{w}))$ is an eigenpair of T, i.e.,

$$T(\boldsymbol{u}, \boldsymbol{w}) = \frac{1}{\lambda}(\boldsymbol{u}, \boldsymbol{w}).$$

By Lemma 2.4, T is a bounded linear operator from $[L_2(\Omega_F)]^2 \times [L_2(\Omega_S)]^2$ to $H^{1+\gamma,1}(\mathrm{div}; \Omega_F) \times [H^{1+\gamma}(\Omega_S)]^2$ with $\gamma > \frac{1}{2}$. Therefore, the operator $T : [L_2(\Omega_F)]^2 \times [L_2(\Omega_S)]^2 \to [L_2(\Omega_F)]^2 \times [L_2(\Omega_S)]^2$ is symmetric, positive and compact. Hence the spectrum of T consists of a sequence of finite-multiplicity eigenvalues $\mu_n > 0$, $n \in \mathbb{N}$, converging to 0.

It was shown in [4] that λ is a positive eigenvalue of (1.1) if and only if $\mu = \frac{1}{\lambda}$ is a positive eigenvalue of the operator T, and the corresponding associated eigenfunctions coincide.

Next, we consider a nonconforming approximation of T. The nonconforming eigensolver for (3.1) is defined as follows:

Problem. Find $\lambda_h \in \mathbb{R}$ and $(\boldsymbol{u}_h, \boldsymbol{w}_h) \in V_h$ such that

$$a_h((\boldsymbol{u}_h, \boldsymbol{w}_h), (\boldsymbol{v}, \boldsymbol{z})) = \lambda_h b((\boldsymbol{u}_h, \boldsymbol{w}_h), (\boldsymbol{v}, \boldsymbol{z})) \quad \text{for all } (\boldsymbol{v}, \boldsymbol{z}) \in V_h, \tag{3.2}$$

where $a_h(\cdot, \cdot)$ is defined by (2.19).

The discrete analog of T is the operator $T_h : [L_2(\Omega_F)]^2 \times [L_2(\Omega_S)]^2 \to V_h \subset [L_2(\Omega_F)]^2 \times [L_2(\Omega_S)]^2$ defined by

$$a_h(T_h(\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{v}, \boldsymbol{z})) = b((\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{v}, \boldsymbol{z})) \quad \text{for all } (\boldsymbol{v}, \boldsymbol{z}) \in V_h.$$

In other words $T_h(\boldsymbol{f}, \boldsymbol{g}) \in V_h$ is the nonconforming finite element approximation of the solution $T(\boldsymbol{f}, \boldsymbol{g})$ of the source problem. Moreover,

$$T_h(\boldsymbol{u}, \boldsymbol{w}) = \frac{1}{\lambda_h}(\boldsymbol{u}, \boldsymbol{w})$$

is equivalent to (3.2).

The following discretization error estimates for the source problem have been derived in Corollary 2.15:

$$|(T - T_h)(\boldsymbol{f}, \boldsymbol{g})| \le Ch^2|(\boldsymbol{f}, \boldsymbol{g})| \tag{3.3}$$

for all $(\boldsymbol{f}, \boldsymbol{g}) \in [L_2(\Omega_F)]^2 \times [L_2(\Omega_S)]^2$. The following theorem can be obtained by applying the classical theory of spectral approximation [1, 14, 22] to the nonconforming eigensolver (3.2). The proof, which is based on estimate (3.3), is identical with that of [12, Theorem 3.1] for the Maxwell eigenvalues.

Theorem 3.1. *Let* $0 \le \lambda_1 \le \lambda_2 \le \cdots$ *be the eigenvalues of* (3.1), *and let* $\lambda = \lambda_j = \lambda_{j+1} = \lambda_{j+m-1}$ *be an eigenvalue with multiplicity* m. *Let* $0 \le \lambda_{h,1} \le \lambda_{h,2} \le \cdots$ *be the eigenvalues obtained by* (3.2). *Then as* $h \downarrow 0$, *we have*

$$|\lambda_{h,\ell} - \lambda| \le C_\lambda h^2 \quad \ell = j, j+1, \dots, j+m-1.$$

Furthermore, if V_λ *is the eigenspace for* λ *and* $V_{h,\lambda}$ *is the space spanned by the discrete eigenfunctions corresponding to* $\lambda_{h,1}, \dots, \lambda_{h,j+m-1}$, *then the gap between* $V_{h,\lambda}$ *and* V_λ *is* $O(h^2)$ *in the* L_2*-norm and* $O(h)$ *in the norm* $\|\cdot\|_h$.

