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A New Rotated Nonconforming Quadrilateral Element

Zhaoliang Meng[1] · **Jintao Cui**[2] · **Zhongxuan Luo**[1,3]

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Abstract In this paper, a new nonparametric nonconforming quadrilateral finite element is introduced. This element takes the four edge mean values as the degrees of the freedom and the finite element space is a subspace of P_2. Different from the other nonparametric elements, the basis functions of this new element can be expressed explicitly without solving linear systems locally, which can be achieved by introducing a new reference quadrilateral. To evaluate the integration, a class of new quadrature formulae with only three equally weighted points on quadrilateral are constructed. Hence the stiffness matrix can be calculated by the same way with the parametric elements. Numerical results are shown to confirm the optimality of the convergence order for the second order elliptic problems and the Stokes problem.

Keywords Nonconforming element · Nonparametric · Finite element · Quadrilateral mesh

Mathematics Subject Classification 65N30

1 Introduction

The nonconforming finite element methods successfully provide stable numerical solutions of many practical fluid flow and solid mechanics problems: see, for instance, [3–5,7,8,19] for the Stokes and Navier–Stokes problems and [1,2,14,16,17] for elasticity related problems. The linear nonconforming finite elements for triangles or tetrahedrons were devised

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✉ Zhaoliang Meng
mzhl@dlut.edu.cn

1 School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China

2 Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hum, Hong Kong, China

3 School of Software, Dalian University of Technology, Dalian 116620, China

by Crouzeix and Raviart in [5] to provide stable finite element pairs for the Stokes problem with optimal convergence properties. Concerning quadrilateral nonconforming elements, Han [9] introduced a rectangular element with number of local degrees of freedom being five, Rannacher–Turek [19] presented the rotated Q_1 nonconforming element; and Douglas et al. [6] introduced the nonconforming finite element using only four values at the midpoints of the edges as degrees of freedom. These three elements all obtain optimal orders of convergence for second-order problems and provide stable finite element pairs for the Stokes problems on rectangular meshes. However, if they are applied to general quadrilateral meshes, the optimality in convergence will be lost. Thus for the general quadrilateral case, an extra term xy should be added [4], in order to recover optimal convergence.

For general quadrilateral meshes, Park and Sheen [18] presented a nonparametric P_1-nonconforming finite element which has the lowest number of degrees of freedom. A similar element was also introduced by Hu and Shi [10]. But without any modification they cannot be used to solve fluid and solid mechanics in a stable manner. There are also other nonparametric lower-order nonconforming elements, for instance, [11,12,19], which all include P_1 plus an extra higher-order polynomial. However, the construction of the basis functions of these elements requires solving at least four 4×4 matrix systems. Furthermore, the computation of the local stiffness matrix is done on the standard reference domain $[-1, 1]^2$ and the bilinear transformation is required. Since the corresponding Jacobian determinate therein is a linear function, higher-degree quadrature formulae need to be employed on general quadrilaterals. Nonparametric nonconforming elements of higher order are also considered in [13,15,21].

In summary, the parametric nonconforming elements will loss convergence order or require extra bubble functions. Although the nonparametric versions can keep optimal convergence order, the basis functions therein can not be given explicitly and higher-degree quadrature formulae will be employed if general quadrilateral is considered.

The main purpose of this paper is to develop a new nonparametric nonconforming element of lower order which can be regarded as an extension of the rotated Q_1 element [19]. The proposed element takes the four edge mean values as the DOFs and the finite element space is locally P_1 plus $\text{Span}\{l_{13}l_{24}\}$, where l_{13} and l_{24} are two linear polynomials vanishing at the vertices V_1, V_3 and V_2, V_4, respectively. Our element has the same DOFs with those studied in [11,19], but has different shape function space. For rectangular mesh, our element is equivalent to that of [19]. But for general quadrilateral mesh, our element is different from the nonparametric version in [19]. Our strategy is to introduce a new reference domain $\tilde{Q}$ which is different from that used in [11,12,18,21], and then define an affine map from the reference element to the physical element. By virtue of the reference element, the basis functions can be expressed explicitly without solving linear systems locally.

