A NONCONFORMING FINITE ELEMENT METHOD FOR A TWO-DIMENSIONAL CURL-CURL AND GRAD-DIV PROBLEM

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Abstract. A numerical method for a two-dimensional curl-curl and grad-div problem is studied in this paper. It is based on a discretization using weakly continuous P_1 vector fields and includes two consistency terms involving the jumps of the vector fields across element boundaries. Optimal convergence rates (up to an arbitrary positive ϵ) in both the energy norm and the L_2 norm are established on graded meshes. The theoretical results are confirmed by numerical experiments.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. In this paper we consider the following curl-curl and grad-div problem:

Find $\boldsymbol{u} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ such that

$$(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) + \gamma(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}) + \alpha(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \tag{1.1}$$

for all $\boldsymbol{v} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$, where $(\cdot,\cdot)$ denotes the inner product of $L_2(\Omega)$ (or $[L_2(\Omega)]^2$), $\alpha \in \mathbb{R}$ and $\gamma > 0$ are constants, $\boldsymbol{f} \in [L_2(\Omega)]^2$, and the spaces $H_0(\text{curl};\Omega)$ and $H(\text{div};\Omega)$ are defined as follows.

$$H(\text{curl};\Omega) = \Big\{ \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in [L_2(\Omega)]^2 : \nabla \times \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \in L_2(\Omega) \Big\},$$
$$H_0(\text{curl};\Omega) = \{ \boldsymbol{v} \in H(\text{curl};\Omega) : \boldsymbol{n} \times \boldsymbol{v} = 0 \text{ on } \partial\Omega \},$$

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where $\boldsymbol{n}$ is the unit outer normal, and

$$H(\mathrm{div};\Omega)=\Big\{\boldsymbol{v}=\begin{bmatrix}v_1\\ v_2\end{bmatrix}\in[L_2(\Omega)]^2:\ \nabla\cdot\boldsymbol{v}=\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}\in L_2(\Omega)\Big\}.$$

Note that $\boldsymbol{n}\times\boldsymbol{v}=0$ on $\partial\Omega$ is equivalent to $\boldsymbol{\tau}\cdot\boldsymbol{v}=0$ on $\partial\Omega$, where $\boldsymbol{\tau}$ is a unit tangent along $\partial\Omega$.

For positive α, the problem (1.1) is uniquely solvable by the Riesz representation theorem applied to the Hilbert space

$$X_N=H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)$$

with the inner product

$$(\boldsymbol{v},\boldsymbol{w})_{X_N}=(\nabla\times\boldsymbol{v},\nabla\times\boldsymbol{w})+(\nabla\cdot\boldsymbol{v},\nabla\cdot\boldsymbol{w})+(\boldsymbol{v},\boldsymbol{w}).$$

Since $H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)$ is compactly embedded in $[L_2(\Omega)]^2$ (cf. [41, 49, 23, 51, 42] and the discussion in Section 2 below), there exists a sequence of nonnegative numbers $0\leq\lambda_{\gamma,1}\leq\lambda_{\gamma,2}\leq\cdots\to\infty$ such that the following eigenproblem has a nontrivial solution $\boldsymbol{w}\in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)$:

$$(\nabla\times\boldsymbol{w},\nabla\times\boldsymbol{v})+\gamma(\nabla\cdot\boldsymbol{w},\nabla\cdot\boldsymbol{v})=\lambda_{\gamma,j}(\boldsymbol{w},\boldsymbol{v})\tag{1.2}$$

for all $\boldsymbol{v}\in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)$.

For $\alpha\leq 0$, the problem (1.1) is well-posed as long as $\alpha\neq-\lambda_{\gamma,j}$ for $j\geq 1$, which we assume to be the case throughout the paper. In particular, in the case where $\alpha=0$ and $\partial\Omega$ is connected, the problem (1.1) is uniquely solvable due to Friedrichs' inequality [42]:

$$\|\boldsymbol{v}\|_{L_2(\Omega)}\leq C\big(\|\nabla\times\boldsymbol{v}\|_{L_2(\Omega)}+\|\nabla\cdot\boldsymbol{v}\|_{L_2(\Omega)}\big)$$

for all $\boldsymbol{v}\in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)$.

When $\nabla\cdot\boldsymbol{f}=0$ and (1.1) is well-posed, the solution $\boldsymbol{u}$ of (1.1) belongs to the space $H(\mathrm{div}^0;\Omega)$ defined by

$$H(\mathrm{div}^0;\Omega)=\{\boldsymbol{v}\in H(\mathrm{div};\Omega):\ \nabla\cdot\boldsymbol{v}=0\},$$

and it is also a solution of the following curl-curl problem:
Find $\boldsymbol{u}\in H_0(\mathrm{curl};\Omega)$ such that

$$(\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v})+\alpha(\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v})\tag{1.3}$$

for all $\boldsymbol{v}\in H_0(\mathrm{curl};\Omega)$.

The curl-curl problem (1.3) appears in semi-discretizations of electric fields in the time-dependent (time-domain) Maxwell equations when $\alpha>0$ and the time-harmonic (frequency-domain) Maxwell equations when $\alpha\leq 0$. When $\alpha=0$, it is also related to electrostatic problems.

The numerical solution of (1.1) by finite element methods has an interesting history. It was realized early on [46, 29, 47, 50] that the solution of the non-elliptic curl-curl problem (1.3) can be obtained by solving the elliptic curl-curl and grad-div problem (1.1) in the case where $\nabla \cdot \boldsymbol{f} = 0$. Since it is difficult to construct finite element subspaces for $H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$, the problem (1.1) was discretized by H^1-conforming vector nodal finite elements [29]. However, the space $[H^1(\Omega)]^2 \cap X_N$ turns out to be a closed subspace of X_N [11, 24]. Therefore any H^1-conforming finite element method for (1.1) must fail if the solution $\boldsymbol{u}$ does not belong to $[H^1(\Omega)]^2$, which happens when Ω is non-convex [11, 6, 26]. Even worse, the solutions obtained by H^1-conforming finite element methods in such situations converge to the wrong solution (the projection of $\boldsymbol{u}$ in $[H^1(\Omega)]^2 \cap X_N$). Consequently the idea of solving (1.3) through (1.1) was abandoned. Instead, the curl-curl problem (1.3) is usually solved by $H(\text{curl})$-conforming edge elements [44, 45, 39, 42, 33, 14].

Nevertheless, the elliptic problem (1.1) remains an attractive alternative approach and successful schemes have been discovered in recent years that either solve (1.1) using nodal H^1 vector finite elements complemented by singular vector fields [13, 5, 38, 3, 4], or solve a regularized version of (1.1) using standard nodal H^1 vector finite elements [27, 28, 21].

In this paper we will show that (1.1) can also be solved by a nonconforming method using weakly continuous piecewise P_1 vector fields, where optimal convergence rates (up to an arbitrarily small $\epsilon > 0$) in both the energy norm and the L_2 norm can be achieved on general polygonal domains, provided that two consistency terms involving the jumps of the vector fields across element boundaries are included in the discretization and properly graded meshes are used. This is a continuation of our previous work in [17, 16], which considerably facilitates the analysis of the new method.

Note that, since we are working in two-dimensions, the problem (1.1) is equivalent to the following problem:

Find $\boldsymbol{u} \in H_0(\text{div};\Omega) \cap H(\text{curl};\Omega)$ such that

$$\gamma(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) + (\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}) + \alpha(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \tag{1.4}$$

for all $\boldsymbol{v} \in H_0(\text{div};\Omega) \cap H(\text{curl};\Omega)$, where

$$H_0(\text{div};\Omega) = \{\boldsymbol{v} \in H(\text{div};\Omega) : \boldsymbol{n} \cdot \boldsymbol{v} = 0 \quad \text{on} \quad \partial\Omega\}.$$

Therefore all the results in this paper (after straight-forward modifications) hold for the problem (1.4), which appears in problems involving magnetic fields and also problems in fluid-structure interaction [37, 10, 9, 12].

The rest of the paper is organized as follows. We discuss the elliptic regularity of (1.1) in Section 2 and introduce the nonconforming finite element method in Section 3. The convergence analysis is given in Section 4, followed by the results of numerical experiments in Section 5. We end the paper with some concluding remarks in Section 6.

2. Regularity of the Curl-Curl and Grad-Div Problem

The regularity of (1.1) is closely related to the regularity of the Laplace operator with homogeneous Dirichlet or Neumann boundary conditions. To explain this connection, we begin by reviewing the relation between the space $H_0(\text{curl};\Omega)\cap H(\text{div};\Omega)$ and Sobolev spaces. For simplicity, we first assume that Ω is simply connected.

Let $\boldsymbol{u}\in H_0(\text{curl};\Omega)\cap H(\text{div};\Omega)$. By a well-known Helmholtz decomposition [34, 42], we have an orthogonal decomposition for $\boldsymbol{u}$ in both $[L_2(\Omega)]^2$ and $H_0(\text{curl};\Omega)$:

$$\boldsymbol{u}=\mathring{\boldsymbol{u}}+\nabla\phi, \tag{2.1}$$

where $\mathring{\boldsymbol{u}}\in H_0(\text{curl};\Omega)\cap H(\text{div}^0;\Omega)$ and $\phi\in H_0^1(\Omega)$.

The function $\phi\in H_0^1(\Omega)$ in (2.1) is the variational solution of the following Dirichlet boundary value problem:

$$\Delta\phi=\nabla\cdot\boldsymbol{u}\quad\text{in }\Omega, \tag{2.2a}$$

$$\phi=0\quad\text{on }\partial\Omega. \tag{2.2b}$$

Since we assume Ω to be simply connected, there exists (cf. [34]) $\psi\in H^1(\Omega)$ such that

$$\nabla\times\psi=\mathring{\boldsymbol{u}}\quad\text{and}\quad\int_\Omega\psi\,dx=0,$$

and we can rewrite (2.1) as

$$\boldsymbol{u}=\nabla\times\psi+\nabla\phi. \tag{2.3}$$

Note that ψ is the unique variational solution with zero mean of the following Neumann boundary value problem:

$$\Delta\psi=-(\nabla\times\boldsymbol{u})\quad\text{in }\Omega, \tag{2.4a}$$

$$\frac{\partial\psi}{\partial n}=0\quad\text{on }\partial\Omega. \tag{2.4b}$$

Since the right-hand side of (2.2a) belongs to $L_2(\Omega)$, the elliptic regularity theory for polygonal domains [35, 32, 36, 43] provides a decomposition

$$\phi=\phi_R+\phi_S, \tag{2.5}$$

where the *regular* part $\phi_R \in H^2(\Omega)$ and the *singular* part ϕ_S is supported near the corners $c_1, \ldots, c_L$ of Ω. More precisely, we can choose a small positive number δ such that the neighborhoods

$$\mathcal{N}_{\ell,2\delta} = \{x \in \Omega : |x - c_\ell| < 2\delta\}$$

are disjoint. Then we can write

$$\phi_S = \sum_{\ell=1}^{L} \chi_\ell(r_\ell) \sum_{\substack{j\in\mathbb{N} \\ j(\pi/\omega_\ell)\in(0,1)}} \kappa_{\ell,j} r_\ell^{j(\pi/\omega_\ell)} \sin\big(j(\pi/\omega_\ell)\theta_\ell\big), \tag{2.6}$$

where (r_ℓ, θ_ℓ) are the polar coordinates at c_ℓ so that the two edges of Ω emanating from c_ℓ are defined by $\theta = 0$ and $\theta = \omega_\ell$, $\chi_\ell(t)$ is a smooth cut-off function that equals 1 for $t < 3\delta/2$ and vanishes for $t > 7\delta/4$, and $\kappa_{\ell,j}$ are constants. Furthermore, we have the following elliptic regularity estimate:

$$\|\phi_R\|_{H^2(\Omega)} + \sum_{\ell=1}^{L} \sum_{\substack{j\in\mathbb{N} \\ j(\pi/\omega_\ell)\in(0,1)}} |\kappa_{\ell,j}| \le C\|\nabla \cdot \boldsymbol{u}\|_{L_2(\Omega)}. \tag{2.7}$$

Here and below we use C with or without subscripts to denote a generic positive constant independent of h that can take different values at different appearances.