4 Numerical Experiments

In this section we report the results of a series of numerical experiments that corroborate the theoretical results obtained in Section 2 and Section 3.

4.1 Numerical Results for the Source Problem

We first examine the convergence behavior of the numerical scheme (2.18) for the source problem (2.4) on graded meshes. The computational domain is depicted in Figure 3, where $\Omega_F = (1,3)^2$ and $\Omega_S = (0,4)^2 \setminus \overline{\Omega}_F$. We take ρ_S, ρ_F, c, μ_S and λ_S all to be 1 in the experiment.

Note that at the corner $c = (3,3)$ of Ω_F, $\omega_F = \frac{\pi}{4}$ and $\omega_S = \frac{3\pi}{4}$. The corresponding singularity index is $\gamma = 0.544483661651611$.

Let (r,θ) be the polar coordinates at the corner $c = (3,3)$ of the interface Γ_I. In view of (2.8) and (2.9), we take

$$\hat{\boldsymbol{w}}(r,\theta) = r^\gamma \begin{pmatrix} -A\cos((\gamma+1)\theta) + \cos((\gamma-1)\theta) \\ A\sin((\gamma+1)\theta) - \Theta\sin((\gamma-1)\theta) \end{pmatrix} \quad \text{and} \quad \tilde{\boldsymbol{u}}(r,\theta) = r^\gamma \begin{pmatrix} E\cos((\gamma+1)\theta) \\ -E\sin((\gamma+1)\theta) \end{pmatrix},$$

where

$$\begin{aligned} A &:= -\Big(\frac{(\gamma-1)(\mu+\lambda)}{(\mu+\lambda)\gamma - (3\mu+\lambda)}\Big)\frac{\sin((\gamma-1)\omega_S)}{\sin((\gamma+1)\omega_S)}, \\ E &:= \frac{2(2\mu+\lambda)}{(\mu+\lambda)\gamma - (3\mu+\lambda)}\frac{\sin((\gamma-1)\omega_S)}{\sin((\gamma+1)\omega_F)}, \\ \Theta &:= \frac{\frac{\mu}{\lambda+2\mu}(\gamma-1) - (\gamma+1)}{\frac{\mu}{\lambda+2\mu}(\gamma+1) - (\gamma-1)}. \end{aligned}$$

It can easily be checked that $\hat{\boldsymbol{w}}$ and $\tilde{\boldsymbol{u}}$ satisfy the compatibility conditions (2.5c) and (2.5d). Moreover, we have $\tilde{\boldsymbol{u}} = \nabla p$, where $p(r,\theta) = E\frac{r^{\gamma+1}}{\gamma+1}\cos(\gamma+1)\theta$.

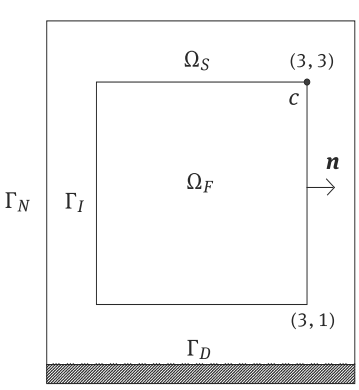

Figure 3: Domains of fluid and solid.