To compute the stiffness matrix and right-hand vector, we derive a family of second-degree quadrature formulae over the reference quadrilateral which have only three quadrature points with equal weights. Hence all the integrations can be done over the reference domain, which is more efficient since the Jacobian determinant is constant and less quadrature points are used.

An outline of this paper is as follows. We devote Sect. 2 to the construction of our quadrilateral nonconforming element. To compute the integrations efficiently, we develop a family of quadrature formulae over the reference domains in Sect. 3. Some numerical examples are given in Sect. 4. Finally, in Sect. 5, we conclude our results.

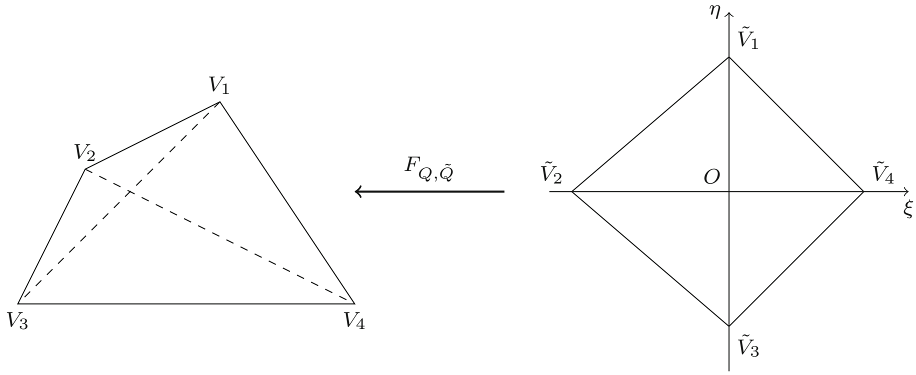

Fig. 1 An affine map from a reference quadrilateral $\tilde{Q}$ to a quadrilateral Q

2 Quadrilateral Nonconforming Element

Let Q be a convex quadrilateral shown as in Fig. 1, where V_1, V_2, V_3, V_4 denote the vertices with counterclockwise indices, E_j designates the edge between V_j to V_{j+1} modulo 4, and M_j is the midpoint of E_j, $j = 1, \ldots, 4$. Let $l_j(x, y)$ denote the linear polynomial which vanishes on the edge E_j for $j = 1, \ldots, 4$. Let l_{13} and l_{24} be the linear polynomials satisfying

$$l_{13}(V_1) = l_{13}(V_3) = l_{24}(V_2) = l_{24}(V_4) = 0, \quad l_{13}(V_4) = l_{24}(V_1) = 1.$$

We define our nonconforming element $(Q, \mathbb{P}(Q), \Phi_Q)$ by

(1) Q is a convex quadrilateral,
(2) $\mathbb{P}(Q) = \text{Span}\{1, x, y, l_{13}l_{24}\}$,
(2) Φ_Q is the degree of freedom vector with components

$$\frac{1}{|E_i|}\int_{E_i} v(x, y)\mathrm{d}s, \quad i = 1, 2, 3, 4, \forall v \in C^0(Q).$$

We will show that Φ_Q is $\mathbb{P}(Q)$-unisolvent. To do this, let $l_{13}(V_2) = h_1, l_{24}(V_3) = h_2$. Note that Q is convex if and only if $h_1 < 0$ and $h_2 < 0$. Thus if we take $\tilde{Q}$ as a reference quadrilateral with four vertices

$$\tilde{V}_1 = (0, 1), \quad \tilde{V}_2 = (h_1, 0), \quad \tilde{V}_3 = (0, h_2), \quad \tilde{V}_4 = (1, 0),$$