Similarly, since the right-hand side of (2.4a) belongs to $L_2(\Omega)$, we have the following decomposition for ψ:

$$\psi = \psi_R + \psi_S, \tag{2.8}$$

where $\psi_R \in H^2(\Omega)$, and

$$\psi_S = \sum_{\ell=1}^{L} \chi_\ell(r_\ell) \sum_{\substack{j\in\mathbb{N} \\ j(\pi/\omega_\ell)\in(0,1)}} \varrho_{\ell,j} r_\ell^{j(\pi/\omega_\ell)} \cos\big(j(\pi/\omega_\ell)\theta_\ell\big). \tag{2.9}$$

Furthermore, the following analog of (2.7) holds:

$$\|\psi_R\|_{H^2(\Omega)} + \sum_{\ell=1}^{L} \sum_{\substack{j\in\mathbb{N} \\ j(\pi/\omega_\ell)\in(0,1)}} |\varrho_{\ell,j}| \le C\|\nabla \times \boldsymbol{u}\|_{L_2(\Omega)}. \tag{2.10}$$

Combining (2.3) and (2.5)–(2.10), we have the following description of $\boldsymbol{u}$. First of all, $\boldsymbol{u} \in [H^1(\Omega_\delta)]^2$, where

$$\Omega_\delta = \{x \in \Omega : |x - c_\ell| > \delta \quad \text{for} \quad 1 \le \ell \le L\},$$

and the following estimate holds:

$$\|\boldsymbol{u}\|_{H^1(\Omega_\delta)} \le C\big(\|\nabla \cdot \boldsymbol{u}\|_{L_2(\Omega)} + \|\nabla \times \boldsymbol{u}\|_{L_2(\Omega)}\big). \tag{2.11}$$

Secondly, in the neighborhood $\mathcal{N}_{\ell,3\delta/2}$ of the corner c_ℓ, we have

$$\boldsymbol{u} = \boldsymbol{u}_R + \boldsymbol{u}_S, \tag{2.12}$$

where $\boldsymbol{u}_R \in [H^1(\mathcal{N}_{\ell,3\delta/2})]^2$,

$$\boldsymbol{u}_S = \sum_{\substack{j\in\mathbb{N} \\ j(\pi/\omega)\in(0,1)}} \nu_{\ell,j} r_\ell^{j(\pi/\omega_\ell)-1} \begin{bmatrix} \sin\big(j(\pi/\omega_\ell)-1\big)\theta_\ell \\ \cos\big(j(\pi/\omega_\ell)-1\big)\theta_\ell \end{bmatrix}, \tag{2.13}$$

and

$$\nu_{\ell,j} = j(\pi/\omega_\ell)(\kappa_{\ell,j} - \varrho_{\ell,j}). \tag{2.14}$$

Moreover, we have the following estimate:

$$\sum_{\ell=1}^{L} \Bigg(\|\boldsymbol{u}_R\|_{H^1(\mathcal{N}_{\ell,3\delta/2})} + \sum_{\substack{j\in\mathbb{N} \\ j(\pi/\omega)\in(0,1)}} |\nu_{\ell,j}| \Bigg) \leq C\big(\|\nabla\times\boldsymbol{u}\|_{L_2(\Omega)} + \|\nabla\cdot\boldsymbol{u}\|_{L_2(\Omega)}\big). \tag{2.15}$$

In particular, it follows from (2.11)–(2.13) and (2.15) that $\boldsymbol{u} \in [H^s(\Omega)]^2$ for any $s \in (1/2, 1]$ such that $s < \min_{1\leq\ell\leq L} \pi/\omega_\ell$, and

$$\|\boldsymbol{u}\|_{H^s(\Omega)} \leq C_s\big(\|\nabla\times\boldsymbol{u}\|_{L_2(\Omega)} + \|\nabla\cdot\boldsymbol{u}\|_{L_2(\Omega)}\big), \tag{2.16}$$

i.e., $H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$ can be embedded into $[H^s(\Omega)]^2$.

Now we turn to the regularity/singularity of the solution $\boldsymbol{u}$ of (1.1). For simplicity we first discuss the case where $\alpha > 0$. From (1.1) we immediately see that

$$\|\boldsymbol{u}\|_{L_2(\Omega)} \leq \alpha^{-1}\|\boldsymbol{f}\|_{L_2(\Omega)}, \tag{2.17}$$

$$\|\nabla\times\boldsymbol{u}\|^2_{L_2(\Omega)} + \gamma\|\nabla\cdot\boldsymbol{u}\|^2_{L_2(\Omega)} \leq \alpha^{-1}\|\boldsymbol{f}\|^2_{L_2(\Omega)}. \tag{2.18}$$

In view of (1.1), the divergence free part $\mathring{\boldsymbol{u}}$ in the Helmholtz decomposition (2.1) satisfies

$$(\nabla\times\mathring{\boldsymbol{u}}, \nabla\times\boldsymbol{v}) + \alpha(\mathring{\boldsymbol{u}}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \tag{2.19}$$

for all $\boldsymbol{v} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega)$, which implies

$$\nabla\times(\nabla\times\mathring{\boldsymbol{u}}) + \alpha\mathring{\boldsymbol{u}} = Q\boldsymbol{f}, \tag{2.20}$$

where Q is the orthogonal projection from $[L_2(\Omega)]^2$ onto $H(\mathrm{div}^0;\Omega)$. Indeed, let $\boldsymbol{\zeta} \in [C_0^\infty(\Omega)]^2$ be a test vector field. Then $\boldsymbol{\zeta} \in H_0(\mathrm{curl};\Omega)$ and $(\boldsymbol{\zeta} - Q\boldsymbol{\zeta}) \in \nabla H_0^1(\Omega) \subset H_0(\mathrm{curl};\Omega)$, which imply that $Q\boldsymbol{\zeta} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega)$. Hence it follows from (2.19) that

$$\begin{aligned}(\nabla\times\mathring{\boldsymbol{u}}, \nabla\times\boldsymbol{\zeta}) + \alpha(\mathring{\boldsymbol{u}}, \boldsymbol{\zeta}) &= (\nabla\times\mathring{\boldsymbol{u}}, \nabla\times[Q\boldsymbol{\zeta} + (\boldsymbol{\zeta} - Q\boldsymbol{\zeta})]) + \alpha(\mathring{\boldsymbol{u}}, Q\boldsymbol{\zeta}) \\ &= (\nabla\times\mathring{\boldsymbol{u}}, \nabla\times Q\boldsymbol{\zeta}) + \alpha(\mathring{\boldsymbol{u}}, Q\boldsymbol{\zeta}) = (\boldsymbol{f}, Q\boldsymbol{\zeta}) = (Q\boldsymbol{f}, \boldsymbol{\zeta}),\end{aligned}$$

which yields (2.20).

We deduce from (2.1), (2.17) and (2.20) that $\nabla\times\boldsymbol{u} = \nabla\times\mathring{\boldsymbol{u}} \in H^1(\Omega)$ and

$$|\nabla \times \boldsymbol{u}|_{H^1(\Omega)} = |\nabla \times \mathring{\boldsymbol{u}}|_{H^1(\Omega)} = \|Q\boldsymbol{f} - \alpha\mathring{\boldsymbol{u}}\|_{L_2(\Omega)} \leq 2\|\boldsymbol{f}\|_{L_2(\Omega)}, \tag{2.21}$$

which together with (1.1) implies that $\nabla \cdot \boldsymbol{u} \in H^1(\Omega)$ and

$$\begin{aligned} |\nabla \cdot \boldsymbol{u}|_{H^1(\Omega)} &\leq \gamma^{-1}\|\boldsymbol{f} - \alpha\boldsymbol{u} - \nabla \times (\nabla \times \boldsymbol{u})\|_{L_2(\Omega)} \\ &\leq 4\gamma^{-1}\|\boldsymbol{f}\|_{L_2(\Omega)}. \end{aligned} \tag{2.22}$$

In particular, it follows from the regularity of $\nabla \times \boldsymbol{u}$ and $\nabla \cdot \boldsymbol{u}$ and the usual variational argument that the boundary value problem corresponding to (1.1) is

$$\nabla \times (\nabla \times \boldsymbol{u}) - \gamma\nabla(\nabla \cdot \boldsymbol{u}) + \alpha\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.23a}$$

$$\boldsymbol{n} \times \boldsymbol{u} = 0 \quad \text{on } \partial\Omega, \tag{2.23b}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{on } \partial\Omega. \tag{2.23c}$$

The regularity/singularity of $\boldsymbol{u}$ can be derived through (2.2)–(2.4) and the elliptic regularity theory for polygonal domains. Since $\nabla\cdot\,\boldsymbol{u} \in H^1(\Omega)$, the regular part ϕ_R in (2.5) now belongs to $H^3(\Omega_\delta)$, and $\phi_R \in H^{3-\epsilon}(\mathcal{N}_{\ell,2\delta})$ for any $\epsilon > 0$ and $1 \leq \ell \leq L$. The singular part ϕ_S is now given by

$$\phi_S = \sum_{\ell=1}^{L} \chi_\ell(r_\ell) \sum_{\substack{j\in\mathbb{N} \\ j(\pi/\omega_\ell)\in(0,2)\setminus\{1\}}} \kappa_{\ell,j} r_\ell^{j(\pi/\omega_\ell)} \sin\big(j(\pi/\omega_\ell)\theta_\ell\big). \tag{2.24}$$

Furthermore, we have the following elliptic regularity estimates:

$$\begin{aligned} \|\phi_R\|_{H^3(\Omega_\delta)} &\leq C\|\nabla \cdot \boldsymbol{u}\|_{H^1(\Omega)} \\ &\leq C\gamma^{-1/2}(\gamma^{-1/2} + \alpha^{-1/2})\|\boldsymbol{f}\|_{L_2(\Omega)}, \end{aligned} \tag{2.25a}$$

$$\begin{aligned} \sum_{\ell=1}^{L} \|\phi_R\|_{H^{3-\epsilon}(\mathcal{N}_{\ell,2\delta})} &\leq C_\epsilon\|\nabla \cdot \boldsymbol{u}\|_{H^1(\Omega)} \\ &\leq C_\epsilon\gamma^{-1/2}(\gamma^{-1/2} + \alpha^{-1/2})\|\boldsymbol{f}\|_{L_2(\Omega)}, \end{aligned} \tag{2.25b}$$

$$\begin{aligned} \sum_{\ell=1}^{L} \sum_{\substack{j\in\mathbb{N} \\ j(\pi/\omega_\ell)\in(0,2)\setminus\{1\}}} |\kappa_{\ell,j}| &\leq C\|\nabla \cdot \boldsymbol{u}\|_{H^1(\Omega)} \\ &\leq C\gamma^{-1/2}(\gamma^{-1/2} + \alpha^{-1/2})\|\boldsymbol{f}\|_{L_2(\Omega)}, \end{aligned} \tag{2.25c}$$

where we have used the estimates (2.18) and (2.22).