Define the cut-off function around the corner $c = (3, 3)$ by

$$\phi(r) = \begin{cases} 1, & r \le 0.25, \\ -16(r-0.75)^3[5 + 15(r-0.75) + 12(r-0.75)^2], & 0.25 \le r \le 0.75, \\ 0, & r \ge 0.75. \end{cases}$$

We take $\widetilde{\boldsymbol{w}} = \hat{\boldsymbol{w}}\phi(r)$ and $\boldsymbol{u} = \nabla(p(r,\theta)\phi(r))$. It is clear that $\boldsymbol{\sigma}(\widetilde{\boldsymbol{w}})\boldsymbol{\eta} = \boldsymbol{0}$ on Γ_N, $\widetilde{\boldsymbol{w}} = \boldsymbol{0}$ on Γ_D and $\nabla \times \boldsymbol{u} = 0$ in Ω_F. Since $[\nabla\phi(r)] \cdot \boldsymbol{n} = 0$, we still have $\widetilde{\boldsymbol{w}} \cdot \boldsymbol{n} = \boldsymbol{u} \cdot \boldsymbol{n}$ on Γ_I. However, $\boldsymbol{\sigma}(\widetilde{\boldsymbol{w}})\boldsymbol{n} \neq (\rho_F c^2 \nabla \cdot \boldsymbol{u})\boldsymbol{n}$ on the part of Γ_I where $0 < \phi(r) < 1$. To fix this problem, we introduce a vector field $\widetilde{\boldsymbol{z}}$ in Ω_S and define $\boldsymbol{w} = \widetilde{\boldsymbol{w}} + \widetilde{\boldsymbol{z}}$.

We take $\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{z}}_a + \tilde{\boldsymbol{z}}_b$, where $\tilde{\boldsymbol{z}}_a$ (respectively, $\tilde{\boldsymbol{z}}_b$) is associated with the edge Γ_a (respectively, Γ_b) that has $(3, 1)$ and $(3, 3)$ (respectively, $(1, 3)$ and $(3, 3)$) as endpoints. The vector field $\tilde{\boldsymbol{z}}_a$ is given by

$$\widetilde{\boldsymbol{z}}_a = \begin{pmatrix} \frac{v_1(x_2)(x_1-3)}{2\mu+\lambda} \\ \frac{v_2(x_2)(x_1-3)}{\mu} \end{pmatrix} \rho(x),$$

where

$$\boldsymbol{v}(x_2) = \begin{pmatrix} v_1(x_2) \\ v_2(x_2) \end{pmatrix} = (\rho_F c^2 \nabla \cdot \boldsymbol{u})\boldsymbol{n} - \boldsymbol{\sigma}(\widetilde{\boldsymbol{w}})\boldsymbol{n} \quad \text{on } \Gamma_a$$

and $\rho(x)$ is the following C^2-function defined on the interval $[3, 4]$:

$$\rho(x) = \begin{cases} -192x^5 + 3120x^4 - 20240x^3 + 65520x^2 - 105840x + 68257, & 3 \le x \le 3.5, \\ 0, & x \ge 3.5. \end{cases}$$

Note that $\rho(3) = 1$.

The vector field $\widetilde{\boldsymbol{z}}_b$ on Ω_S is defined by symmetry. Then $(\boldsymbol{u}, \boldsymbol{w})$ satisfies all the boundary conditions and compatibility conditions in (2.4) and we can take it to be an exact solution for the source problem (2.4).

We solve the source problem by the numerical scheme (2.18) on graded meshes, where the grading parameter $\mu = \gamma \approx 0.54448366$ at corners $(1, 1)$, $(1, 3)$, $(3, 1)$ and $(3, 3)$. The first three levels of triangulations are depicted in Figure 4. The errors in the L_2-norm $|(\cdot,\cdot)|$ on $[L_2(\Omega_F)]^2 \times [L_2(\Omega_S)]^2$ and the energy norm $\|(\cdot,\cdot)\|_h$ are tabulated in Table 1. The benefit of the graded meshes is observed.