then there exists a unique affine transformation $\mathcal{F}_{\tilde{Q},Q} : \tilde{Q} \longrightarrow Q$ such that $\mathcal{F}_{\tilde{Q},Q}(\tilde{V}_i) = V_i, i = 1, 2, 3, 4$. Note the inverse of $\mathcal{F}_{\tilde{Q},Q}$ can be written as: $(\xi, \eta) = \mathcal{F}^{-1}_{\tilde{Q},Q}(x, y) = (l_{13}(x, y), l_{24}(x, y))$ where $(x, y) \in Q$ and $(\xi, \eta) \in \tilde{Q}$. Since $\mathcal{F}_{\tilde{Q},Q}$ is an affine transformation, we can show the unisolvency on $\tilde{Q}$. Furthermore let $\tilde{l}_{13} = l_{13} \circ \mathcal{F}_{\tilde{Q},Q}$ and $\tilde{l}_{24} = l_{24} \circ \mathcal{F}_{\tilde{Q},Q}$, then $\tilde{l}_{13}(\xi, \eta) = \xi, \tilde{l}_{24}(\xi, \eta) = \eta$. Similarly, we denote the four edges of $\tilde{Q}$ by $\tilde{E}_j$, $j = 1, 2, 3, 4$. Set

$$\tilde{\mathbb{P}} = \left\{\tilde{u} : \tilde{Q} \longrightarrow \mathbb{R} \middle| \tilde{u} \circ \mathcal{F}^{-1}_{\tilde{Q},Q} \in \mathbb{P}(Q)\right\}.$$

Obviously $\tilde{\mathbb{P}} = \{1, \xi, \eta, \xi\eta\}$.

Theorem 1 *The set* $\{\frac{1}{|\tilde{E}_j|}\int_{\tilde{E}_j} f(\xi, \eta)\mathrm{d}\tilde{s}, j = 1, 2, 3, 4\}$ *is* $\tilde{\mathbb{P}}$*-unisolvent.*

Proof By a simple calculation, we have

$$\frac{1}{|\tilde{E}_4|}\int_{\tilde{E}_4} \xi^i \eta^j \mathrm{d}\tilde{s} = \frac{i!j!}{(i+j+1)!}, \qquad \frac{1}{|\tilde{E}_1|}\int_{\tilde{E}_1} \xi^i \eta^j \mathrm{d}\tilde{s} = \frac{i!j!}{(i+j+1)!}h_1^i,$$
$$\frac{1}{|\tilde{E}_2|}\int_{\tilde{E}_2} \xi^i \eta^j \mathrm{d}\tilde{s} = \frac{i!j!}{(i+j+1)!}h_1^i h_2^j, \quad \frac{1}{|\tilde{E}_3|}\int_{\tilde{E}_3} \xi^i \eta^j \mathrm{d}\tilde{s} = \frac{i!j!}{(i+j+1)!}h_2^j.$$

Denote $1, \xi, \eta, \xi\eta$ by $\tilde{\phi}_j$, $j = 1, 2, 3, 4$, respectively and also define $A = (a_{ij})_{4\times 4}$ by $a_{ij} = \frac{1}{|\tilde{E}_i|}\int_{\tilde{E}_i} \tilde{\phi}_j \mathrm{d}\tilde{s}$. Then

$$A = \begin{bmatrix} 1 & \frac{1}{2}h_1 & \frac{1}{2} & \frac{1}{6}h_1 \\ 1 & \frac{1}{2}h_1 & \frac{1}{2}h_2 & \frac{1}{6}h_1 h_2 \\ 1 & \frac{1}{2} & \frac{1}{2}h_2 & \frac{1}{6}h_2 \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

with $\det(A) = \frac{1}{24}(1-h_1)^2(1-h_2)^2 \neq 0$, since $h_1 < 0$ and $h_2 < 0$. □