Similarly, since $\nabla \times \boldsymbol{u} \in H^1(\Omega)$, the regular part ψ_R in (2.8) now belongs to $H^3(\Omega_\delta)$, and $\psi_R \in H^{3-\epsilon}(\mathcal{N}_{\ell,2\delta})$ for any $\epsilon > 0$ and $1 \leq \ell \leq L$. The singular part ψ_S is now given by

$$\psi_S = \sum_{\ell=1}^{L} \chi_\ell(r_\ell) \sum_{\substack{j\in\mathbb{N} \\ j(\pi/\omega_\ell)\in(0,2)\setminus\{1\}}} \varrho_{\ell,j} r_\ell^{j(\pi/\omega_\ell)} \cos\big(j(\pi/\omega_\ell)\theta_\ell\big). \tag{2.26}$$

Furthermore, the following analog of (2.25) holds:

$$\begin{aligned} \|\psi_R\|_{H^3(\Omega_\delta)} &\leq C\|\nabla \times \boldsymbol{u}\|_{H^1(\Omega)} \\ &\leq C(1+\alpha^{-1/2})\|\boldsymbol{f}\|_{L_2(\Omega)}, \end{aligned} \tag{2.27a}$$

$$\begin{aligned} \sum_{\ell=1}^{L} \|\psi_R\|_{H^{3-\epsilon}(\mathcal{N}_{\ell,2\delta})} &\leq C_\epsilon\|\nabla \times \boldsymbol{u}\|_{H^1(\Omega)} \\ &\leq C_\epsilon(1+\alpha^{-1/2})\|\boldsymbol{f}\|_{L_2(\Omega)}, \end{aligned} \tag{2.27b}$$

$$\begin{aligned} \sum_{\ell=1}^{L} \sum_{\substack{j\in\mathbb{N} \\ j(\pi/\omega_\ell)\in(0,2)\setminus\{1\}}} |\varrho_{\ell,j}| &\leq C\|\nabla \times \boldsymbol{u}\|_{H^1(\Omega)} \\ &\leq C(1+\alpha^{-1/2})\|\boldsymbol{f}\|_{L_2(\Omega)}, \end{aligned} \tag{2.27c}$$

where we have used the estimates (2.18) and (2.21).

Combining (2.3), (2.5), (2.8) and (2.24)–(2.27), we can describe the regularity/singularity of the solution $\boldsymbol{u}$ of (1.1) as follows. We have $\boldsymbol{u} \in [H^2(\Omega_\delta)]^2$ and the following estimate is valid:

$$\|\boldsymbol{u}\|_{H^2(\Omega_\delta)} \leq C[1+\gamma^{-1}+\alpha^{-1/2}(1+\gamma^{-1/2})]\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{2.28}$$

In the neighborhood $\mathcal{N}_{\ell,3\delta/2}$ of the corner c_ℓ, we have

$$\boldsymbol{u} = \boldsymbol{u}_R + \boldsymbol{u}_S, \tag{2.29}$$

where $\boldsymbol{u}_R \in [H^{2-\epsilon}(\mathcal{N}_{\ell,3\delta/2})]^2$ for any $\epsilon > 0$,

$$\boldsymbol{u}_S = \sum_{\substack{j\in\mathbb{N} \\ j(\pi/\omega_\ell)\in(0,2)\setminus\{1\}}} \nu_{\ell,j} r_\ell^{j(\pi/\omega_\ell)-1} \begin{bmatrix} \sin\big(j(\pi/\omega_\ell)-1\big)\theta_\ell \\ \cos\big(j(\pi/\omega_\ell)-1\big)\theta_\ell \end{bmatrix}, \tag{2.30}$$

and the constants $\nu_{\ell,j}$ are related to $\kappa_{\ell,j}$ and $\varrho_{\ell,j}$ by (2.14). Moreover, we have the following corner regularity estimates:

$$\begin{aligned} &\sum_{\ell=1}^{L} \|\boldsymbol{u}_R\|_{H^{2-\epsilon}(\mathcal{N}_{\ell,3\delta/2})} \\ &\qquad \leq C_\epsilon[1+\gamma^{-1}+\alpha^{-1/2}(1+\gamma^{-1/2})]\|\boldsymbol{f}\|_{L_2(\Omega)}, \end{aligned} \tag{2.31a}$$

$$\sum_{\ell=1}^{L} \sum_{\substack{j\in\mathbb{N} \\ j(\pi/\omega_\ell)\in(0,2)\setminus\{1\}}} |\nu_{\ell,j}| \\ \leq C[1+\gamma^{-1}+\alpha^{-1/2}(1+\gamma^{-1/2})]\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{2.31b}$$

Remark 2.1. Note that the description of the regularity/singularity of the solution of the reduced time-harmonic Maxwell equations given in [17] is the same as (2.28)–(2.31), only with all $\kappa_{\ell,j}$'s equal to 0.

We have derived the regularity/singularity of $\boldsymbol{u}$ under the assumption that Ω is simply connected. Since the regularity/singularity is a local behavior, the preceding results remain valid for general polygonal domains by a standard partition of unity argument.

For $\alpha \leq 0$, the problem (1.1) is well-posed as long as $\alpha \neq -\lambda_{\gamma,j}$, where $0 \leq \lambda_{\gamma,1} \leq \lambda_{\gamma,2} \leq \cdots \to \infty$ are the eigenvalues defined by (1.2), in which case we can replace (2.17) and (2.18) by

$$\|\nabla\times\boldsymbol{u}\|^2_{L_2(\Omega)} + \gamma\|\nabla\cdot\boldsymbol{u}\|_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} \leq C_\alpha\|\boldsymbol{f}\|^2_{L_2(\Omega)}. \tag{2.32}$$

Hence the results for $\alpha > 0$ remain valid for $\alpha \leq 0$ provided $\alpha \neq -\lambda_{\gamma,j}$ for $j \geq 1$, except that the dependence of the estimates on α is no longer explicit.

3. The Nonconforming Finite Element Method

Let $\mathcal{T}_h$ be a family of simplicial triangulations of Ω with mesh-parameter $h = \max_{T\in\mathcal{T}_h} h_T$, where h_T is the diameter of the triangle T. To recover optimal *a priori* error estimates in the presence of singularities, the triangulation $\mathcal{T}_h$ is graded around the corners $c_1,\ldots,c_L$ of Ω with the property that

$$C_1 h_T \leq h\Phi_\mu(T) \leq C_2 h_T, \tag{3.1}$$

where

$$\Phi_\mu(T) = \prod_{\ell=1}^{L} |c_\ell - c_T|^{1-\mu_\ell}. \tag{3.2}$$

Here c_T is the center of T and the positive constants C_1 and C_2 are independent of h.

The grading parameters $\mu_1,\ldots,\mu_L$ are chosen according to

$$\begin{aligned} &\mu_\ell = 1 && \text{if} \quad \omega_\ell \leq \frac{\pi}{2}, \\ &\mu_\ell < \frac{\pi}{2\omega_\ell} && \text{if} \quad \omega_\ell > \frac{\pi}{2}. \end{aligned} \tag{3.3}$$

In other words, grading is needed around any corner whose angle is larger than a right angle, which is different from the grading strategy for the Laplace operator, where grading is needed only around re-entrant corners. This is due to the fact that the singularity of (1.1) is one order more severe than the singularity of the Laplace operator (cf. (2.6), (2.9) and (2.30)).

The construction of $\mathcal{T}_h$ is described for example in [1, 2, 15, 8]. Note that $\mathcal{T}_h$ satisfies the minimum angle condition for any given grading parameters.

Let V_h be the space of weakly continuous P_1 vector fields associated with $\mathcal{T}_h$ whose tangential components vanish at the midpoints of the boundary edges in $\mathcal{T}_h$. More precisely, let $\mathcal{E}_h$ (resp. $\mathcal{E}_h^b$ and $\mathcal{E}_h^i$) be the set of the edges (resp. boundary edges and interior edges) of $\mathcal{T}_h$. Then

$$\begin{aligned} V_h = \{\boldsymbol{v} \in [L_2(\Omega)]^2 :\, & \boldsymbol{v}_T = \boldsymbol{v}\big|_T \in [P_1(T)]^2 \quad \forall\, T \in \mathcal{T}_h, \\ & \boldsymbol{v} \text{ is continuous at the midpoint of any } e \in \mathcal{E}_h, \\ & \boldsymbol{n} \times \boldsymbol{v} \text{ vanishes at the midpoint of any } e \in \mathcal{E}_h^b.\}. \end{aligned}$$

The advantage of using weakly continuous P_1 vector fields is due to the fact that it is easy to define, for any $s > 1/2$, a weak interpolation operator $\Pi_T : [H^s(T)]^2 \longrightarrow [P_1(T)]^2$ as follows:

$$(\Pi_T \boldsymbol{\zeta})(m_{e_j}) = \frac{1}{|e_j|} \int_{e_j} \boldsymbol{\zeta}\, ds \quad \text{for } 1 \le j \le 3,$$

where e_1, e_2 and e_3 are the edges of T, and m_e and $|e|$ denote the midpoint and length of the edge e. In view of the midpoint rule, we can also write

$$\int_{e_j} \Pi_T \boldsymbol{\zeta}\, ds = \int_{e_j} \boldsymbol{\zeta}\, ds \qquad \text{for } 1 \le j \le 3. \tag{3.4}$$

Furthermore, the operator Π_T satisfies a standard error estimate [30]:

$$\|\boldsymbol{\zeta} - \Pi_T \boldsymbol{\zeta}\|_{L_2(T)} + h_T^{\min(s,1)} |\boldsymbol{\zeta} - \Pi_T \boldsymbol{\zeta}|_{H^{\min(s,1)}(T)} \le C_T h_T^s |\boldsymbol{\zeta}|_{H^s(T)} \tag{3.5}$$

for all $\boldsymbol{\zeta} \in [H^s(T)]^2$ and $s \in (1/2, 2]$, where the positive constant C_T depends on the minimum angle of T (and also on s when s tends to 1/2).

