MULTIGRID METHODS FOR THE SYMMETRIC INTERIOR PENALTY METHOD ON GRADED MESHES

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ABSTRACT. The symmetric interior penalty (SIP) method on graded meshes and its fast solution by multigrid methods are studied in this paper. We obtain quasi-optimal error estimates in both the energy norm and the L_2 norm for the SIP method, and prove uniform convergence of the W-cycle multigrid algorithm for the resulting discrete problem. The performance of these methods is illustrated by numerical results.

1. Introduction

Interior penalty methods [7, 36, 2, 32] are prototypical discontinuous Galerkin methods [3] for elliptic boundary value problems. They are useful in handling hanging nodes, problems with constraints [8, 15], higher order problems [25, 18], and parameter dependent problems [29, 4, 37]. However, very little attention has been paid to interior penalty methods on graded meshes, which are needed for overcoming singularities due to nonsmooth boundary and/or abrupt change of boundary conditions [5, 6, 1].

The goal of this paper is two-fold. First we will investigate discretization errors of interior penalty methods on graded meshes, and secondly, we will study the convergence of multigrid methods for the resulting discrete problem. For simplicity, we will consider the model problem of finding $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall\, v \in H_0^1(\Omega), \tag{1.1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$ with reentrant corners and f belongs to the weighted Sobolev space $L_{2,\mu}(\Omega)$ (cf. (1.4) below). We will carry out the analysis of the symmetric interior penalty (SIP) method [36, 2] on graded meshes for (1.1).

The results of this paper can of course be extended to more complicated problems [8, 25, 18] where interior penalty methods have distinct advantages. Our multigrid analysis complements existing ones for interior penalty methods [26, 20, 35, 19, 23, 16], and it is

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also relevant for other nonconforming methods where graded meshes play a crucial role [15, 13, 14].

The rest of the paper is organized as follows. The discretization error estimates for the symmetric interior penalty method on graded meshes are established in Section 2. Descriptions of several multigrid algorithms are given in Section 3, followed by the convergence analysis of the W-cycle multigrid algorithm in Section 4. Numerical results that corroborate the theoretical results are reported in Section 5.

In the remaining part of this section we briefly recall the elliptic regularity results for (1.1). Let $\omega_1,\ldots,\omega_L$ be the interior angles at the corners $c_1,\ldots,c_L$ of the bounded polygonal domain Ω. Let the parameters $\mu_1,\ldots,\mu_\ell$ be chosen according to

$$\begin{cases} \mu_\ell = 1 & \omega_\ell < \pi \\ \dfrac{1}{2} < \mu_\ell < \dfrac{\pi}{\omega_\ell} & \omega_\ell > \pi \end{cases} \tag{1.2}$$

and the weight function ϕ_μ be defined by

$$\phi_\mu(x) = \prod_{\ell=1}^{L} |x - c_\ell|^{1-\mu_\ell}. \tag{1.3}$$

The weighted Sobolev space $L_{2,\mu}(\Omega)$ is defined by

$$L_{2,\mu}(\Omega) = \{f \in L_{2,\mathrm{loc}}(\Omega) : \|f\|^2_{L_{2,\mu}(\Omega)} = \int_\Omega \phi_\mu^2(x) f^2(x)\,dx < \infty\}. \tag{1.4}$$

Note that $L_2(\Omega) \subset L_{2,\mu}(\Omega)$ and

$$\|f\|_{L_{2,\mu}(\Omega)} \le C_\Omega \|f\|_{L_2(\Omega)} \qquad \forall\, f \in L_2(\Omega), \tag{1.5}$$

where C_Ω denotes a generic positive constant depending only on Ω.

Sobolev's inequality implies that

$$\int_\Omega |fv|\,dx \le C_\Omega \|f\|_{L_{2,\mu}(\Omega)} \|v\|_{H^1(\Omega)} \qquad \forall\, v \in H^1(\Omega). \tag{1.6}$$

Hence the model problem (1.1) has a unique solution u for any $f \in L_{2,\mu}(\Omega)$. Moreover u has the following properties.

(i) The second order weak derivatives of u belong to $L_{2,\mu}$ and they satisfy

$$\|\partial^2 u/\partial x_i \partial x_j\|^2_{L_{2,\mu}(\Omega)} = \int_\Omega \phi_\mu^2(x)(\partial^2 u/\partial x_i \partial x_j)^2(x)\,dx \le C_\Omega \|f\|^2_{L_{2,\mu}(\Omega)} \tag{1.7}$$

for $1 \le i, j \le 2$.

(ii) Let $\delta > 0$ be small enough so that the neighborhoods $\Omega_{\ell,\delta} = \{x \in \Omega : |x - c_\ell| < \delta\}$ around the corners c_ℓ for $1 \le \ell \le L$ are disjoint. At a reentrant corner c_ℓ where $\omega_\ell > \pi$, we have $u \in H^{1+\mu_\ell}(\Omega_{\ell,\delta})$ and

$$\|u\|_{H^{1+\mu_\ell}(\Omega_{\ell,\delta})} \le C_\Omega \|f\|_{L_{2,\mu}(\Omega)}. \tag{1.8}$$

(iii) u is continuous on $\bar{\Omega}$.

The regularity of u away from the corners follows from the standard elliptic regularity theory. The elliptic regularity of u near a corner c_ℓ can be obtained through the change of coordinates

$$(x_1, x_2) = e^t(\cos\theta, \sin\theta),$$

where the local Euclidean coordinates (x_1, x_2) centered at c_ℓ are chosen so that the two edges emanating from c_ℓ are represented by $\theta = 0$ and $\theta = \omega_\ell$.

Let $\hat{u}(t,\theta) = \psi(x)u(x)$, where ψ is a smooth cut-off function that equals 1 near 0. Then $U(t,\theta) = e^{-\mu_\ell t}\hat{u}(t,\theta) \in H^2(\mathbb{S})$, where $\mathbb{S}$ is the infinite strip $\mathbb{R} \times (0, \omega_\ell)$, and

$$\|U\|_{H^{1+\mu_\ell}(\mathbb{S})} \leq C_{\omega_\ell}\|U\|_{H^2(\mathbb{S})} \leq C_\Omega \|f\|_{L_{2,\mu}(\Omega)}. \tag{1.9}$$

The estimates (1.7) and (1.8) follow from (1.9) and a change of coordinates. The continuity of u away from the reentrant corners follows from the usual Sobolev inequality, while the continuity of u at a reentrant corner c_ℓ follows from the Sobolev inequality on $\mathbb{S}$ and a change of coordinates.

Details can be found in [30, 22, 31].