Furthermore, we can explicitly present the expression of the basis functions as follows:

$$\begin{aligned}
\tilde{\psi}_1(\xi,\eta) &= C_{\tilde{Q}}\left(-h_2 + 2h_2\xi + 2\eta - 6\xi\eta\right),\\
\tilde{\psi}_2(\xi,\eta) &= C_{\tilde{Q}}\left(1 - 2\xi - 2\eta + 6\xi\eta\right),\\
\tilde{\psi}_3(\xi,\eta) &= C_{\tilde{Q}}\left(-h_1 + 2\xi + 2h_1\eta - 6\xi\eta\right),\\
\tilde{\psi}_4(\xi,\eta) &= C_{\tilde{Q}}\left(h_1 h_2 - 2h_2\xi - 2h_1\eta + 6\xi\eta\right),
\end{aligned}$$

where

$$C_{\tilde{Q}} = \frac{1}{(1-h_1)(1-h_2)} = \frac{1}{2\cdot \mathrm{area}(\tilde{Q})}.$$

Now turn to consider the physical quadrilateral Q. The basis functions are given by

$$\begin{aligned}
\psi_1(x,y) &= C_{\tilde{Q}}\left(-h_2 + 2h_2 l_{13} + 2l_{24} - 6l_{13}l_{24}\right),\\
\psi_2(x,y) &= C_{\tilde{Q}}\left(1 - 2l_{13} - 2l_{24} + 6l_{13}l_{24}\right),\\
\psi_3(x,y) &= C_{\tilde{Q}}\left(-h_1 + 2l_{13} + 2h_1 l_{24} - 6l_{13}l_{24}\right),\\
\psi_4(x,y) &= C_{\tilde{Q}}\left(h_1 h_2 - 2h_2 l_{13} - 2h_1 l_{24} + 6l_{13}l_{24}\right).
\end{aligned}$$

Let $\Omega \in \mathbb{R}^2$ be a simply connected polygonal domain. Let $(\mathcal{T}_h)_{h>0}$ be a family of shape regular quadrilateral triangulation of Ω with $h = \max_{Q\in\mathcal{T}_h} \mathrm{diam}(Q)$. Denote by $\mathcal{E}_h$, $\mathcal{E}_h^i$ and $\mathcal{E}_h^b$ the sets of edges, inner edges and boundary edges, respectively.

Now we can define the global nonparametric nonconforming element spaces as follows

$$\begin{aligned}
\mathcal{NC}_h &= \{v_h \in L^2(\Omega) : v_h|_Q \in \mathbb{P}(Q) \ \text{ for all } Q \in \mathcal{T}_h, (1, [v_h]_E)_E = 0, \quad \text{for all } E \in \mathcal{E}_h^i\},\\
\mathcal{NC}_{h,0} &= \{v_h \in \mathcal{NC}_h : (1, v_h)_E = 0, \quad \text{for all } E \in \mathcal{E}_h^b\},
\end{aligned}$$

where $(\cdot,\cdot)_E$ is the standard inner product over $L^2(E)$, and $[\cdot]_E$ stands for the jump of a function across the side E.

According to the definition of our element space, it has the orthogonal property. So it can be used to solve second-order elliptic problems and get the optimal convergence order.

Furthermore, the new element can be used as a stable family of mixed finite element for the velocity fields, combined with piecewise constant element for pressure, in solving the Navier–Stokes equations.

Now let us turn to the computational aspect. Usually, the computation of quadrilateral element is done on the standard reference domain $[-1, 1]^2$ and the bilinear transformation is required. Since the corresponding Jacobian determinant is a linear function, if general quadrilateral is considered, higher-degree quadrature formulae need to be employed. In comparison, for the new element developed above, all the computation can be done efficiently on our reference element $\tilde{Q}$. We will elaborate the ideas below in detail.