Since $H_0(\text{curl};\Omega) \cap H(\text{div};\Omega) \subset [H^s(\Omega)]^2$ for some $s > 1/2$, we can define a global interpolation operator

$$\Pi_h : H_0(\text{curl};\Omega) \cap H(\text{div};\Omega) \longrightarrow V_h$$

by piecing together the local interpolation operators:

$$(\Pi_h \boldsymbol{v})_T = \Pi_T \boldsymbol{v}_T \qquad \forall\, T \in \mathcal{T}_h. \tag{3.6}$$

Let $\nabla_h\times$ and $\nabla_h\cdot$ be the piecewise curl and div operator defined by

$$(\nabla_h \times \boldsymbol{v})_T = \nabla \times (\boldsymbol{v}_T) \qquad \forall\, T \in \mathcal{T}_h, \tag{3.7}$$

$$(\nabla_h \cdot \boldsymbol{v})_T = \nabla \cdot (\boldsymbol{v}_T) \qquad \forall\, T \in \mathcal{T}_h. \tag{3.8}$$

Observe that (3.4) and Green's theorem imply, for any $\boldsymbol{v} \in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)$ and $T \in \mathcal{T}_h$,

$$\int_T \nabla \times (\Pi_T \boldsymbol{v})\, dx = \int_T \nabla \times \boldsymbol{v}\, dx,$$
$$\int_T \nabla \cdot (\Pi_T \boldsymbol{v})\, dx = \int_T \nabla \cdot \boldsymbol{v}\, dx,$$

which, in view of (3.6)–(3.8), means that

$$\nabla_h \times (\Pi_h \boldsymbol{v}) = \Pi_0^h(\nabla \times \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega), \tag{3.9}$$

$$\nabla_h \cdot (\Pi_h \boldsymbol{v}) = \Pi_0^h(\nabla \cdot \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega), \tag{3.10}$$

where Π_0^h is the orthogonal projection from $L_2(\Omega)$ onto the space of piecewise constant functions associated with $\mathcal{T}_h$. These commutative diagram relations indicate that we have good control over $\nabla_h \times (\Pi_h \boldsymbol{u})$ and $\nabla_h \cdot (\Pi_h \boldsymbol{u})$ simultaneously, which explains why weakly continuous P_1 vector fields can be used to solve problems involving the space $H(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)$.

Remark 3.1. Π_h is also the interpolation operator used in [17, 16].

Let $e \in \mathcal{E}_h^i$ be shared by the two triangles $T_{e,1}, T_{e,2} \in \mathcal{T}_h$ and $\boldsymbol{n}_1$ (resp. $\boldsymbol{n}_2$) be the unit normal of e pointing towards the outside of $T_{e,1}$ (resp. $T_{e,2}$). We define, on e,

$$[\![\boldsymbol{n} \times \boldsymbol{v}]\!] = \boldsymbol{n}_1 \times (\boldsymbol{v}_{T_{e,1}}\big|_e) + \boldsymbol{n}_2 \times (\boldsymbol{v}_{T_{e,2}}\big|_e), \tag{3.11a}$$

$$[\![\boldsymbol{n} \cdot \boldsymbol{v}]\!] = \boldsymbol{n}_1 \cdot (\boldsymbol{v}_{T_{e,1}}\big|_e) + \boldsymbol{n}_2 \cdot (\boldsymbol{v}_{T_{e,2}}\big|_e). \tag{3.11b}$$

For an edge $e \in \mathcal{E}_h^b$, we take $\boldsymbol{n}_e$ to be the unit normal of e pointing towards the outside of Ω and define

$$[\![\boldsymbol{n} \times \boldsymbol{v}]\!] = \boldsymbol{n}_e \times (\boldsymbol{v}\big|_e). \tag{3.12}$$

The nonconforming finite element method for (1.1) is:

Find $\boldsymbol{u}_h \in V_h$ such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in V_h, \tag{3.13}$$

where

$$
\begin{aligned}
a_h(\boldsymbol{w},\boldsymbol{v}) &= (\nabla_h \times \boldsymbol{w}, \nabla_h \times \boldsymbol{v}) + \gamma(\nabla_h \cdot \boldsymbol{w}, \nabla_h \cdot \boldsymbol{v}) + \alpha(\boldsymbol{w},\boldsymbol{v}) \\
&\quad + \sum_{e\in\mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \int_e [\![\boldsymbol{n}\times\boldsymbol{w}]\!]\,[\![\boldsymbol{n}\times\boldsymbol{v}]\!]\,ds \\
&\quad + \sum_{e\in\mathcal{E}_h^i} \frac{[\Phi_\mu(e)]^2}{|e|} \int_e [\![\boldsymbol{n}\cdot\boldsymbol{w}]\!][\![\boldsymbol{n}\cdot\boldsymbol{v}]\!]ds,
\end{aligned} \tag{3.14}
$$

and the edge weight $\Phi_\mu(e)$ is defined by

$$
\Phi_\mu(e) = \Pi_{\ell=1}^L |c_\ell - m_e|^{1-\mu_\ell}. \tag{3.15}
$$

Note that, by comparing (3.2) and (3.15), we have

$$
C_1\Phi_\mu(e) \le \Phi_\mu(T) \le C_2\Phi_\mu(e) \quad \text{if} \quad e \subset \partial T, \tag{3.16}
$$

where the positive constants C_1 and C_2 are independent of h. This relation is important for the derivation of optimal *a priori* error estimates.

Remark 3.2. The last two terms on the right-hand side of (3.14) involving the tangential and normal jumps of the weakly continuous P_1 vector fields are crucial for the convergence of the scheme. Unlike the nonconforming P_1 finite element method for the Stokes problem or the membrane problem, a naive discretization of (1.1) with only the first three terms does not converge (see the numerical results in Table 5.3 below). This is one of the reasons why classical nonconforming finite element methods have not been pursued in computational electromagnetics (see also the comments on page 200 of [42]). The crucial difference is that the piecewise $H(\text{curl})\cap H(\text{div})$ semi-norm, unlike the piecewise H^1 semi-norm, is too weak to control the jumps even with the weak continuity of the vector fields in V_h. Hence the two terms involving the jumps must be included in the discretization to control the consistency error.

4. Convergence Analysis

We will measure the discretization error in the L_2 norm and the mesh-dependent energy norm $\|\cdot\|_h$ defined by

$$
\begin{aligned}
\|\boldsymbol{v}\|_h^2 = &\|\nabla_h \times \boldsymbol{v}\|_{L_2(\Omega)}^2 + \gamma\|\nabla_h \cdot \boldsymbol{v}\|_{L_2(\Omega)}^2 + \|\boldsymbol{v}\|_{L_2(\Omega)}^2 \\
&+ \sum_{e\in\mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \|[\![\boldsymbol{n}\times\boldsymbol{v}]\!]\|_{L_2(e)}^2
\end{aligned} \tag{4.1}
$$

$$+\sum_{e\in\mathcal{E}_h^i}\frac{[\Phi_\mu(e)]^2}{|e|}\|[\![\boldsymbol{n}\cdot\boldsymbol{v}]\!]\|_{L_2(e)}^2.$$

Note that we have suppressed the dependence of the norm on γ to keep the notation simple.

Observe that $a_h(\cdot,\cdot)$ is bounded by the energy norm, i.e.,

$$|a_h(\boldsymbol{w},\boldsymbol{v})|\le(|\alpha|+1)\|\boldsymbol{w}\|_h\|\boldsymbol{v}\|_h \tag{4.2}$$

for all $\boldsymbol{v},\boldsymbol{w}\in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)+V_h$.

For $\alpha>0$, $a_h(\cdot,\cdot)$ is also coercive with respect to $\|\cdot\|_h$, i.e.,

$$a_h(\boldsymbol{v},\boldsymbol{v})\ge\min(1,\alpha)\|\boldsymbol{v}\|_h^2$$

for all $\boldsymbol{v}\in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)+V_h$. In this case the discrete problem is well-posed and we have the following abstract error estimate, whose proof is identical with the proof of Lemma 3.5 in [16].

Lemma 4.1. *Let α be positive, $\beta=\min(1,\alpha)$, $\boldsymbol{u}\in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)$ be the solution of* (1.1)*, and $\boldsymbol{u}_h$ satisfy the discrete problem* (3.13)*. It holds that*

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|_h\le\Big(\frac{1+\alpha+\beta}{\beta}\Big)&\inf_{\boldsymbol{v}\in V_h}\|\boldsymbol{u}-\boldsymbol{v}\|_h\\&+\frac{1}{\beta}\sup_{\boldsymbol{w}\in V_h\setminus\{\boldsymbol{0}\}}\frac{a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w})}{\|\boldsymbol{w}\|_h}.\end{aligned}\tag{4.3}$$

For $\alpha\le0$, we have a Gårding (in)equality:

$$a_h(\boldsymbol{v},\boldsymbol{v})+(|\alpha|+1)(\boldsymbol{v},\boldsymbol{v})=\|\boldsymbol{v}\|_h^2 \tag{4.4}$$

for all $\boldsymbol{v}\in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)+V_h$. In this case the discrete problem is indefinite and the following lemma provides an abstract error estimate for the scheme (3.13) under the assumption that it has a solution. Its proof, which is based on (4.2) and (4.4), is identical with the proof of Lemma 3.6 in [16].

Lemma 4.2. *Let $\alpha\le0$ and $\boldsymbol{u}\in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)$ satisfy* (1.1)*. Assume that the discrete problem* (3.13) *has a solution $\boldsymbol{u}_h$. Then we have*

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|_h\le(2|\alpha|+3)&\inf_{\boldsymbol{v}\in V_h}\|\boldsymbol{u}-\boldsymbol{v}\|_h\\&+\sup_{\boldsymbol{w}\in V_h\setminus\{\boldsymbol{0}\}}\frac{a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w})}{\|\boldsymbol{w}\|_h}+(|\alpha|+1)\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)}.\end{aligned}\tag{4.5}$$

From here on we consider α and γ to be fixed and drop the dependence on these constants in our estimates. We also assume in Lemmas 4.8 and 4.9 below that the discrete problem (3.13) has a solution $\boldsymbol{u}_h$ when $\alpha \leq 0$

Remark 4.3. The first term on the right-hand side of (4.3) and (4.5) measures the approximation property of V_h with respect to the energy norm. The second term measures the consistency error. The third term on the right-hand side of (4.5) addresses the indefiniteness of the problem when $\alpha \leq 0$.

Since the description of the regularity/singularity of the solution of the reduced time-harmonic Maxwell equations in [17] is identical with the description of the regularity/singularity of the solution $\boldsymbol{u}$ of (1.1) and the interpolation operator Π_h defined in Section 3 is also the one employed in [17], we can use in our analysis the following two results from that paper (cf. Lemma 5.1 and Lemma 5.2 of [17]), which were obtained using (2.28)–(2.31), (3.1), (3.3), (3.5), (3.15) and (3.16).