2. Analysis of the SIP Method on Graded Meshes

On a convex polygonal domain Ω, the solution u of (1.1) belongs to $H^2(\Omega)$ when $f \in L_{2,\mu}(\Omega)$ and the convergence of the SIP method using piecewise P_1 polynomials and quasi-uniform meshes is quasi-optimal [2, 32, 17]. This is no longer the case when Ω is nonconvex because $u \notin H^2(\Omega)$ in general [27, 22, 31].

To compensate for the lack of H^2 regularity in the presence of reentrant corners, we use a triangulation $\mathcal{T}_h$ of Ω with the following property:

$$C_1 h_T \leq \Phi_\mu(T)h \leq C_2 h_T \qquad \forall\, T \in \mathcal{T}_h, \tag{2.1}$$

where $h_T = \operatorname{diam} T$, $h = \max_{T\in\mathcal{T}_h} h_T$ is the mesh parameter, and $\Phi_\mu(T)$ is defined by

$$\Phi_\mu(T) = \prod_{\ell=1}^{L} |c_\ell - c_T|^{1-\mu_\ell}. \tag{2.2}$$

Here the grading parameters $\mu_1, \ldots, \mu_L$ are chosen according to (1.2) and c_T is the center of T. From here on we use C (with or without subscript) to denote a generic positive constant independent of the mesh parameter that can take different values at different occurrences, and we will denote the relation (2.1) by $h_T \approx \Phi_\mu(T)h$.

The construction of graded meshes that satisfy (2.1) can be found for example in [1, 11]. (See also the description at the beginning of Section 3 below.) Note that,

$$h_T \approx h^{1/\mu_\ell} \quad \text{if the corner } c_\ell \text{ is a vertex of } T \in \mathcal{T}_h, \tag{2.3}$$

and for a given set of grading parameters $\mu_1, \ldots, \mu_L$, the triangulation $\mathcal{T}_h$ satisfies the minimum angle condition.

Let V_h be the space of discontinuous P_1 finite element functions defined by

$$V_h = \{v \in L_2(\Omega) : v_T = v\big|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h\},$$

and denote by $\mathcal{E}_h$ the set of edges of $\mathcal{T}_h$.

For $f \in L_{2,\mu}(\Omega)$, the symmetric interior penalty (SIP) method [36, 2] for (1.1) is: Find $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_h, \tag{2.4}$$

where

$$a_h(w, v) = \sum_{T\in\mathcal{T}_h} \int_T \nabla w \cdot \nabla v\, dx - \sum_{e\in\mathcal{E}_h} \int_e \big(\{\{\nabla w\}\} \cdot [[v]] + \{\{\nabla v\}\} \cdot [[w]]\big) ds + \eta \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \int_e [[w]] \cdot [[v]]\, ds. \tag{2.5}$$

Here $\eta > 0$ is a penalty parameter, $|e|$ denotes the length of the edge e, and the mean $\{\{\nabla v\}\}$ and jump $[[v]]$ are defined as follows.

Let $e \in \mathcal{E}_h$ be an interior edge shared by two triangles $T_\pm \in \mathcal{T}_h$, $v_\pm = v\big|_{T_\pm}$, and $\boldsymbol{n}_\pm$ be the unit normals of e pointing towards the outside of $T_\pm$. We define, on e,

$$\{\{\nabla v\}\} = \frac{\nabla v_+ + \nabla v_-}{2} \qquad \text{and} \qquad [[v]] = v_+\boldsymbol{n}_+ + v_-\boldsymbol{n}_-.$$

Let $e \in \mathcal{E}_h$ be a boundary edge. Then $e \subset \partial T$ for a $T \in \mathcal{T}_h$. We define on e

$$\{\{\nabla v\}\} = \nabla v_T \quad \text{and} \quad [[v]] = v_T\boldsymbol{n},$$

where $v_T = v\big|_T$ and $\boldsymbol{n}$ is the unit normal of e pointing towards the outside of Ω.

Remark 2.1. Note that the right-hand side of (2.4) is well-defined because $\phi_\mu^{-1} \in L_2(\Omega)$ and $V_h \subset L_\infty(\Omega)$.

It is well-known that the SIP method is consistent in the sense that

$$a_h(u, v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_h, \tag{2.6}$$

where u is the solution of (1.1). From the Cauchy-Schwarz inequality we can see that the variational form $a_h(\cdot,\cdot)$ is bounded, namely,

$$a_h(w, v) \le \|w\|_h \|v\|_h \qquad \forall\, v, w \in H^s(\Omega) + V_h \tag{2.7}$$

for any $s > 3/2$, where

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^1(T)}^2 + \eta^{-1} \sum_{e\in\mathcal{E}_h} |e|\, \|\{\{\nabla v\}\}\|_{L_2(e)}^2 + 2\eta \sum_{e\in\mathcal{E}_h} |e|^{-1} \|[[v]]\|_{L_2(e)}^2. \tag{2.8}$$

The SIP method is also coercive on V_h if $\eta \geq \eta_* > 0$, where η_* is a constant depending only on the minimum angle of $\mathcal{T}_h$. Consequently, we have the quasi-optimal error estimate

$$\|u - u_h\|_h \leq C \inf_{v \in V_h} \|u - v\|_h, \tag{2.9}$$

where the constant C depends only on the minimum angle of $\mathcal{T}_h$ and the lower bound η_* for the penalty parameter.

Note that under the condition $\eta \geq \eta_*$ we have

$$\|v\|_h^2 \approx a_h(v, v) \qquad \forall\, v \in V_h, \tag{2.10}$$

and $a_h(\cdot, \cdot)$ is an inner product on V_h.

Details concerning (2.6)–(2.10) can be found for example in [17].

To turn the abstract error estimate (2.9) into a concrete estimate, we need an interpolation operator. Let $\Pi_h : C(\bar{\Omega}) \longrightarrow V_h$ be the nodal interpolation operator for the conforming P_1 finite element, i.e., $\Pi_h u \in V_h \cap H^1(\Omega)$ agrees with u at the vertices of the triangles of $\mathcal{T}_h$. The following lemma provides an interpolation error estimate for Π_h.