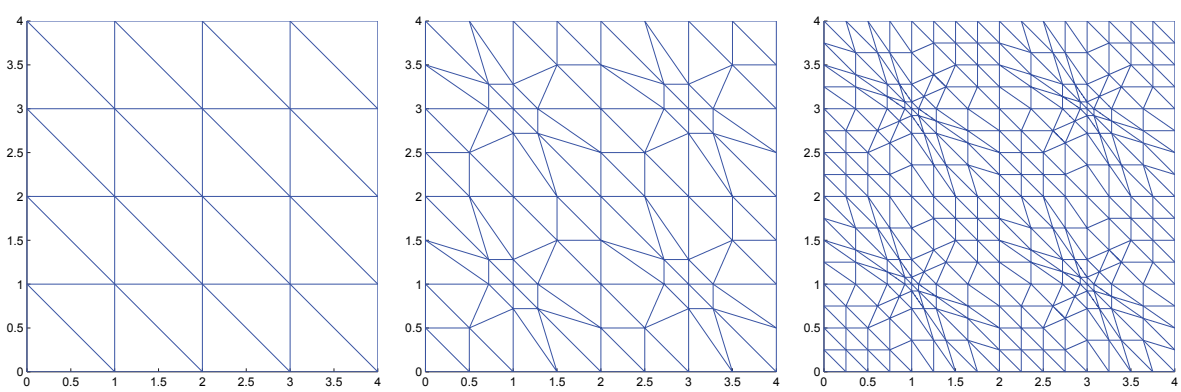

Figure 4: The first three levels of triangulations with graded meshes.

h	$\frac{\|(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{w}})-(\boldsymbol{u}_h,\boldsymbol{w}_h)\|_h}{\|(\boldsymbol{f},\boldsymbol{g})\|}$	Order	$\frac{\|(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{w}})-(\boldsymbol{u}_h,\boldsymbol{w}_h)\|}{\|(\boldsymbol{f},\boldsymbol{g})\|}$	Order
$\frac{1}{8}$	1.82 E−1	–	6.86 E−2	–
$\frac{1}{16}$	9.56 E−2	0.93	1.36 E−2	2.32
$\frac{1}{32}$	5.16 E−2	0.88	3.71 E−3	1.88
$\frac{1}{64}$	3.37 E−2	0.81	1.02 E−3	1.86

Table 1: Convergence of the scheme with graded meshes.

4.2 Numerical Results for the Eigenproblem

In this subsection we examine the convergence behavior of the numerical scheme (3.2) for the eigenproblem (3.1). The first three numerical experiments are conducted on the domain depicted in Figure 1, where $\Omega_F = (0.25, 1.25)^2$ and $\Omega_S = (0, 1.5)^2 \setminus \overline{\Omega}_F$. The same set-up was used in [6].

In the first numerical experiment, we consider the case where Ω_S is made with steel and Ω_F is filled with air. In this case, the physical parameters in (1.1) are taken to be $\rho_F = 1\,\mathrm{kg/m^3}$, $\rho_S = 7700\,\mathrm{kg/m^3}$, $c = 340\,\mathrm{m/s}$, $M = 1.44 \times 10^{11}$ Pa and $\nu = 0.35$. They are identical to the parameters in the third numerical experiment in [6]. We use uniform meshes and compute the eigenfrequencies by (3.2). The computed eigenvalues are reported in Table 2. Since we do not know the analytical eigenvalues, the order of convergence is computed in terms of the "exact" eigenvalues obtained by extrapolating the approximate solutions for $h = \frac{1}{128}$ and $\frac{1}{256}$. The results are very close to the ones reported in [6, Table 3]. Since the density of air is much smaller than the density of steel, this is a small perturbation of an uncoupled problem with a rigid cavity where the eigenfunctions are smooth (cf. [6]). Hence the order of convergence for eigenvalues is close to 2 with uniform meshes.

In the second experiment, we replace air with water, which corresponds to the choices of $\rho_F = 1000\,\mathrm{kg/m^3}$ and $c = 1430\,\mathrm{m/s}$. They are identical to the choices in the fourth experiment in [6]. The results obtained on uniform meshes are tabulated in Table 3, where the extrapolated eigenvalues obtained on graded meshes in Table 4 are used as "exact" solutions. In this case the order of magnitude of solid and fluid densities are similar so the interaction is much stronger. Therefore the order of convergence of the computed eigenvalues is lower than the case of a steel-air interaction in Table 2. Again they are very close to the results reported in [6, Table 4].

In the third experiment, we investigate the case of water in a steel cavity as in the second set of experiments, but with graded meshes. The meshes are graded around the corners $(0.25, 0.25)$, $(0.25, 1.25)$, $(1.25, 0.25)$, $(1.25, 1.25)$ with grading parameter $\mu = \gamma \approx 0.54448366$. The results are tabulated in Table 4, where the "exact" eigenvalues are obtained by extrapolating the approximate solutions for $h = \frac{1}{128}$ and $\frac{1}{256}$.