Lemma 4.4. *Let $\boldsymbol{u} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$ be the solution of* (1.1). *We have the following interpolation error estimate*:

$$\|\boldsymbol{u} - \Pi_h \boldsymbol{u}\|_{L_2(\Omega)} \leq C_\epsilon h^{2-\epsilon} \|\boldsymbol{f}\|_{L_2(\Omega)} \qquad \text{for any} \quad \epsilon > 0. \tag{4.6}$$

Lemma 4.5. *Let $\boldsymbol{u} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$ be the solution of* (1.1). *We have the following interpolation error estimate*:

$$\sum_{e \in \mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \|[\![\boldsymbol{u} - \Pi_h \boldsymbol{u}]\!]\|^2_{L_2(e)} \leq C_\epsilon h^{2-\epsilon} \|\boldsymbol{f}\|^2_{L_2(\Omega)} \tag{4.7}$$

for any $\epsilon > 0$, where $[\![\boldsymbol{u} - \Pi_h \boldsymbol{u}]\!]$ is the jump of $\boldsymbol{u} - \Pi_h \boldsymbol{u}$ across the interior edges of $\mathcal{T}_h$ and $[\![\boldsymbol{u} - \Pi_h \boldsymbol{u}]\!] = \boldsymbol{u} - \Pi_h \boldsymbol{u}$ on the boundary edges of $\mathcal{T}_h$.

The following result gives the approximation property of V_h.

Lemma 4.6. *Let $u \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$ be the solution of* (1.1). *For any $\epsilon > 0$ there exists a positive constant C_ϵ independent of h and $\boldsymbol{f}$ such that:*

$$\inf_{\boldsymbol{v} \in V_h} \|\boldsymbol{u} - \boldsymbol{v}\|_h \leq \|\boldsymbol{u} - \Pi_h \boldsymbol{u}\|_h \leq C_\epsilon h^{1-\epsilon} \|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{4.8}$$

Proof. According to (4.1), we have

$$\begin{aligned} \|\boldsymbol{u} - \Pi_h \boldsymbol{u}\|_h^2 = \|\nabla_h \times (\boldsymbol{u} - \Pi_h \boldsymbol{u})\|^2_{L_2(\Omega)} + \gamma \|\nabla_h \cdot (\boldsymbol{u} - \Pi_h \boldsymbol{u})\|^2_{L_2(\Omega)} \\ + \|\boldsymbol{u} - \Pi_h \boldsymbol{u}\|^2_{L_2(\Omega)} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{e\in\mathcal{E}_h}\frac{[\Phi_\mu(e)]^2}{|e|}\|[\![\boldsymbol{n}\times(\boldsymbol{u}-\Pi_h\boldsymbol{u})]\!]\|^2_{L_2(e)} \\
&\quad+\sum_{e\in\mathcal{E}_h^i}\frac{[\Phi_\mu(e)]^2}{|e|}\|[\![\boldsymbol{n}\cdot(\boldsymbol{u}-\Pi_h\boldsymbol{u})]\!]\|^2_{L_2(e)}.
\end{aligned}
\tag{4.9}
$$

The third term on the right-hand side of (4.9) has been estimated in Lemma 4.4, and the last two terms can be estimated using Lemma 4.5. Therefore it only remains to estimate the first two terms.

It follows from (2.21), (2.22), (3.9), (3.10) and a standard interpolation error estimate [22, 20] that

$$
\begin{aligned}
\|\nabla_h\times(\boldsymbol{u}-\Pi_h\boldsymbol{u})\|^2_{L_2(\Omega)} &= \|\nabla\times\boldsymbol{u}-\Pi_h^0(\nabla\times\boldsymbol{u})\|^2_{L_2(\Omega)} \\
&\le Ch^2|\nabla\times\boldsymbol{u}|^2_{H^1(\Omega)}\le Ch^2\|\boldsymbol{f}\|^2_{L_2(\Omega)},
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
\gamma\|\nabla_h\cdot(\boldsymbol{u}-\Pi_h\boldsymbol{u})\|^2_{L_2(\Omega)} &= \gamma\|\nabla\cdot\boldsymbol{u}-\Pi_h^0(\nabla\cdot\boldsymbol{u})\|^2_{L_2(\Omega)} \\
&\le Ch^2|\nabla\cdot\boldsymbol{u}|^2_{H^1(\Omega)}\le Ch^2\|\boldsymbol{f}\|^2_{L_2(\Omega)}.
\end{aligned}
\tag{4.11}
$$

The estimate (4.8) follows from (4.9)–(4.11) and Lemmas 4.4–4.5. □

Next we turn to the consistency error. The following lemma, which is identical with Lemma 5.3 in [17], is useful for estimating terms involving the jumps of the weakly continuous P_1 vector fields across edges.

Lemma 4.7. *It holds that*

$$
\sum_{e\in\mathcal{E}_h}|e|\,[\Phi_\mu(e)]^{-2}\|\eta-\bar{\eta}_{T_e}\|^2_{L_2(e)}\le Ch^2|\eta|^2_{H^1(\Omega)}\qquad\forall\,\eta\in H^1(\Omega),
$$

where $\bar{\eta}_{T_e}=\int_{T_e}\eta\,dx/|T_e|$ *is the mean of* η *over* T_e, *one of the triangles in* $\mathcal{T}_h$ *that has* e *as an edge.*

The following result gives an optimal bound for the consistency error.

Lemma 4.8. *Let* $\boldsymbol{u}\in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)$ *be the solution of* (1.1), *and* $\boldsymbol{u}_h\in V_h$ *satisfy* (3.13). *Then we have*

$$
\sup_{\boldsymbol{w}\in V_h\setminus\{0\}}\frac{a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w})}{\|\boldsymbol{w}\|_h}\le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}.
\tag{4.12}
$$

Proof. Let $\boldsymbol{w}\in V_h$ be arbitrary. Since the strong form of (1.1) is given by (2.23a), we have, by (3.11), (3.12), (3.14) and integration by parts,

$$
\begin{aligned}
a_h(\boldsymbol{u},\boldsymbol{w}) &= \sum_{T\in\mathcal{T}_h}\int_T(\nabla\times\boldsymbol{u})(\nabla\times\boldsymbol{w})dx \\
&\quad+\sum_{T\in\mathcal{T}_h}\gamma\int_T(\nabla\cdot\boldsymbol{u})(\nabla\cdot\boldsymbol{w})dx+\alpha(\boldsymbol{u},\boldsymbol{w}) \qquad (4.13)\\
&= (\boldsymbol{f},\boldsymbol{w})+\sum_{e\in\mathcal{E}_h}\int_e(\nabla\times\boldsymbol{u})[\![\boldsymbol{n}\times\boldsymbol{w}]\!]ds+\sum_{e\in\mathcal{E}_h^i}\gamma\int_e(\nabla\cdot\boldsymbol{u})[\![\boldsymbol{n}\cdot\boldsymbol{w}]\!]ds.
\end{aligned}
$$

Note that the last sum on the right-hand side of (4.13) involves only the interior edges because of (2.23c).

Subtracting (3.13) from (4.13), we find

$$
\begin{aligned}
a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w}) &= \sum_{e\in\mathcal{E}_h}\int_e(\nabla\times\boldsymbol{u})[\![\boldsymbol{n}\times\boldsymbol{w}]\!]ds \qquad (4.14)\\
&\quad+\sum_{e\in\mathcal{E}_h^i}\gamma\int_e(\nabla\cdot\boldsymbol{u})[\![\boldsymbol{n}\cdot\boldsymbol{w}]\!]ds.
\end{aligned}
$$

Since $\boldsymbol{n}\times\boldsymbol{w}$ is continuous at the midpoints of the interior edges and vanishes at the midpoints of the boundary edges, we can write, using the midpoint rule,

$$
\sum_{e\in\mathcal{E}_h}\int_e(\nabla\times\boldsymbol{u})[\![\boldsymbol{n}\times\boldsymbol{w}]\!]ds=\sum_{e\in\mathcal{E}_h}\int_e(\nabla\times\boldsymbol{u}-\overline{(\nabla\times\boldsymbol{u})}_{T_e})[\![\boldsymbol{n}\times\boldsymbol{w}]\!]ds \qquad (4.15)
$$

where $\overline{(\nabla\times\boldsymbol{u})}_{T_e}$ is the mean of $\nabla\times\boldsymbol{u}$ on T_e, one of the triangles in $\mathcal{T}_h$ that has e as an edge. It then follows from the Cauchy-Schwarz inequality, (2.21), (4.1) and Lemma 4.7 that

$$
\begin{aligned}
&\sum_{e\in\mathcal{E}_h}\int_e(\nabla\times\boldsymbol{u})[\![\boldsymbol{n}\times\boldsymbol{w}]\!]ds \\
&\quad\le\Big\{\sum_{e\in\mathcal{E}_h}|e|[\Phi_\mu(e)]^{-2}\|(\nabla\times\boldsymbol{u}-\overline{(\nabla\times\boldsymbol{u})}_{T_e})\|^2_{L_2(e)}\Big\}^{1/2} \qquad (4.16)\\
&\qquad\times\Big\{\sum_{e\in\mathcal{E}_h}\frac{[\Phi_\mu(e)]^2}{|e|}\|[\![\boldsymbol{n}\times\boldsymbol{w}]\!]\|^2_{L_2(e)}\Big\}^{1/2}\\
&\quad\le C\big(h|\nabla\times\boldsymbol{u}|_{H^1(\Omega)}\big)\|\boldsymbol{w}\|_h\le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}\|\boldsymbol{w}\|_h,
\end{aligned}
$$

and similarly,

$$
\sum_{e\in\mathcal{E}_h^i}\gamma\int_e(\nabla\cdot\boldsymbol{u})[\![\boldsymbol{n}\cdot\boldsymbol{w}]\!]ds\le Ch\|\boldsymbol{f}\|_{L_2(\Omega)}\|\boldsymbol{w}\|_h. \qquad (4.17)
$$

The estimate (4.12) follows from (4.14), (4.16) and (4.17). ☐

We now derive an L_2 error estimate by a duality argument.

Lemma 4.9. *Let $\boldsymbol{u} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$ be the solution of* (1.1) *and $\boldsymbol{u}_h \in V_h$ satisfy* (3.13). *Then we have*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L_2(\Omega)} \le C_\epsilon\big(h^{2-\epsilon}\|\boldsymbol{f}\|_{L_2(\Omega)} + h^{1-\epsilon}\|\boldsymbol{u} - \boldsymbol{u}_h\|_h\big) \tag{4.18}$$

for any $\epsilon > 0$.