Lemma 2.2. *Let $f \in L_{2,\mu}(\Omega)$ and $u \in H_0^1(\Omega)$ satisfy (1.1). Then*

$$\|u - \Pi_h u\|_h \leq Ch\|f\|_{L_{2,\mu}(\Omega)}. \tag{2.11}$$

Proof. It follows from (2.8) and the definition of the mean of the gradient that

$$\begin{aligned} \|u - \Pi_h u\|_h^2 &= \sum_{T \in \mathcal{T}_h} |u - \Pi_h u|_{H^1(T)}^2 + \eta^{-1} \sum_{e \in \mathcal{E}_h} |e| \, \|\{\!\{\nabla(u - \Pi_h u)\}\!\}\|_{L_2(e)}^2 \\ &\leq C \sum_{T \in \mathcal{T}_h} \Big(|u - \Pi_h u|_{H^1(T)}^2 + |\partial T| \, \|\nabla(u - \Pi_h u)\|_{L_2(\partial T)}^2\Big). \end{aligned} \tag{2.12}$$

Let $\mathcal{T}_{h,\ell}$ be the collection of triangles in $\mathcal{T}_h$ that touch a corner c_ℓ of Ω. We can divide the triangles in $\mathcal{T}_h$ into two disjoint families $\mathcal{T}_h'$ and $\mathcal{T}_h''$ where

$$\mathcal{T}_h' = \bigcup_{\omega_\ell > \pi} \mathcal{T}_{h,\ell} \quad \text{and} \quad \mathcal{T}_h'' = \mathcal{T}_h \setminus \mathcal{T}_h'.$$

For the triangles away from the reentrant corners, we derive from (1.3), (1.7), (2.1), (2.2), a standard interpolation error estimate [21, 17], and the trace theorem with scaling that

$$\begin{aligned} \sum_{T \in \mathcal{T}_h''} \Big(|u - \Pi_h u|_{H^1(T)}^2 + |\partial T| \, \|\nabla(u - \Pi_h u)\|_{L_2(T)}^2\Big) &\leq C \sum_{T \in \mathcal{T}_h''} h_T^2 |u|_{H^2(T)}^2 \\ &\leq C \sum_{T \in \mathcal{T}_h''} h^2 [\Phi_\mu(T)]^2 \sum_{i,j=1}^{2} \|\partial^2 u / \partial x_i \partial x_j\|_{L_2(T)}^2 \\ &\leq Ch^2 \sum_{i,j=1}^{2} \sum_{T \in \mathcal{T}_h''} \|\phi_\mu^2 (\partial^2 u / \partial x_i \partial x_j)\|_{L_2(T)}^2 \leq Ch^2 \|f\|_{L_{2,\mu}(\Omega)}^2. \end{aligned} \tag{2.13}$$

For the triangles touching a reentrant corner, we can apply an interpolation error estimate for fractional order Sobolev spaces [24] together with (1.8), (2.3) and the trace theorem with scaling to obtain

$$\begin{aligned}\sum_{T\in\mathcal{T}_h'}\Big(|u-\Pi_h u|^2_{H^1(T)}+|\partial T|\,\|\nabla(u-\Pi_h u)\|^2_{L_2(\partial T)}\Big)&\leq C\sum_{\omega_\ell>\pi}\sum_{T\in\mathcal{T}_{h,\ell}}h_T^{2\mu_\ell}|u|^2_{H^{1+\mu_\ell}(T)}\\&\leq Ch^2\sum_{\omega_\ell>\pi}|u|^2_{H^{1+\mu_\ell}(\Omega_{\ell,\delta})}\leq Ch^2\|f\|^2_{L_{2,\mu}(\Omega)}.\end{aligned}\tag{2.14}$$

(Without loss of generality we may assume $h<\delta$.)

The estimate (2.11) follows from (2.12)–(2.14). □

Theorem 2.3. *Let $f\in L_{2,\mu}(\Omega)$, u be the solution of* (1.1), *and u_h be the solution of the SIP method associated with a triangulation $\mathcal{T}_h$ that satisfies* (2.1). *We have the following error estimate*:

$$\|u-u_h\|_{L_2(\Omega)}+h\|u-u_h\|_h\leq Ch^2\|f\|_{L_{2,\mu}(\Omega)}.\tag{2.15}$$

Proof. The estimate

$$\|u-u_h\|_h\leq Ch\|f\|_{L_{2,\mu}(\Omega)}$$

follows immediately from (2.9) and (2.11).

In view of (2.4) and (2.6), we have the following Galerkin orthogonality:

$$a_h(u-u_h,v)=0\qquad\forall\,v\in V_h.\tag{2.16}$$

The L_2 error estimate can then be established using a standard duality argument.

Let $\zeta\in H^1_0(\Omega)$ satisfy

$$\int_\Omega\nabla v\cdot\nabla\zeta\,dx=\int_\Omega v(u-u_h)\,dx\qquad\forall\,v\in H^1_0(\Omega).\tag{2.17}$$

It follows from elliptic regularity, (1.5) and Lemma 2.2 (applied to ζ) that

$$\|\zeta-\Pi_h\zeta\|_h\leq Ch\|u-u_h\|_{L_2(\Omega)}.\tag{2.18}$$

Note that we can rewrite (2.17) as

$$a_h(v,\zeta)=\int_\Omega v(u-u_h)\,dx\qquad\forall\,v\in H^1_0(\Omega),$$

and that the consistency of the SIP method implies

$$a_h(v,\zeta)=\int_\Omega v(u-u_h)\,dx\qquad\forall\,v\in V_h.$$

Hence we have, by (2.7), (2.16) and (2.18),

$$\begin{aligned}\|u-u_h\|^2_{L_2(\Omega)}&=\int_\Omega u(u-u_h)\,dx-\int_\Omega u_h(u-u_h)\,dx\\&=a_h(u,\zeta)-a_h(u_h,\zeta)\\&=a_h(u-u_h,\zeta-\Pi_h\zeta)\end{aligned}$$

$$\leq \|u-u_h\|_h \|\zeta - \Pi_h\zeta\|_h \leq Ch\|u-u_h\|_h \|u-u_h\|_{L_2(\Omega)},$$

which implies

$$\|u-u_h\|_{L_2(\Omega)} \leq Ch\|u-u_h\|_h \leq Ch^2\|f\|_{L_{2,\mu}(\Omega)}.$$

□

3. Multigrid Methods

Let $\mathcal{T}_0$ be an initial triangulation of Ω with the property that any triangle in $\mathcal{T}_0$ can have at most one vertex that is a reentrant corner. The triangulations $\mathcal{T}_k$ $(k \geq 1)$ are then created recursively as follows. Given $\mathcal{T}_k$, we divide each triangle $T \in \mathcal{T}_k$ into four triangles according to the following rules to obtain $\mathcal{T}_{k+1}$.