Let $l_{13}(x, y) = a_1 x + b_1 y + c_1$ and $l_{24}(x, y) = a_2 x + b_2 y + c_2$. Then the derivatives of the basis function can be obtained as follows:

$$\frac{\partial \psi_i}{\partial x} = \frac{\partial \tilde{\psi}_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \tilde{\psi}_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = a_1 \frac{\partial \tilde{\psi}_i}{\partial \xi} + a_2 \frac{\partial \tilde{\psi}_i}{\partial \eta}$$
$$\frac{\partial \psi_i}{\partial y} = \frac{\partial \tilde{\psi}_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial \tilde{\psi}_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = b_1 \frac{\partial \tilde{\psi}_i}{\partial \xi} + b_2 \frac{\partial \tilde{\psi}_i}{\partial \eta}$$

The Jacobian matrix can be simply calculated by

$$J = \frac{\partial (x, y)}{\partial (\xi, \eta)} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}^{-1}.$$

We see that J is a constant matrix and its determinant

$$|J| = 1/(a_1 b_2 - a_2 b_1)$$

is simply the ratio of the area of the element Q to that of the element $\tilde{Q}$.

Thus the components of the stiffness matrix can be expressed by

$$\begin{aligned}
&\int_Q \left(\frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) \mathrm{d}x \mathrm{d}y \\
&= \int_{\tilde{Q}} \left(a_1 \frac{\partial \tilde{\psi}_i}{\partial \xi} + a_2 \frac{\partial \tilde{\psi}_i}{\partial \eta} \right) \left(a_1 \frac{\partial \tilde{\psi}_j}{\partial \xi} + a_2 \frac{\partial \tilde{\psi}_j}{\partial \eta} \right) |J| \mathrm{d}\xi \mathrm{d}\eta \\
&\quad + \int_{\tilde{Q}} \left(b_1 \frac{\partial \tilde{\psi}_i}{\partial \xi} + b_2 \frac{\partial \tilde{\psi}_i}{\partial \eta} \right) \left(b_1 \frac{\partial \tilde{\psi}_j}{\partial \xi} + b_2 \frac{\partial \tilde{\psi}_j}{\partial \eta} \right) |J| \mathrm{d}\xi \mathrm{d}\eta, \quad 1 \le i, j \le 4.
\end{aligned}$$

For our element, $\psi_i(\xi, \eta)$ is a polynomial of degree two, and hence the integration above can be computed exactly by using Eq. (1) or quadrature formula of degree two of the next section. In next section, we will derive the quadrature formula over $\tilde{Q}$, which only involves three equally weighted quadrature points. It also provides an efficient way to compute the load vector.

Remark 1 If the stiffness matrix is computed by mapping from a reference element $\widehat{Q} = [-1, 1]^2$ onto the given element Q, the use of a 2×2 Gauss rule is at least necessary. In fact, let $(x, y) = F(\xi, \eta) = (F_1(\xi, \eta), F_2(\xi, \eta))$ denote the bilinear mapping from $\widehat{Q}$ to Q and $|\widehat{J}(\xi, \eta)|$ denote the Jacobian determinant. Note that the determinant $|\widehat{J}(\xi, \eta)|$ is always a linear function of the coordinates. The component of the stiffness matrix can be computed via

$$\begin{aligned}
&\int_Q \left(\frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) \mathrm{d}x\mathrm{d}y \\
&= \int_{\widehat{Q}} \left(\frac{\partial \psi_i}{\partial x}(F_1(\xi,\eta), F_2(\xi,\eta)) \frac{\partial \psi_j}{\partial x}(F_1(\xi,\eta), F_2(\xi,\eta)) \right. \\
&\quad \left. + \frac{\partial \psi_i}{\partial y}(F_1(\xi,\eta), F_2(\xi,\eta)) \frac{\partial \psi_j}{\partial y}(F_1(\xi,\eta), F_2(\xi,\eta)) \right) |\widehat{J}(\xi,\eta)| \mathrm{d}\xi \mathrm{d}\eta.
\end{aligned}$$

Since $\frac{\partial \psi_i}{\partial x}(x, y)$ is a linear polynomial with respect to x and y and $F_i(\xi, \eta)$ is a bilinear polynomial with respect to ξ and η, $\frac{\partial \psi_i}{\partial x}(F_1(\xi, \eta), F_2(\xi, \eta))$ must be a bilinear polynomial with respect to ξ and η. Hence the integrand is a bicubic polynomial and at least a 2×2 Gauss rule is used. Furthermore, one must evaluate the value of $|\widehat{J}(\xi, \eta)|$ at every quadrature point since it is not a constant for a given Q.