Proof. Let $\boldsymbol{z} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$ satisfy

$$(\nabla \times \boldsymbol{v}, \nabla \times \boldsymbol{z}) + \gamma(\nabla \cdot \boldsymbol{v}, \nabla \cdot \boldsymbol{z}) + \alpha(\boldsymbol{v}, \boldsymbol{z}) = \big(\boldsymbol{v}, (\boldsymbol{u} - \boldsymbol{u}_h)\big) \tag{4.19}$$

for all $\boldsymbol{v} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$. Note that the strong form of (4.19) is

$$\nabla \times (\nabla \times \boldsymbol{z}) - \gamma\nabla(\nabla \cdot \boldsymbol{z}) + \alpha \boldsymbol{z} = \boldsymbol{u} - \boldsymbol{u}_h, \tag{4.20}$$

and we have the following analog of (2.21) and (2.22):

$$|\nabla \times \boldsymbol{z}|_{H^1(\Omega)} + |\nabla \cdot \boldsymbol{z}|_{H^1(\Omega)} \le C\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L_2(\Omega)}. \tag{4.21}$$

Furthermore we can write (4.19) as

$$a_h(\boldsymbol{v}, \boldsymbol{z}) = (\boldsymbol{v}, (\boldsymbol{u} - \boldsymbol{u}_h)) \qquad \forall\, \boldsymbol{v} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega). \tag{4.22}$$

It follows from (4.20), (4.22), and integration by parts that the following analog of (4.13) holds:

$$\begin{aligned} a_h(\boldsymbol{u}_h, \boldsymbol{z}) &= \sum_{T \in \mathcal{T}_h} \int_T (\nabla \times \boldsymbol{u}_h)(\nabla \times \boldsymbol{z})dx \\ &\qquad + \sum_{T \in \mathcal{T}_h} \gamma \int_T (\nabla \cdot \boldsymbol{u}_h)(\nabla \cdot \boldsymbol{z})dx + \alpha(\boldsymbol{u}_h, \boldsymbol{z}) \\ &= (\boldsymbol{u}_h, (\boldsymbol{u} - \boldsymbol{u}_h)) + \sum_{e \in \mathcal{E}_h} \int_e [\![\boldsymbol{n} \times \boldsymbol{u}_h]\!](\nabla \times \boldsymbol{z})ds \\ &\qquad + \sum_{e \in \mathcal{E}_h^i} \gamma \int_e [\![\boldsymbol{n} \cdot \boldsymbol{u}_h]\!](\nabla \cdot \boldsymbol{z})ds \end{aligned} \tag{4.23}$$

Combining (4.22) and (4.23), we find

$$
\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)}^2 &= (\boldsymbol{u},\boldsymbol{u}-\boldsymbol{u}_h)-(\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{u}_h)\\
&= a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{z})+\sum_{e\in\mathcal{E}_h}\int_e [\![\boldsymbol{n}\times\boldsymbol{u}_h]\!](\nabla\times\boldsymbol{z})ds \\
&\quad +\sum_{e\in\mathcal{E}_h^i}\gamma\int_e [\![\boldsymbol{n}\cdot\boldsymbol{u}_h]\!](\nabla\cdot\boldsymbol{z})ds,
\end{aligned}
\tag{4.24}
$$

and we will estimate the three terms on the right-hand side of (4.24) separately.

We can write the first term as

$$
a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{z}) = a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{z}-\Pi_h\boldsymbol{z})+a_h(\boldsymbol{u}-\boldsymbol{u}_h,\Pi_h\boldsymbol{z}). \tag{4.25}
$$

From (4.2) and Lemma 4.6 (applied to $\boldsymbol{z}$) we immediately have the following estimate:

$$
\begin{aligned}
a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{z}-\Pi_h\boldsymbol{z}) &\le C\|\boldsymbol{u}-\boldsymbol{u}_h\|_h\|\boldsymbol{z}-\Pi_h\boldsymbol{z}\|_h\\
&\le C_\epsilon h^{1-\epsilon}\|\boldsymbol{u}-\boldsymbol{u}_h\|_h\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)}.
\end{aligned}
\tag{4.26}
$$

Using (4.14) we can rewrite the second term on the right-hand side of (4.25) as

$$
\begin{aligned}
a_h(\boldsymbol{u}-\boldsymbol{u}_h,\Pi_h\boldsymbol{z}) &= \sum_{e\in\mathcal{E}_h}\int_e(\nabla\times\boldsymbol{u})[\![\boldsymbol{n}\times(\Pi_h\boldsymbol{z})]\!]ds\\
&\quad+\sum_{e\in\mathcal{E}_h^i}\int_e\gamma(\nabla\cdot\boldsymbol{u})[\![\boldsymbol{n}\cdot(\Pi_h\boldsymbol{z})]\!]ds.
\end{aligned}
\tag{4.27}
$$

Following the notation introduced in (4.15), the first term on the right-hand side of (4.27) can be written as

$$
\begin{aligned}
&\sum_{e\in\mathcal{E}_h}\int_e(\nabla\times\boldsymbol{u})[\![\boldsymbol{n}\times(\Pi_h\boldsymbol{z})]\!]ds\\
&\qquad=\sum_{e\in\mathcal{E}_h}\int_e(\nabla\times\boldsymbol{u}-\overline{(\nabla\times\boldsymbol{u})}_{T_e})[\![\boldsymbol{n}\times(\Pi_h\boldsymbol{z})]\!]ds\\
&\qquad=\sum_{e\in\mathcal{E}_h}\int_e(\nabla\times\boldsymbol{u}-\overline{(\nabla\times\boldsymbol{u})}_{T_e})[\![\boldsymbol{n}\times(\Pi_h\boldsymbol{z}-\boldsymbol{z})]\!]ds,
\end{aligned}
$$

since $\boldsymbol{n}\times(\Pi_h\boldsymbol{z})$ is continuous at the midpoint of any edge $e\in\mathcal{E}_h^i$ and vanishes at the midpoint of any edge $e\in\mathcal{E}_h^b$, and $[\![\boldsymbol{n}\times\boldsymbol{z}]\!]=0$. It then follows from the Cauchy-Schwarz inequality, (2.21), Lemma 4.5

(applied to $\boldsymbol{z}$) and Lemma 4.7 that

$$
\begin{aligned}
&\sum_{e\in\mathcal{E}_h}\int_e(\nabla\times\boldsymbol{u})[\![\boldsymbol{n}\times(\Pi_h\boldsymbol{z})]\!]ds\\
&\qquad\le\Big(\sum_{e\in\mathcal{E}_h}|e|[\Phi_\mu(e)]^{-2}\|(\nabla\times\boldsymbol{u}-\overline{(\nabla\times\boldsymbol{u})}_{T_e})\|^2_{L_2(e)}\Big)^{1/2}\\
&\qquad\qquad\times\Big(\sum_{e\in\mathcal{E}_h}\frac{[\Phi_\mu(e)]^2}{|e|}\|[\![\boldsymbol{n}\times(\Pi_h\boldsymbol{z}-\boldsymbol{z})]\!]\|^2_{L_2(e)}\Big)^{1/2}\\
&\qquad\le C_\epsilon\,(h|\nabla\times\boldsymbol{u}|_{H^1(\Omega)})(h^{1-\epsilon}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)})\\
&\qquad\le C_\epsilon h^{2-\epsilon}\|\boldsymbol{f}\|_{L_2(\Omega)}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)}.
\end{aligned}
\tag{4.28}
$$

Similarly, the second term on the right-hand side of (4.27) satisfies the following estimate:

$$
\sum_{e\in\mathcal{E}_h^i}\gamma\int_e(\nabla\cdot\boldsymbol{u})[\![\boldsymbol{n}\cdot(\Pi_h\boldsymbol{z})]\!]ds\le C_\epsilon h^{2-\epsilon}\|\boldsymbol{f}\|_{L_2(\Omega)}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)}.
\tag{4.29}
$$

Combining (4.25)–(4.29), we have

$$
\begin{aligned}
&a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{z})\\
&\qquad\le C_\epsilon\big(h^{2-\epsilon}\|\boldsymbol{f}\|_{L_2(\Omega)}+h^{1-\epsilon}\|\boldsymbol{u}-\boldsymbol{u}_h\|_h\big)\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)}.
\end{aligned}
\tag{4.30}
$$

We now consider the second term on the right-hand side of (4.24). Since $\boldsymbol{n}\times\boldsymbol{u}_h$ is continuous at the midpoints of interior edges and vanishes at the midpoints of boundary edges, and $[\![\boldsymbol{n}\times\boldsymbol{u}]\!]=0$, we can write, following the notation introduced in (4.15),

$$
\begin{aligned}
&\sum_{e\in\mathcal{E}_h}\int_e[\![\boldsymbol{n}\times\boldsymbol{u}_h]\!](\nabla\times\boldsymbol{z})ds\\
&\qquad=\sum_{e\in\mathcal{E}_h}\int_e[\![\boldsymbol{n}\times\boldsymbol{u}_h]\!](\nabla\times\boldsymbol{z}-\overline{(\nabla\times\boldsymbol{z})}_{T_e})ds\\
&\qquad=\sum_{e\in\mathcal{E}_h}\int_e[\![\boldsymbol{n}\times(\boldsymbol{u}_h-\boldsymbol{u})]\!](\nabla\times\boldsymbol{z}-\overline{(\nabla\times\boldsymbol{z})}_{T_e})ds.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, (4.1), (4.21) and Lemma 4.7, we obtain

$$
\begin{aligned}
&\sum_{e\in\mathcal{E}_h}\int_e[\![\boldsymbol{n}\times\boldsymbol{u}_h]\!](\nabla\times\boldsymbol{z})ds\\
&\qquad\le\Big(\sum_{e\in\mathcal{E}_h}|e|[\Phi_\mu(e)]^{-2}\|\nabla\times\boldsymbol{z}-\overline{(\nabla\times\boldsymbol{z})}_{T_e}\|^2_{L_2(e)}\Big)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big(\sum_{e\in\mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \|[\![\boldsymbol{n}\times(\boldsymbol{u}_h-\boldsymbol{u})]\!]\|_{L_2(e)}^2 \Big)^{1/2} \qquad (4.31)\\
&\leq C\big(h|\nabla\times\boldsymbol{z}|_{H^1(\Omega)}\big)\|\boldsymbol{u}-\boldsymbol{u}_h\|_h\\
&\leq Ch\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)}\|\boldsymbol{u}-\boldsymbol{u}_h\|_h.
\end{aligned}
$$

Similarly, we have the following bound on the third term on the right-hand side of (4.24):

$$
\sum_{e\in\mathcal{E}_h^i} \gamma \int_e [\![\boldsymbol{n}\cdot(\Pi_h\boldsymbol{u}_h)]\!](\nabla\cdot\boldsymbol{z})ds \leq Ch\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)}\|\boldsymbol{u}-\boldsymbol{u}_h\|_h. \qquad (4.32)
$$

The estimate (4.18) follows from (4.24) and (4.30)–(4.32). □

In the case where $\alpha > 0$, the following theorem is an immediate consequence of Lemma 4.1, Lemma 4.6, Lemma 4.8 and Lemma 4.9.