- If none of the reentrant corners is a vertex of T, then we divide T uniformly by connecting the midpoints of the edges of T.
- If a reentrant corner c_ℓ is a vertex of T and the other two vertices of T are denoted by p_1 and p_2, then we divide T by connecting the points m, g_1 and g_2 (cf. Figure 3.1). Here m is the midpoint of the edge p_1p_2 and g_1 (resp. g_2) is the point on the edge $c_\ell p_1$ (resp. $c_\ell p_2$) such that

$$\frac{|c_\ell - g_i|}{|c_\ell - p_i|} = 2^{-(1/\mu_\ell)} \qquad \text{for} \quad i = 1, 2,$$

where μ_ℓ is the grading factor chosen according to (1.2).

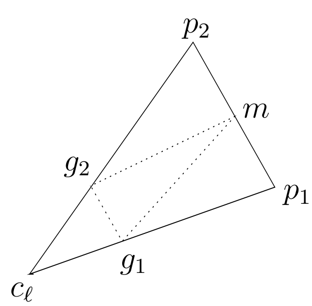

Figure 3.1. Refinement of a triangle at a reentrant corner

The triangulations $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ for an L-shaped domain are depicted in Figure 3.2, where the grading factor at the reentrant corner is taken to be 2/3.

It is easy to check that the nested triangulations $\mathcal{T}_k$ satisfy (2.1). We will denote $\max_{T\in\mathcal{T}_k} h_T$ by h_k. The mesh parameters on two consecutive levels are equivalent, i.e., there exists a positive constant C independent of k such that

$$h_k \leq h_{k-1} \leq Ch_k \qquad \text{for } k \geq 1. \tag{3.1}$$

Remark 3.1. The refinement procedure is identical with the one in [11].

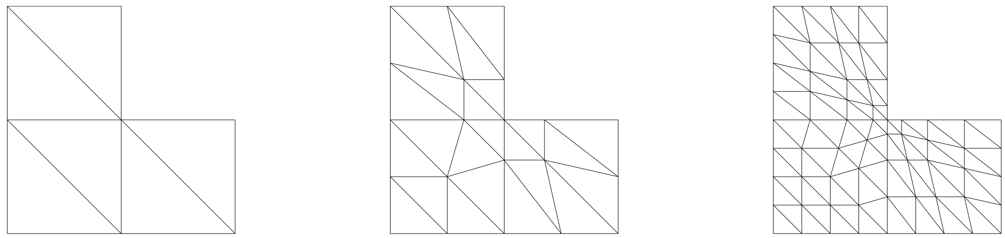

FIGURE 3.2. The triangulations $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ for an L-shaped domain

Let V_k be the discontinuous P_1 finite element space associated with $\mathcal{T}_k$. The k-th level SIP method for (1.1) is:
Find $u_k \in V_k$ such that

$$a_k(u_k, v) = \int_\Omega f v \, dx \qquad \forall\, v \in V_k, \tag{3.2}$$

where

$$\begin{aligned} a_k(w, v) = \sum_{T \in \mathcal{T}_k} \int_T \nabla w \cdot \nabla v \, dx - \sum_{e \in \mathcal{E}_k} \int_e \big(\{\!\{\nabla w\}\!\} \cdot [\![v]\!] + \{\!\{\nabla v\}\!\} \cdot [\![w]\!]\big) ds \\ + \eta \sum_{e \in \mathcal{E}_k} \frac{1}{|e|} \int_e [\![w]\!] \cdot [\![v]\!] \, ds, \end{aligned} \tag{3.3}$$

and $\mathcal{E}_k$ is the set of the edges of $\mathcal{T}_k$.

The analog of $\|\cdot\|_h$ is denoted by $\|\cdot\|_k$, i.e.,

$$\|v\|_k^2 = \sum_{T \in \mathcal{T}_k} |v|_{H^1(T)}^2 + \eta^{-1} \sum_{e \in \mathcal{E}_k} |e| \, \|\{\!\{\nabla v\}\!\}\|_{L_2(e)}^2 + 2\eta \sum_{e \in \mathcal{E}_k} |e|^{-1} \|[\![v]\!]\|_{L_2(e)}^2.$$

Note that (2.10) becomes

$$\|v\|_k^2 \approx a_k(v, v) \qquad \forall\, v \in V_k, \tag{3.4}$$

and (2.11) is translated into

$$\|u - \Pi_k u\|_k \le C h_k \|f\|_{L_{2,\mu}(\Omega)}, \tag{3.5}$$

where $\Pi_k : C(\bar{\Omega}) \longrightarrow V_k$ is the nodal interpolation operator for the Lagrange P_1 element, i.e., $\Pi_k u \in V_k \cap H_0^1(\Omega)$ agrees with u at the vertices of the triangles of $\mathcal{T}_k$. Furthermore, the norms $\|\cdot\|_k$ and $\|\cdot\|_{k-1}$ are equivalent for functions that are piecewise smooth on $\mathcal{T}_{k-1}$, i.e.,

$$\|w\|_k \approx \|w\|_{k-1} \qquad \forall\, w \in H^s(\Omega) + V_{k-1}, \tag{3.6}$$

where $s > 3/2$.

We can rewrite (3.2) as

$$A_k u_k = f_k \tag{3.7}$$

where $A_k : V_k \longrightarrow V_k'$ and $f_k \in V_k'$ are defined by

$$\langle A_k w, v\rangle = a_k(w,v) \qquad \forall\, v, w \in V_k, \tag{3.8}$$
$$\langle f_k, v\rangle = \int_\Omega f v\, dx \qquad \forall\, v \in V_k.$$

Here $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $V_k' \times V_k$. Equations of the form (3.7) can be solved by multigrid algorithms [28, 33, 10, 34, 17].

There are two key ingredients in the design of a multigrid algorithm. We need intergrid transfer operators to move functions between grids and a good smoother to damp out the highly oscillatory part of the error. Since the finite element spaces are nested, we can take the coarse-to-fine intergrid transfer operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$ to be the natural injection and define the fine-to-coarse intergrid transfer operator $I_k^{k-1} : V_k' \longrightarrow V_{k-1}'$ to be the transpose of I_{k-1}^k with respect to the canonical bilinear forms, i.e.,

$$\langle I_k^{k-1}\alpha, v\rangle = \langle \alpha, I_{k-1}^k v\rangle \qquad \forall\, \alpha \in V_k',\ v \in V_{k-1}. \tag{3.9}$$

In order to define the smoother, we first introduce an operator $B_k : V_k \longrightarrow V_k'$ defined by

$$\langle B_k w, v\rangle = \sum_{T\in\mathcal{T}_k}\sum_{m\in\mathcal{M}_T} w(m)v(m) \qquad \forall\, v, w \in V_k, \tag{3.10}$$

where $\mathcal{M}_T$ is the set of the midpoints of the three edges of T. It is easy to see from (3.3), (3.8), and (3.10) that we can choose a (constant) damping factor λ so that the spectral radius $\rho(\lambda B_k^{-1}A_k)$ satisfies

$$\rho(\lambda B_k^{-1}A_k) < 1 \qquad \text{for} \quad k \geq 0. \tag{3.11}$$

Given any $g \in V_k'$, we will use a preconditioned Richardson relaxation scheme for the equation

$$A_k z = g \tag{3.12}$$

as the smoother, namely,

$$z_{\text{new}} = z_{\text{old}} + \lambda B_k^{-1}(g - A_k z_{\text{old}}). \tag{3.13}$$