3 The Construction of Quadrature Formulae on the Quadrilateral

In this section, we will consider the construction of quadrature formulae on the quadrilateral.

Let $\tilde{Q}$ be a quadrilateral with vertices $\tilde{V}_1 = (0, 1)$, $\tilde{V}_2 = (h_1, 0)$, $\tilde{V}_3 = (0, h_2)$, $\tilde{V}_4 = (1, 0)$. We divide $\tilde{Q}$ into four triangles and denote them by $\tilde{T}_1, \tilde{T}_2, \tilde{T}_3$ and $\tilde{T}_4$ (see Fig. 2). Then

$$\int_{\tilde{Q}} \xi^i \eta^j \mathrm{d}\xi \mathrm{d}\eta = \int_{\tilde{T}_1} \xi^i \eta^j \mathrm{d}\xi \mathrm{d}\eta + \int_{\tilde{T}_2} \xi^i \eta^j \mathrm{d}\xi \mathrm{d}\eta + \int_{\tilde{T}_3} \xi^i \eta^j \mathrm{d}\xi \mathrm{d}\eta + \int_{\tilde{T}_4} \xi^i \eta^j \mathrm{d}\xi \mathrm{d}\eta.$$

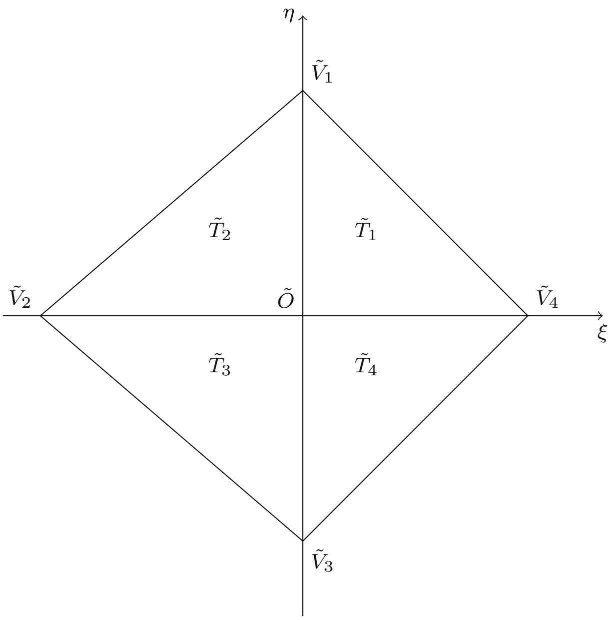

Fig. 2 Reference quadrilateral

According to the result from [20], we have

$$\int_{\tilde{T}_1} \xi^i \eta^j \mathrm{d}\xi \mathrm{d}\eta = \frac{i!j!}{(2+i+j)!}.$$

For the integration on $\tilde{T}_2$, let $\xi = h_1 t$ and $\eta = s$, then

$$\int_{\tilde{T}_2} \xi^i \eta^j \mathrm{d}\xi \mathrm{d}\eta = -\int_{\tilde{T}_1} (h_1 t)^i s^j (h_1) \mathrm{d}s \mathrm{d}t = -\frac{i!j!}{(2+i+j)!} h_1^{i+1}.$$

And similarly,

$$\begin{aligned}\int_{\tilde{T}_3} \xi^i \eta^j \mathrm{d}\xi \mathrm{d}\eta &= \frac{i!j!}{(2+i+j)!} h_1^{i+1} h_2^{j+1}, \\ \int_{\tilde{T}_4} \xi^i \eta^j \mathrm{d}\xi \mathrm{d}\eta &= -\frac{i!j!}{(2+i+j)!} h_2^{j+1}.\end{aligned}$$