Theorem 4.10. *Let α be positive. The following discretization error estimates hold for the solution $\boldsymbol{u}_h$ of* (3.13):

$$
\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}_h\|_h &\leq C_\epsilon h^{1-\epsilon}\|\boldsymbol{f}\|_{L_2(\Omega)} &&\text{for any } \epsilon>0,\\
\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)} &\leq C_\epsilon h^{2-\epsilon}\|\boldsymbol{f}\|_{L_2(\Omega)} &&\text{for any } \epsilon>0.
\end{aligned}
$$

In the case where $\alpha \leq 0$, we have the following convergence theorem for the scheme (3.13). The proof, which is based on Lemma 4.2, Lemma 4.6, Lemma 4.8, Lemma 4.9 and the approach of Schatz for indefinite problems [48], is identical with the proof of Theorem 4.5 in [16].

Theorem 4.11. *Assume that $-\alpha \geq 0$ is not one of the eigenvalues $\lambda_{\gamma,j}$ defined by* (1.2). *There exists a positive number h_* such that the discrete problem* (3.13) *is uniquely solvable for all $h \leq h_*$, in which case the following discretization error estimates are valid*:

$$
\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}_h\|_h &\leq C_\epsilon h^{1-\epsilon}\|\boldsymbol{f}\|_{L_2(\Omega)} &&\text{for any}\quad \epsilon>0,\\
\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)} &\leq C_\epsilon h^{2-\epsilon}\|\boldsymbol{f}\|_{L_2(\Omega)} &&\text{for any}\quad \epsilon>0.
\end{aligned}
$$

5. Numerical Results

In this section we report the results of a series of numerical experiments that confirm our theoretical results. We take γ to be 1 in all the experiments. Besides the errors in the L_2 norm $\|\cdot\|_{L_2(\Omega)}$ and the energy norm $\|\cdot\|_h$, we also include the errors in the semi-norms $|\cdot|_{\mathrm{curl}}$ and $|\cdot|_{\mathrm{div}}$ defined by

$$
|\boldsymbol{v}|_{\mathrm{curl}} = \|\nabla_h\times\boldsymbol{v}\|_{L_2(\Omega)}, \qquad |\boldsymbol{v}|_{\mathrm{div}} = \|\nabla_h\cdot\boldsymbol{v}\|_{L_2(\Omega)}.
$$

In the first experiment we examine the convergence behavior of our numerical scheme on the square domain $(0, 0.5)^2$ with uniform meshes, where the exact solution $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \begin{bmatrix} \Big(\dfrac{x^3}{3} - \dfrac{x^2}{4}\Big)(y^2 - 0.5y)\sin(ky) \\ \Big(\dfrac{y^3}{3} - \dfrac{y^2}{4}\Big)(x^2 - 0.5x)\cos(kx) \end{bmatrix}. \tag{5.1}$$

The results are tabulated in Table 5.1 for $\alpha = k^2$ and $k = 0$, 1 and 10, and in Table 5.2 for $\alpha = -k^2$ and for $k = 1$ and 10. They show that the scheme (3.13) is second order accurate in the L_2 norm and first order accurate in the energy norm, which agrees with the error estimates in Theorem 4.10 and Theorem 4.11.

Table 5.1. Convergence of the scheme on the square $(0, 0.5)^2$ for $\alpha = k^2$, with uniform meshes and exact solution given by (5.1)

h	$\frac{\\|\boldsymbol{u}-\boldsymbol{u}_h\\|_{L_2(\Omega)}}{\\|\boldsymbol{u}\\|_{L_2(\Omega)}}$	order	$\frac{\\|\boldsymbol{u}-\boldsymbol{u}_h\\|_h}{\\|\boldsymbol{u}\\|_h}$	order	$\frac{\lvert\boldsymbol{u}-\boldsymbol{u}_h\rvert_{\mathrm{curl}}}{\lvert\boldsymbol{u}\rvert_{\mathrm{curl}}}$	order	$\frac{\lvert\boldsymbol{u}-\boldsymbol{u}_h\rvert_{\mathrm{div}}}{\lvert\boldsymbol{u}\rvert_{\mathrm{div}}}$	order
$k = 0$								
1/10	6.45E−02	–	3.46E−01	–	1.89E−01	–	2.13E−01	–
1/20	1.38E−02	2.23	1.70E−01	1.03	9.49E−02	0.99	1.08E−01	0.99
1/40	3.20E−03	2.11	8.37E−02	1.01	4.75E−02	1.00	5.40E−02	1.00
1/80	7.73E−04	2.05	4.17E−02	1.01	2.37E−02	1.00	2.70E−02	1.00
$k = 1$								
1/10	5.49E−02	–	3.23E−01	–	1.74E−01	–	2.13E−01	–
1/20	1.20E−02	2.19	1.59E−01	1.02	8.71E−02	0.99	1.07E−01	0.99
1/40	2.83E−03	2.09	7.92E−02	1.01	4.36E−02	1.00	5.35E−02	1.00
1/80	6.87E−04	2.04	3.94E−02	1.01	2.18E−02	1.00	2.67E−02	1.00
$k = 10$								
1/10	1.77E−01	–	6.54E−01	–	3.90E−01	–	4.08E−01	–
1/20	3.93E−02	2.17	3.37E−01	0.96	1.98E−01	0.98	1.99E−01	1.04
1/40	8.90E−03	2.14	1.67E−01	1.01	9.92E−02	1.00	9.81E−02	1.02
1/80	2.12E−03	2.07	8.34E−02	1.01	4.96E−02	1.00	4.89E−02	1.01

In the second experiment we check the behavior of the scheme without the consistency terms (cf. Remark 3.2). The results in Table 5.3 show that these terms are necessary for the convergence of the proposed scheme.

The goal of the third experiment is to demonstrate the convergence behavior of our scheme on the L-shaped domain $(-0.5, 0.5)^2 \setminus [0, 0.5]^2$. The exact solution is chosen to be

$$\boldsymbol{u} = \nabla \times \Big(r^{2/3}\cos\Big(\frac{2}{3}\theta - \frac{\pi}{3}\Big)\phi(r/0.5)\Big), \tag{5.2}$$

Table 5.2. Convergence of the scheme on the square $(0,0.5)^2$ for $\alpha = -k^2$, with uniform meshes and exact solution given by (5.1)

h	$\frac{\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)}}{\|\boldsymbol{u}\|_{L_2(\Omega)}}$	order	$\frac{\|\boldsymbol{u}-\boldsymbol{u}_h\|_h}{\|\boldsymbol{u}\|_h}$	order	$\frac{\vert\boldsymbol{u}-\boldsymbol{u}_h\vert_{\mathrm{curl}}}{\vert\boldsymbol{u}\vert_{\mathrm{curl}}}$	order	$\frac{\vert\boldsymbol{u}-\boldsymbol{u}_h\vert_{\mathrm{div}}}{\vert\boldsymbol{u}\vert_{\mathrm{div}}}$	order
$k=1$								
1/10	5.59E−02	–	3.24E−01	–	1.74E−01	–	2.13E−01	–
1/20	1.21E−02	2.20	1.59E−01	1.02	8.71E−02	0.99	1.07E−01	0.99
1/40	2.86E−03	2.09	7.92E−02	1.01	4.36E−02	1.00	5.35E−02	1.00
1/80	6.94E−04	2.04	3.94E−02	1.01	2.18E−02	1.00	2.67E−02	1.00
$k=10$								
1/10	4.42E−01	–	8.79E−01	–	4.10E−01	–	4.66E−01	–
1/20	5.94E−02	2.89	3.50E−01	1.33	1.99E−01	1.05	2.00E−01	1.22
1/40	1.26E−02	2.24	1.69E−01	1.05	9.92E−02	1.00	9.82E−02	1.03
1/80	2.96E−03	2.09	8.34E−02	1.02	4.96E−02	1.00	4.89E−02	1.01

Table 5.3. Errors of the scheme without the consistency terms on the square $(0,0.5)^2$, with uniform meshes and exact solution given by (5.1) with $k=1$

h	$\frac{\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)}}{\|\boldsymbol{u}\|_{L_2(\Omega)}}$	$\frac{\vert\boldsymbol{u}-\boldsymbol{u}_h\vert_{\mathrm{curl}}}{\vert\boldsymbol{u}\vert_{\mathrm{curl}}}$	$\frac{\vert\boldsymbol{u}-\boldsymbol{u}_h\vert_{\mathrm{div}}}{\vert\boldsymbol{u}\vert_{\mathrm{div}}}$
$\alpha=1$			
1/10	4.14E+01	4.57E−01	5.53E−01
1/20	4.18E+01	4.41E−01	5.48E−01
1/40	4.18E+01	4.36E−01	5.46E−01
1/80	4.18E+01	4.35E−01	5.46E−01
$\alpha=-1$			
1/10	4.14E+01	4.57E−01	5.53E−01
1/20	4.18E+01	4.41E−01	5.48E−01
1/40	4.18E+01	4.36E−01	5.46E−01
1/80	4.18E+01	4.35E−01	5.46E−01

where (r,θ) are the polar coordinates at the origin and the cut-off function is given by

$$\phi(r) = \begin{cases} 1 & r \le 0.25 \\ -16(r-0.75)^3 & \\ \quad \times\big[5+15(r-0.75)+12(r-0.75)^2\big] & 0.25 \le r \le 0.75 \\ 0 & r \ge 0.75 \end{cases}.$$

The meshes are graded around the re-entrant corner with the grading parameter 1/3. The results are tabulated in Table 5.4 and they agree with the error estimates for our scheme. That is, the scheme is second order accurate in the L_2 norm and first order accurate in the energy norm. Since the divergence of the exact solution is zero, the absolute errors instead of the relative errors in divergence are included in Table 5.4.

Table 5.4. Convergence of the scheme on the L-shaped domain $(-0.5, 0.5)^2 \backslash [0, 0.5]^2$, with graded meshes and the exact solution given by (5.2)

h	$\frac{\lVert \boldsymbol{u}-\boldsymbol{u}_h \rVert_{L_2(\Omega)}}{\lVert \boldsymbol{u} \rVert_{L_2(\Omega)}}$	order	$\frac{\lVert \boldsymbol{u}-\boldsymbol{u}_h \rVert_h}{\lVert \boldsymbol{u} \rVert_h}$	order	$\frac{\lvert \boldsymbol{u}-\boldsymbol{u}_h \rvert_{\mathrm{curl}}}{\lvert \boldsymbol{u} \rvert_{\mathrm{curl}}}$	order	$\lvert \boldsymbol{u}-\boldsymbol{u}_h \rvert_{\mathrm{div}}$	order
$\alpha = 0$								
1/4	9.93E+01	–	1.32E+01	–	4.93E+00	–	1.84E−02	–
1/8	3.24E+01	1.62	6.70E−00	0.97	3.59E−00	0.46	1.85E−02	–
1/16	3.29E−00	3.30	2.24E−00	1.58	7.79E−01	2.20	3.61E−03	2.36
1/32	6.91E−01	2.25	1.11E−00	1.01	4.33E−01	0.84	2.14E−03	0.76
1/64	1.71E−01	2.01	5.54E−01	1.00	2.34E−01	0.90	6.49E−04	1.72
$\alpha = 1$								
1/4	7.57E+01	–	1.01E+01	–	3.52E−00	–	1.65E−02	–
1/8	2.82E+01	1.43	6.07E−00	0.74	3.05E−00	0.20	1.80E−02	–
1/16	3.23E−00	3.13	2.21E−00	1.46	7.73E−01	1.98	3.59E−03	2.33
1/32	6.84E−01	2.23	1.10E−00	1.00	4.33E−01	0.84	2.13E−03	0.75
1/64	1.67E−01	2.04	5.54E−01	1.00	2.34E−01	0.89	6.47E−04	1.73
$\alpha = -1$								
1/4	1.46E+02	–	1.90E+01	–	7.77E−00	–	2.22E−02	–
1/8	3.85E+01	1.92	7.58E−00	1.32	4.40E−00	0.82	1.91E−02	0.22
1/16	3.37E−00	3.51	2.25E−00	1.75	7.87E−01	2.49	3.63E−03	2.39
1/32	6.99E−01	2.27	1.11E−00	1.03	4.34E−01	0.86	2.14E−03	0.76
1/64	1.77E−01	1.98	5.54E−01	1.00	2.34E−01	0.90	6.51E−04	1.72

6. Concluding Remarks

The results in this paper and [17, 16, 18] have firmly established the feasibility of using nonconforming finite element methods in computational electromagnetics.