Remark 3.2. The dual space of $L_{2,\mu}(\Omega)$ is the space $L_{2,-\mu}(\Omega)$ consisting of measurable functions ϱ such that

$$\|\varrho\|^2_{L_{2,-\mu}(\Omega)} = \int_\Omega \phi_\mu^{-2}(x)\varrho^2(x)\, dx < \infty. \tag{3.14}$$

The weighted norm $\|\cdot\|_{L_{2,-\mu}(\Omega)}$ is connected to the operator B_k in (3.13) through the relation

$$\langle B_k v, v\rangle = \sum_{T\in\mathcal{T}_k}\sum_{m\in\mathcal{M}_T} [v(m)]^2 \approx h_k^{-2}\|v\|^2_{L_{2,-\mu}(\Omega)} \qquad \forall\, v \in V_k, \tag{3.15}$$

which follows from (1.3), (2.1) and (2.2).

We are now ready to describe multigrid algorithms for (3.12).

Algorithm 3.3. Let $g \in V_k'$ and $z_0 \in V_k$ be an initial guess. The multigrid V-cycle algorithm for (3.12) with m_1 (resp. m_2) pre-smoothing (resp. post-smoothing) steps produces an approximate solution $MG_V(k, g, z_0, m_1, m_2)$. For $k = 0$, $MG_V(k, g, z_0, m_1, m_2) = A_0^{-1}g$. For $k \geq 1$, $MG_V(k, g, z_0, m_1, m_2)$ is computed recursively as follows.

Pre-smoothing

Apply m_1 steps of (3.13) starting with z_0 to obtain z_{m_1}.

Coarse Grid Correction

Let $r_{k-1} = I_k^{k-1}(g - A_k z_{m_1}) \in V_{k-1}'$ be the coarse grid residual. Apply the $(k-1)$-st level algorithm to the coarse grid residual equation

$$A_{k-1}e_{k-1} = r_{k-1}$$

with initial guess 0 to obtain the correction $q = MG_V(k-1, r_{k-1}, 0, m_1, m_2)$ and define

$$z_{m_1+1} = z_{m_1} + I_{k-1}^k q.$$

Post-smoothing

Apply m_2 steps of (3.13) starting with z_{m_1+1} to obtain $z_{m_1+m_2+1}$.

Final Output

$$MG_V(k, g, z_0, m_1, m_2) = z_{m_1+m_2+1}$$

Algorithm 3.4. Let $g \in V_k'$ and $z_0 \in V_k$ be an initial guess. The W-cycle algorithm computes an approximate solution $MG_W(k, g, z_0, m_1, m_2)$ of (3.12). It differs from algorithm 3.3 in the coarse grid correction step, where the coarse grid algorithm is applied twice. More precisely, the correction $q \in V_{k-1}$ is computed by

$$\begin{aligned} q' &= MG_W(k-1, r_{k-1}, 0, m_1, m_2), \\ q &= MG_W(k-1, r_{k-1}, q', m_1, m_2). \end{aligned}$$

Algorithm 3.5. Let $g \in V_k'$ and $z_0 \in V_k$ be an initial guess. The F-cycle algorithm computes an approximate solution $MG_F(k, g, z_0, m_1, m_2)$ of (3.12). It differs from algorithm 3.3 and algorithm 3.4 in the coarse grid correction step, where the coarse grid algorithm is applied once followed by a V-cycle algorithm. More precisely, the correction $q \in V_{k-1}$ is computed by

$$\begin{aligned} q' &= MG_F(k-1, r_{k-1}, 0, m_1, m_2), \\ q &= MG_V(k-1, r_{k-1}, q', m_1, m_2). \end{aligned}$$

4. Convergence Analysis of the W-cycle Multigrid Algorithm

We will analyze the W-cycle multigrid algorithm in this section and provide numerical results for W-cycle, F-cycle and V-cycle algorithms in Section 5. The convergence analysis of the V-cycle and F-cycle algorithm, which relies on the additive multigrid theory [12], will be carried out elsewhere.

Let $E_k : V_k \longrightarrow V_k$ be the error propagation operator for the k-th level W-cycle algorithm. We have the following well-known recursive relation [28, 17]:

$$E_k = R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k E_{k-1}^2 P_k^{k-1}) R_k^{m_1}, \tag{4.1}$$

where Id_k is the identity operator on V_k, the operator $R_k : V_k \longrightarrow V_k$ which measures the effect of one smoothing step is defined by

$$R_k = Id_k - \lambda B_k^{-1} A_k, \tag{4.2}$$

and the operator $P_k^{k-1} : V_k \longrightarrow V_{k-1}$ is the transpose of I_{k-1}^k with respect to the variational forms, i.e.,

$$a_{k-1}(P_k^{k-1} w, v) = a_k(w, I_{k-1}^k v) \qquad \forall\, v \in V_{k-1},\ w \in V_k. \tag{4.3}$$

Note that, for $z \in V_{k-1} \cap H_0^1(\Omega)$, we have

$$a_{k-1}(P_k^{k-1} I_{k-1}^k z, v) = a_k(I_{k-1}^k z, I_{k-1}^k v) = a_{k-1}(z, v) \qquad \forall\, v \in V_{k-1}$$

which implies

$$P_k^{k-1} I_{k-1}^k z = z \qquad \forall\, z \in V_{k-1} \cap H_0^1(\Omega).$$

It follows that

$$\begin{aligned} a_k(I_{k-1}^k z, (Id_k - I_{k-1}^k P_k^{k-1}) v) &= a_k(I_{k-1}^k z, v) - a_k(P_k^{k-1} I_{k-1}^k z, P_k^{k-1} v) \\ &= a_k(z, P_k^{k-1} v) - a_k(z, P_k^{k-1} v) = 0 \qquad \forall\, z \in V_{k-1} \cap H_0^1(\Omega),\ v \in V_k. \end{aligned} \tag{4.4}$$

The key to the convergence analysis of the W-cycle algorithm is a good estimate for the operator $R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1}) R_k^{m_1}$, which is the error-propagation operator for the two-grid algorithm.