Mode	$h = \frac{1}{64}$	$h = \frac{1}{128}$	$h = \frac{1}{256}$	Orders			"Exact"
1	695.115	675.990	669.280	1.58	1.65	2.00	667.044
2	1067.768	1068.055	1068.118	2.07	2.13	2.00	1068.139
3	1068.188	1068.168	1068.162	1.88	1.91	2.00	1068.160
4	1509.982	1510.439	1510.549	2.11	2.04	2.00	1510.585
5	2134.651	2135.673	2135.911	2.13	2.07	2.00	2135.991
6	2134.843	2135.927	2136.186	2.08	2.03	2.00	2136.274
7	2350.401	2304.084	2290.394	0.44	1.82	2.00	2285.831
8	2384.996	2387.605	2388.235	1.52	2.04	2.00	2388.445
9	2387.387	2388.197	2388.388	1.52	2.07	2.00	2388.451

Table 2: Air in a rectangular steel cavity: the eigenvalues computed with uniform meshes.

Mode	$h = \frac{1}{64}$	$h = \frac{1}{128}$	$h = \frac{1}{256}$	Orders		"Exact"
1	671.958	653.327	646.805	1.46	1.38	642.733
2	2203.156	2159.534	2146.480	1.72	1.66	2140.444
3	3511.318	3439.433	3413.737	1.45	1.38	3397.769
4	3951.314	3900.298	3880.075	1.31	1.27	3865.819
5	4239.285	4219.354	4213.419	1.77	1.81	4211.056
6	4726.272	4705.872	4697.966	1.34	1.30	4692.539
7	5189.372	5165.067	5157.320	1.60	1.50	5153.047
8	5522.207	5449.036	5426.561	1.67	1.61	5415.586
9	6291.773	6275.419	6268.471	1.25	1.28	6263.591

Table 3: Water in a rectangular steel cavity: the eigenvalues computed with uniform meshes.

Mode	$h = \frac{1}{64}$	$h = \frac{1}{128}$	$h = \frac{1}{256}$	Orders		"Exact"	
1	664.078	648.593	644.198	1.74	1.86	2.00	642.733
2	2199.476	2156.226	2144.390	1.78	1.90	2.00	2140.444
3	3479.186	3420.264	3403.393	1.67	1.86	2.00	3397.769
4	3919.545	3880.839	3869.574	1.68	1.84	2.00	3865.819
5	4242.217	4219.668	4213.209	1.74	1.86	2.00	4211.056
6	4713.047	4698.381	4693.999	1.62	1.81	2.00	4692.539
7	5182.508	5160.915	5155.014	1.85	1.90	2.00	5153.047
8	5505.496	5439.868	5421.657	1.80	1.89	2.00	5415.586
9	6283.721	6269.711	6265.121	1.30	1.72	2.00	6263.591

Table 4: Water in a rectangular steel cavity: the eigenvalues computed with graded meshes.

The benefit of graded meshes is observed, but the asymptotic order of convergence in Theorem 3.1 has not yet been reached.

Remark 4.1. Since the extrapolated eigenvalues are computed by using the approximate solutions corresponding to $h = \frac{1}{128}$ and $\frac{1}{256}$, the last order of convergence in Tables 2 and 4 is exactly 2.

The set-up for the last two experiments is depicted in Figure 5, where Ω_F is an L-shaped domain with vertices $(0.25, 0.25)$, $(1.25, 0.25)$, $(0.75, 0.75)$, $(1.25, 0.75)$, $(0.25, 1.25)$, $(0.75, 1.25)$ and $\Omega_S = [0, 1.5]^2 \setminus \Omega_F$.