We have only treated the source problem in this paper, but the scheme can also be applied to the eigenproblem (1.2). In fact, it follows from the L_2 error estimate in Theorem 4.10 that the classical theory of spectral approximation [40, 7] can be invoked to provide a straightforward convergence analysis for the solution of the eigenproblem by the scheme in this paper. Note that the eigenvalues defined by (1.2) are closely related to Maxwell eigenvalues [25], and by choosing γ large enough the scheme in this paper can be used to compute Maxwell eigenvalues. This will be further investigated in [19].

Since the problem (1.1) resembles a second order elliptic boundary value problem, many of the fast solvers developed for second order elliptic boundary value problems using nonconforming/interior penalty methods can be adopted for the scheme (3.13). The error analysis in this paper provides the foundation for the study of multigrid methods for Maxwell's equations using nonconforming finite elements, which will be carried out in [31].

References

[1] Th. Apel. *Anisotropic Finite Elements*. Teubner, Stuttgart, 1999.

[2] Th. Apel, A.-M. Sändig, and J.R. Whiteman. Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains. *Math. Methods Appl. Sci.*, 19:63–85, 1996.

[3] F. Assous, P. Ciarlet, Jr., S. Labrunie, and J. Segré. Numerical solution to the time-dependent Maxwell equations in axisymmetric singular domains: the singular complement method. *J. Comput. Phys.*, 191:147–176, 2003.

[4] F. Assous, P. Ciarlet Jr., E. Garcia, and J. Segré. Time-dependent Maxwell's equations with charges in singular geometries. *Comput. Methods Appl. Mech. Engrg.*, 196:665–681, 2006.

[5] F. Assous, P. Ciarlet, Jr., S. Labrunie, and S. Lohrengel. The singular complement method. In N. Debit, M. Garbey, R. Hoppe, D. Keyes, Y. Kuznetsov, and J. Périaux, editors, *Domain Decomposition Methods in Science and Engineering*, pages 161–189. CIMNE, Barcelona, 2002.

[6] F. Assous, P. Ciarlet, Jr., and E. Sonnendrücker. Resolution of the Maxwell equation in a domain with reentrant corners. *M2AN Math. Model. Numer. Anal.*, 32:359–389, 1998.

[7] I. Babuška and J. Osborn. Eigenvalue Problems. In P.G. Ciarlet and J.L. Lions, editors, *Handbook of Numerical Analysis II*, pages 641–787. North-Holland, Amsterdam, 1991.

[8] C. Băcuţă, V. Nistor, and L.T. Zikatanov. Improving the rate of convergence of 'high order finite elements' on polygons and domains with cusps. *Numer. Math.*, 100:165–184, 2005.

[9] K.J. Bathe, C. Nitikitpaiboon, and X. Wang. A mixed displacement-based finite element formulation for acoustic fluid-structure interaction. *Comput. Struct.*, 56:225–237, 1995.

[10] A. Bermúdez and R. Rodríguez. Finite element computation of the vibration modes of a fluid-solid system. *Comput. Methods Appl. Mech. Engrg.*, 119:355–370, 1994.

[11] M. Birman and M. Solomyak. L^2-theory of the Maxwell operator in arbitrary domains. *Russ. Math. Surv.*, 42:75–96, 1987.

[12] D. Boffi and L. Gastaldi. On the "$-$grad div $+\, s$ curl rot" operator. In *Computational fluid and solid mechanics, Vol. 1, 2* (*Cambridge, MA, 2001*), pages 1526–1529. Elsevier, Amsterdam, 2001.

[13] A.-S. Bonnet-Ben Dhia, C. Hazard, and S. Lohrengel. A singular field method for the solution of Maxwell's equations in polyhedral domains. *SIAM J. Appl. Math.*, 59:2028–2044, 1999.

[14] A. Bossavit. Discretization of electromagnetic problems: the "generalized finite differences" approach. In P.G. Ciarlet, W.H.A. Schilders, and E.J.W. Ter Maten, editors, *Handbook of numerical analysis. Vol. XIII*, Handb. Numer. Anal., XIII, pages 105–197. North-Holland, Amsterdam, 2005.

[15] S.C. Brenner and C. Carstensen. Finite Element Methods. In E. Stein, R. de Borst, and T.J.R. Hughes, editors, *Encyclopedia of Computational Mechanics*, pages 73–118. Wiley, Weinheim, 2004.

[16] S.C. Brenner, F. Li, and L.-Y. Sung. A locally divergence-free interior penalty method for two dimensional curl-curl problems. *SIAM J. Numer. Anal.*, 46:1190–1211, 2008.

[17] S.C. Brenner, F. Li, and L.-Y. Sung. A locally divergence-free nonconforming finite element method for the time-harmonic Maxwell equations. *Math. Comp.*, 76:573–595, 2007.

[18] S.C. Brenner, F. Li, and L.-Y. Sung. A nonconforming penalty method for a two dimensional curl-curl problem. *preprint*, 2007.

[19] S.C. Brenner, F. Li, and L.-Y. Sung. Parameter free nonconforming Maxwell eigensolvers without spurious eigenmodes. (in preparation).

[20] S.C. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods* (*Second Edition*). Springer-Verlag, New York-Berlin-Heidelberg, 2002.

[21] P. Ciarlet, Jr. Augmented formulations for solving Maxwell equations. *Comput. Methods Appl. Mech. Engrg.*, 194:559–586, 2005.

[22] P.G. Ciarlet. *The Finite Element Method for Elliptic Problems.* North-Holland, Amsterdam, 1978.

[23] M. Costabel. A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains. *Math. Methods Appl. Sci.*, 12:36–368, 1990.

[24] M. Costabel. A coercive bilinear form for Maxwell's equations. *J. Math. Anal. Appl.*, 157:527–541, 1991.

[25] M. Costabel and M. Dauge. Maxwell and Lamé eigenvalues on polyhedra. *Math. Methods Appl. Sci.*, 22:243–258, 1999.

[26] M. Costabel and M. Dauge. Singularities of electromagnetic fields in polyhedral domains. *Arch. Ration. Mech. Anal.*, 151:221–276, 2000.

[27] M. Costabel and M. Dauge. Weighted regularization of Maxwell equations in polyhedral domains. *Numer. Math.*, 93:239–277, 2002.

[28] M. Costabel, M. Dauge, and C. Schwab. Exponential convergence of hp-FEM for Maxwell equations with weighted regularization in polygonal domains. *Math. Models Methods Appl. Sci.*, 15:575–622, 2005.

[29] J.L. Coulomb. Finite element three dimensional magnetic field computation. *IEEE Trans. Magnetics*, 17:3241–3246, 1981.

[30] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations I. *RAIRO Anal. Numér.*, 7:33–75, 1973.

[31] J. Cui. *Nonconforming Multigrid Methods for Maxwell's Equations.* PhD thesis, Louisiana State University, (in preparation).

[32] M. Dauge. *Elliptic Boundary Value Problems on Corner Domains,* Lecture Notes in Mathematics 1341. Springer-Verlag, Berlin-Heidelberg, 1988.

[33] L. Demkowicz. Finite Element Methods for Maxwell Equations. In E. Stein, R. de Borst, and T.J.R. Hughes, editors, *Encyclopedia of Computational Mechanics*, pages 723–737. Wiley, Weinheim, 2004.

[34] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms.* Springer-Verlag, Berlin, 1986.

[35] P. Grisvard. *Elliptic Problems in Non Smooth Domains.* Pitman, Boston, 1985.

[36] P. Grisvard. *Singularities in Boundary Value Problems.* Masson, Paris, 1992.

[37] M.A. Hamdi, Y. Ousset, and G. Verchery. A displacement method for the analysis of vibrations of coupled fluid-structure systems. *Internat. J. Numer. Methods Engrg.*, 13:139–150, 1978.

[38] C. Hazard and S. Lohrengel. A singular field method for Maxwell's equations: Numerical aspects for 2D magnetostatics. *SIAM J. Numer. Anal.*, 40:1021–1040, 2002.

[39] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numer.*, 11:237–339, 2002.
[40] T. Kato. *Perturbation Theory of Linear Operators*. Springer-Verlag, Berlin, 1966.
[41] R. Leis. Zur Theorie elektromagnetischer Schwingungen in anisotopen inhomgenen Medien. *Math. Z.*, 106:213–224, 1968.
[42] P. Monk. *Finite Element Methods for Maxwell's Equations*. Oxford University Press, New York, 2003.
[43] S.A. Nazarov and B.A. Plamenevsky. *Elliptic Problems in Domains with Piecewise Smooth Boundaries*. de Gruyter, Berlin-New York, 1994.
[44] J.-C. Nédélec. Mixed finite elements in $\mathbf{R}^3$. *Numer. Math.*, 35:315–341, 1980.
[45] J.-C. Nédélec. A new family of mixed finite elements in $\mathbf{R}^3$. *Numer. Math.*, 50:57–81, 1986.
[46] P. Neittaanmäki and R. Picard. Error estimates for the finite element approximation to a Maxwell-type boundary value problem. *Numer. Funct. Anal. Optimiz.*, 2:267–285, 1980.
[47] B.M.A. Rahman and J.B. Davies. Finite element analysis of optical and microwave waveguide problems. *IEEE Trans. Microwave Theory Tech.*, 32:20–28, 1984.
[48] A. Schatz. An observation concerning Ritz-Galerkin methods with indefinite bilinear forms. *Math. Comp.*, 28:959–962, 1974.
[49] C. Weber. A local compactness theorem for Maxwell's equations. *Math. Meth. Appl. Sci.*, 2:12–25, 1980.
[50] T. Weiland. On the unique numerical solution of Maxwellian eigenvalue problems in three dimensions. *Part. Accel.*, 17:227–242, 1985.
[51] K.J. Witsch. A remark on a compactness result in electromagnetic theory. *Math. Meth. Appl. Sci,*, 16:123–129, 1993.

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