We will follow the approach of [9, 38] in the analysis below. Let the mesh-dependent norms $|||v|||_{j,k}$ for $j = 0, 1, 2$ and $k \geq 1$ be defined by

$$|||v|||_{j,k} = \sqrt{\langle B_k (B_k^{-1} A_k)^j v, v \rangle} \qquad \forall\, v \in V_k,\ k \geq 1. \tag{4.5}$$

In particular, we have, in view of (3.4) and (3.14),

$$|||v|||_{0,k}^2 = \langle B_k v, v \rangle \approx h_k^{-2} \|v\|_{L_{2,-\mu}(\Omega)}^2 \qquad \forall\, v \in V_k, \tag{4.6}$$

$$|||v|||_{1,k}^2 = \langle A_k v, v \rangle = a_k(v, v) \qquad \forall\, v \in V_k. \tag{4.7}$$

Also the Cauchy-Schwarz inequality implies that

$$|||v|||_{2,k} = \max_{w \in V_k \setminus \{0\}} \frac{\langle A_k v, w \rangle}{|||w|||_{0,k}} \qquad \forall\, v \in V_k. \tag{4.8}$$

The *smoothing properties* in the following lemma are simple consequences of (3.11), (4.2) and (4.5). Their proofs are standard [28, 17].

Lemma 4.1. *There exists a positive constant C independent of k such that*

$$\|R_k v\|_{1,k} \le \|v\|_{1,k} \qquad \forall\, v \in V_k,\ k \ge 1, \tag{4.9}$$

$$\|R_k^m v\|_{1,k} \le C(1+m)^{-1/2}\|v\|_{0,k} \qquad \forall\, v \in V_k,\ k \ge 1, \tag{4.10}$$

$$\|R_k^m v\|_{2,k} \le C(1+m)^{-1/2}\|v\|_{1,k} \qquad \forall\, v \in V_k,\ k \ge 1. \tag{4.11}$$

The following lemma gives a preliminary approximation property.

Lemma 4.2. *There exists a positive constant C independent of k such that*

$$\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{0,k} \le C\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{1,k} \qquad \forall\, v \in V_k,\ k \ge 1. \tag{4.12}$$

Proof. We will prove (4.12) by a duality argument.

Let $v \in V_k$ be arbitrary and

$$\chi = \phi_\mu^{-2}(Id_k - I_{k-1}^k P_k^{k-1})v.$$

According to (1.4) and (3.14), we have

$$\|\chi\|_{L_{2,\mu}(\Omega)} = \|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{L_{2,-\mu}(\Omega)}. \tag{4.13}$$

Let $\xi \in H_0^1(\Omega)$ satisfy

$$\int_\Omega \nabla\xi \cdot \nabla v\, dx = \int_\Omega \chi v\, dx \qquad \forall\, v \in H_0^1(\Omega).$$

It follows from the consistency of the SIP method that

$$a_k(\xi, v) = \int_\Omega \chi v\, dx \qquad \forall\, v \in V_k. \tag{4.14}$$

Furthermore, we have, by (3.1), (3.5) (applied to ξ), (3.6) and (4.13),

$$\begin{aligned} \|\xi - I_{k-1}^k \Pi_{k-1}\xi\|_k &\le C\|\xi - \Pi_{k-1}\xi\|_{k-1} \\ &\le Ch_{k-1}\|\chi\|_{L_{2,\mu}(\Omega)} \le Ch_k\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{L_{2,-\mu}(\Omega)}. \end{aligned} \tag{4.15}$$

Combining (2.7), (3.4), (3.14), (3.15), (4.4), (4.6), (4.14) and (4.15), we find

$$\begin{aligned} \|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{0,k}^2 &= \langle B_k(Id_k - I_{k-1}^k P_k^{k-1})v, (Id_k - I_{k-1}^k P_k^{k-1})v\rangle \\ &\approx h_k^{-2}\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{L_{2,-\mu}(\Omega)}^2 \\ &= h_k^{-2}\int_\Omega \phi_\mu^{-2}[(Id_k - I_{k-1}^k P_k^{k-1})v]^2 dx \\ &= h_k^{-2}\int_\Omega \chi(Id_k - I_{k-1}^k P_k^{k-1})v\, dx \\ &= h_k^{-2} a_k(\xi, (Id_k - I_{k-1}^k P_k^{k-1})v) \\ &= h_k^{-2} a_k(\xi - I_{k-1}^k \Pi_{k-1}\xi, (Id_k - I_{k-1}^k P_k^{k-1})v) \\ &\le Ch_k^{-2}\|\xi - I_{k-1}^k \Pi_{k-1}\xi\|_k \|(Id_k - I_{k-1}^k P_k^{k-1})v\|_k \\ &\approx Ch_k^{-1}\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{L_{2,-\mu}(\Omega)} \|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{1,k} \end{aligned} \tag{4.16}$$

$$\approx C\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{0,k}\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{1,k},$$

which implies (4.12). □

The approximation property for the convergence analysis is provided by the next lemma.

Lemma 4.3. *There exists a positive constant C independent of k such that*

$$\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{0,k} \le C\|v\|_{2,k} \qquad \forall\, v \in V_k,\ k \ge 1. \tag{4.17}$$

Proof. Since $a_k(\cdot,\cdot)$ is an inner product on V_k, we have by (4.7) and duality,

$$\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{1,k} = \sup_{w\in V_k\setminus\{0\}} \frac{a_k((Id_k - I_{k-1}^k P_k^{k-1})v, w)}{\|w\|_{1,k}}. \tag{4.18}$$

Using (4.3), (4.8) and (4.12), the numerator on the right-hand side of (4.18) can be estimated as follows:

$$\begin{aligned} a_k((Id_k - I_{k-1}^k P_k^{k-1})v, w) &= a_k(v, (Id_k - I_{k-1}^k P_k^{k-1})w) \\ &\le \|v\|_{2,k}\|(Id_k - I_{k-1}^k P_k^{k-1})w\|_{0,k} \le C\|v\|_{2,k}\|w\|_{1,k}, \end{aligned}$$

which together with (4.18) implies

$$\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{1,k} \le C\|v\|_{2,k} \qquad \forall\, v \in V_k,\ k \ge 1. \tag{4.19}$$

The estimate (4.17) follows from (4.12) and (4.19). □

Combining (4.10), (4.11) and (4.17), we immediately have the following theorem on the two-grid algorithm.