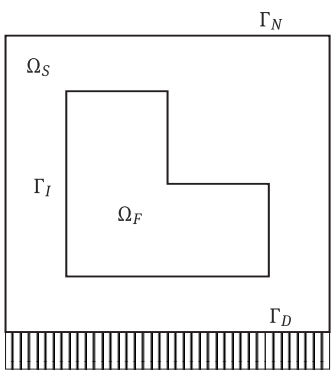

Figure 5: The domain of fluid-structure interaction. The outer dimensions are $1.5 \times 1.5\,\mathrm{m}^2$ and the inner domain is L-shaped.

In the fourth experiment the solid is made with steel and Ω_F is filled with water so that $\rho_F = 1000\,\mathrm{kg/m^3}$, $\rho_S = 7700\,\mathrm{kg/m^3}$, $c = 1430\,\mathrm{m/s}$, $M = 1.44 \times 10^{11}$ Pa and $\nu = 0.35$. The results obtained on uniform meshes are tabulated in Table 5. The extrapolated eigenvalues obtained on graded meshes in Table 6 are used as "exact" solutions in Table 5.

Mode	$h = \frac{1}{64}$	$h = \frac{1}{128}$	$h = \frac{1}{256}$	Orders		"Exact"
1	860.551	806.439	786.546	1.47	1.52	775.913
2	2284.585	2185.225	2152.371	1.62	1.66	2137.153
3	3816.121	3685.666	3639.322	1.50	1.52	3614.502
4	4125.524	3974.467	3924.566	1.63	1.61	3900.743
5	4811.590	4732.613	4707.209	1.70	1.66	4695.887
6	4533.367	5027.816	5264.461	1.19	1.36	5413.846
7	6899.529	6795.979	6744.889	1.13	1.28	6709.262
8	7657.389	7348.713	7241.233	1.52	1.51	7183.215
9	8283.759	8148.368	8098.794	1.45	1.47	8070.865

Table 5: Water in an L-shaped steel cavity: the eigenvalues computed with uniform meshes.

Mode	$h = \frac{1}{64}$	$h = \frac{1}{128}$	$h = \frac{1}{256}$	Orders		"Exact"	
1	796.096	781.464	777.301	1.73	1.86	2.00	775.913
2	2179.638	2147.821	2139.820	1.75	1.99	2.00	2137.153
3	3657.074	3623.638	3616.786	1.89	2.22	2.00	3614.502
4	3961.971	3917.148	3904.844	1.84	1.90	2.00	3900.743
5	4730.336	4705.406	4698.267	1.71	1.86	2.00	4695.887
6	5345.490	5396.282	5409.456	1.83	1.96	2.00	5413.846
7	5779.461	6447.883	6643.917	1.33	1.83	2.00	6709.262
8	7334.446	7222.159	7192.952	1.60	1.95	2.00	7183.215
9	8131.221	8086.949	8074.885	1.76	1.90	2.00	8070.865

Table 6: Water in an L-shaped steel cavity: the eigenvalues computed with graded meshes.

In the final experiment we replace the uniform meshes in the fourth experiment with meshes graded around the corners $(0.25, 0.25)$, $(0.25, 1.25)$, $(1.25, 0.25)$, $(0.75, 1.25)$, $(1.25, 0.75)$ with grading parameter $\mu = \gamma \approx 0.54448366$. The results are tabulated in Table 6. The benefit of graded meshes is again observed.

5 Conclusions

In this paper, we introduce a nonconforming finite element method for the acoustic fluid-structure interaction problem. The approximation spaces are weakly continuous P_1 vector fields for the fluid and standard piecewise linear polynomials for the solid. Optimal estimates in both the energy and the L_2-norms are obtained and validated by our numerical experiments. By excluding the pure rotational motions from the weak formulation of the problem, we guarantee that our discrete solution does not possess any spurious eigenvalues. Furthermore, this exclusion simplifies our analysis.

We note that it is also possible to use locally curl-free vector fields in the construction of the finite element space on the fluid side, in which case the penalty term for the curl of the fluid displacement in (2.19) is not needed. Such an approach was carried out for the Maxwell equations in [9, 10].

The approach in this paper can also be applied to the acoustic fluid-structure interaction problem in three dimensions. The convergence analysis in the case of quasi-uniform meshes can be carried out as in Sections 2.4 and 2.5. On the other hand the construction of graded meshes in three dimensions with the desired properties would be much more challenging.

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