Theorem 4.4. *There exists a positive constant C_{TG} independent of k such that*

$$\|R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1})R_k^{m_1}v\|_{1,k} \le C_{TG}[(1+m_1)(1+m_2)]^{-1/2}\|v\|_{1,k} \quad \forall\, v \in V_k,\ k \ge 1. \tag{4.20}$$

Proof.

$$\begin{aligned} \|R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1})R_k^{m_1}v\|_{1,k} &\le C(1+m_2)^{-1/2}\|(Id_k - I_{k-1}^k P_k^{k-1})R_k^{m_1}v\|_{0,k} \\ &\le C(1+m_2)^{-1/2}\|R_k^{m_1}v\|_{2,k} \\ &\le C(1+m_2)^{-1/2}(1+m_1)^{-1/2}\|v\|_{1,k} \end{aligned}$$

□

To go from the two-grid estimate (4.20) to an estimate for the W-cycle multigrid algorithm, we need the following lemma on the stability of I_{k-1}^k and P_k^{k-1}.

Lemma 4.5. *There exists a positive constant C_{CF} independent of k such that*

$$\|I_{k-1}^k v\|_{1,k} \le C_{CF}\|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}, \tag{4.21}$$

$$\|P_k^{k-1}v\|_{1,k-1} \le C_{CF}\|v\|_{1,k} \qquad \forall\, v \in V_k. \tag{4.22}$$

Proof. The estimate (4.21) follows from (3.4), (3.6) and (4.7):

$$\begin{aligned}\|I_{k-1}^k v\|_{1,k}^2 &= a_k(I_{k-1}^k v, I_{k-1}^k v)\\ &\leq C\|I_{k-1}^k v\|_k^2 \leq C\|v\|_{k-1}^2 \leq C a_{k-1}(v,v) = C_{CF}^2 \|v\|_{1,k-1}^2.\end{aligned}$$

The estimate (4.22) then follows from (4.3) (4.7), (4.21) and duality.

$$\begin{aligned}\|P_k^{k-1} v\|_{1,k-1} &= \max_{w\in V_{k-1}\setminus\{0\}} \frac{a_{k-1}(P_k^{k-1}v, w)}{\|w\|_{1,k-1}}\\ &= \max_{w\in V_{k-1}\setminus\{0\}} \frac{a_k(v, I_{k-1}^k w)}{\|w\|_{1,k-1}} \leq C_{CF}\|v\|_{1,k}.\end{aligned}$$

□

Theorem 4.6. *Given any $C_* > C_{TG}$, there exists a positive integer m_* independent of k such that*

$$\|E_k v\|_{1,k} \leq C_*[(1+m_1)(1+m_2)]^{-1/2}\|v\|_{1,k} \qquad \forall\, v \in V_k,\ k \geq 0, \tag{4.23}$$

provided $m_1 + m_2 \geq m_*$.

Proof. We will prove (4.23) by mathematical induction. The case $k = 0$ holds for any m_* since $A_0 z = g$ is solved exactly.

Assume $k \geq 1$ and (4.23) is valid for $k-1$. Let $v \in V_k$ be arbitrary. In view of (4.1), we have

$$E_k v = R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1})R_k^{m_1} v + R_k^{m_2}(I_{k-1}^k E_{k-1}^2 P_k^{k-1})R_k^{m_1} v.$$

We obtain, from (4.20),

$$\|R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1})R_k^{m_1} v\|_{1,k} \leq C_{TG}[(1+m_1)(1+m_2)]^{-1/2}\|v\|_{1,k},$$

and from (4.9), (4.21), (4.22) and the induction hypothesis,

$$\|R_k^{m_2} I_{k-1}^k E_{k-1}^2 P_k^{k-1} R_k^{m_1} v\|_{1,k} \leq C_{CF}^2 C_*^2 [(1+m_1)(1+m_2)]^{-1}\|v\|_{1,k}.$$

It follows that

$$\|E_k v\|_{1,k} \leq \big(C_{TG}[(1+m_1)(1+m_2)]^{-1/2} + C_{CF}^2 C_*^2[(1+m_1)(1+m_2)]^{-1}\big)\|v\|_{1,k}. \tag{4.24}$$

If we choose $m_* > 0$ so that

$$m_*^{-1/2} \leq \frac{C_* - C_{TG}}{C_{CF}^2 C_*^2},$$

then for $m_1 + m_2 \geq m_*$ we have

$$\begin{aligned} &C_{TG}[(1+m_1)(1+m_2)]^{-1/2} + C_{CF}^2 C_*^2[(1+m_1)(1+m_2)]^{-1}\\ &\quad\leq (C_{TG} + C_{CF}^2 C_*^2[(1+m_1)(1+m_2)]^{-1/2})[(1+m_1)(1+m_2)]^{-1/2}\\ &\quad\leq (C_{TG} + C_{CF}^2 C_*^2 m_*^{-1/2})[(1+m_1)(1+m_2)]^{-1/2}\\ &\quad\leq C_*[(1+m_1)(1+m_2)]^{-1/2},\end{aligned}$$

which together with (4.24) implies that (4.23) is also valid for k. □

It follows from Theorem 4.6 that the W-cycle multigrid is a contraction with contraction number independent of grid levels provided the number of smoothing steps are sufficiently large. Furthermore, the contraction number decreases at the rate of $1/m$ for the symmetric W-cycle algorithm where $m_1 = m_2 = m$. Numerical results show that this is also the case for the V-cycle and F-cycle algorithms.

5. Numerical Results

In this section we report results of several numerical experiments for the model problem (1.1) on the L-shaped domain $(-1,1)^2 \setminus [0,1]^2$. The triangulations $\mathcal{T}_0, \mathcal{T}_1, \ldots$, are created by the refinement procedure described at the beginning of Section 3, where $\mathcal{T}_0$ has six elements and the grading parameter at the reentrant corner is taken to be 2/3 (cf. Figure 3.2). The mesh parameter of $\mathcal{T}_k$ is $h_k = 2^{-k}$.

In the first set of experiments we take the exact solution to be

$$u(x,y) = (1-x^2)(1-y^2)r^{2/3}\sin\Big(\frac{2}{3}\big(\theta - \frac{\pi}{2}\big)\Big),$$

where (r,θ) are the polar coordinates at the origin. We computed the solution u_k of (3.2) with $\eta = 10$, 100 and 1000. The energy errors $a_k(\Pi_k u - u_k, \Pi_k u - u_k)^{1/2}$ and the L_2 errors $\|\Pi_k u - u_k\|_{L_2(\Omega)}$ for $\eta = 10$ and $0 \le k \le 7$ are presented in Table 5.1. The predicted convergence rates in (2.15) are clearly visible.

	Energy Error	L_2 Error
$k=0$	1.16 E+0	3.09 E−1
$k=1$	6.17 E−1	6.36 E−2
$k=2$	2.90 E−1	1.16 E−2
$k=3$	1.37 E−1	2.41 E−3
$k=4$	6.63 E−2	5.76 E−4
$k=5$	3.29 E−2	1.52 E−4
$k=6$	1.65 E−2	4.24 E−5
$k=7$	8.33 E−3	1.19 E−5

Table 5.1. Energy errors and L_2 errors for the L-shaped domain ($\eta = 10$)

We also plotted the energy error versus the mesh size in the log-log scale in Figure 5.1 for $\eta = 10$, 100 and 1000. The energy error decreases as η increases, which indicates that the constant C in (2.15) can indeed be chosen to be independent of η, as long as it is sufficiently large.

In the second set of experiments we computed the contraction numbers of the W-cycle, F-cycle and V-cycle algorithms on the graded meshes $\mathcal{T}_1, \ldots, \mathcal{T}_7$. We used $\eta = 10$ as the penalty parameter, $\lambda = 1/40$ as the damping factor in (3.13), and m pre-smoothing and m post-smoothing steps. The results are presented in Tables 5.2–5.3. It is observed that the W-cycle algorithm is a contraction for $m \ge 1$, the F-cycle algorithm is a contraction for

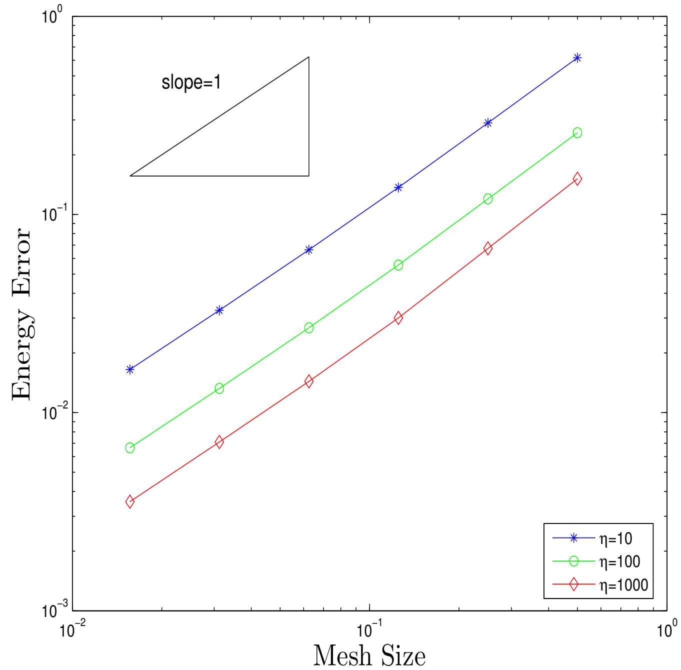

FIGURE 5.1. Comparison of Energy Errors for $\eta = 10$, 100 and 1000

$m \geq 4$, and the V-cycle algorithm is a contraction for $m \geq 8$. Furthermore, when $m \geq 4$, the W-cycle and the F-cycle have similar contraction numbers.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=1$	0.85	0.89	0.88	0.89	0.90	0.90	0.90
$m=2$	0.74	0.80	0.83	0.83	0.83	0.84	0.84
$m=3$	0.64	0.73	0.74	0.78	0.78	0.78	0.78
$m=4$	0.57	0.67	0.72	0.73	0.75	0.75	0.75
$m=5$	0.51	0.63	0.69	0.71	0.71	0.72	0.73
$m=6$	0.45	0.59	0.66	0.69	0.70	0.71	0.71
$m=7$	0.41	0.56	0.64	0.67	0.68	0.69	0.69
$m=8$	0.38	0.53	0.62	0.65	0.66	0.68	0.68
$m=9$	0.35	0.51	0.60	0.64	0.66	0.66	0.66
$m=10$	0.32	0.49	0.59	0.62	0.65	0.65	0.66

TABLE 5.2. Contraction numbers of the W-cycle algorithm on the L-shaped domain ($\eta = 10$)

Finally, the asymptotic behavior of the contraction number with respect to the number of smoothing steps for $k = 6$ is depicted in Figure 5.2. The log-log graphs confirm that the contraction number decreases at the rate of m^{-1}, as predicted by Theorems 4.4 and 4.6.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=4$	0.57	0.67	0.72	0.73	0.75	0.75	0.75
$m=5$	0.51	0.63	0.69	0.71	0.71	0.73	0.73
$m=6$	0.45	0.59	0.66	0.69	0.70	0.71	0.71
$m=7$	0.41	0.56	0.64	0.67	0.68	0.69	0.69
$m=8$	0.38	0.53	0.62	0.65	0.66	0.68	0.68
$m=9$	0.35	0.51	0.60	0.64	0.66	0.66	0.66
$m=10$	0.32	0.49	0.59	0.62	0.65	0.65	0.66
$m=11$	0.30	0.47	0.57	0.61	0.64	0.64	0.64
$m=12$	0.28	0.45	0.56	0.60	0.62	0.63	0.63

TABLE 5.3. Contraction numbers of the F-cycle algorithm on the L-shaped domain with ($\eta = 10$)

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=8$	0.38	0.55	0.65	0.69	0.72	0.73	0.74
$m=9$	0.35	0.53	0.64	0.68	0.71	0.71	0.72
$m=10$	0.32	0.51	0.62	0.67	0.69	0.71	0.71
$m=11$	0.30	0.49	0.60	0.64	0.67	0.69	0.70
$m=12$	0.28	0.47	0.59	0.62	0.66	0.69	0.69
$m=13$	0.27	0.46	0.57	0.61	0.65	0.68	0.68
$m=14$	0.25	0.44	0.56	0.61	0.64	0.66	0.67

TABLE 5.4. Contraction numbers of the V-cycle algorithm on the L-shaped domain ($\eta = 10$)

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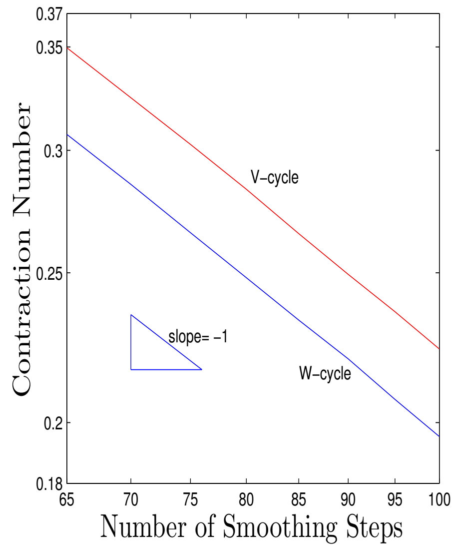

FIGURE 5.2. Asymptotic behavior of the contraction number with respect to the number of smoothing steps

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