Collecting above results, we have

$$\begin{aligned}\int_{\tilde{Q}} \xi^i \eta^j \mathrm{d}\xi \mathrm{d}\eta &= \frac{i!j!}{(2+i+j)!} \left(1 - h_1^{i+1} - h_2^{j+1} + h_1^{i+1} h_2^{j+1}\right) \\ &= \frac{i!j!}{(2+i+j)!} (1 - h_1^{i+1})(1 - h_2^{j+1}). \end{aligned} \tag{1}$$

By applying Gram–Schmidt orthogonal process, we can get the orthogonal polynomial of degree one as follows:

$$w_1(\xi, \eta) = \xi - \frac{1}{3}(1 + h_1), \quad w_2(\xi, \eta) = \eta - \frac{1}{3}(1 + h_2).$$

Thus the quadrature formula of degree one can be expressed as follows:

$$\int_{\tilde{Q}} f(\xi, \eta) \mathrm{d}\xi \mathrm{d}\eta = S_{\tilde{Q}} f\left(\frac{1}{3}(1 + h_1), \frac{1}{3}(1 + h_2)\right), \quad \forall f(\xi, \eta) \in P_1,$$

where $S_{\tilde{Q}} = (1 - h_1)(1 - h_2)/2$ is the area of $\tilde{Q}$.

Now let us turn to the construction of the quadrature formulae of degree two. The basic idea can be found in [20, pp. 79–88]. For convenience, let

$$\mathcal{L}(f) = \frac{1}{S_{\tilde{Q}}} \int_{\tilde{Q}} f(\xi, \eta) \mathrm{d}\xi \mathrm{d}\eta, \quad \forall f \in L^2(\tilde{Q}).$$

Obviously $\mathcal{L}(1) = 1$.

First of all, take the orthonormal polynomials of degree one as follows:

$$\begin{aligned} p_1 &= C_1^{-1/2} \cdot w_1, \\ p_2 &= C_2^{-1/2} \cdot \left((1 + h_1)(1 + h_2) w_1 + 2(1 - h_1 + h_1^2) w_2\right) \\ &= C_2^{-1/2} \cdot \left((1 + h_1)(1 + h_2)\xi + 2(1 - h_1 + h_1^2)\eta - (1 + h_1^2)(1 + h_2)\right), \end{aligned}$$

where C_1 and C_2 are normalized constants such that

$$\mathcal{L}\left(\left(p_i(\xi, \eta) p_j(\xi, \eta)\right)\right) = \delta_{ij}, \quad i, j = 1, 2.$$

By detailed computation, we get

$$C_1 = (1 - h_1 + h_1^2)/18,$$
$$C_2 = 3C_1\left((1 + h_1^2)(1 + h_2^2) - 2(1 + h_1 h_2)(h_1 + h_2)\right).$$

It is well known that there must exist a numerical integration formula of degree two with three equally weighted points with respect to $\mathcal{L}$, i.e., the weight for the points is $1/3$. Suppose we have a second-degree formula

$$\mathcal{L}(f) = \frac{1}{3}\left(f(u^{(1)}) + f(u^{(2)}) + f(u^{(3)})\right), \quad \forall f \in P_2, \tag{2}$$

where $u^{(k)}$, $k = 1, 2, 3$ are the quadrature points. To enforce polynomial exactness of degree 2, it suffices to require (2) to be exact for $1, p_1, p_2, p_1^2, p_1 p_2, p_2^2$. By the orthogonality of $p_i(\xi, \eta)$, we have

$$\mathcal{L}(p_1) = \mathcal{L}(p_2) = 0, \quad \mathcal{L}(p_1 p_2) = 0, \quad \text{and} \quad \mathcal{L}(p_1^2) = \mathcal{L}(p_2^2) = 1.$$

Then the above conditions require, respectively

$$\begin{aligned} &p_i(u^{(1)}) + p_i(u^{(2)}) + p_i(u^{(3)}) = 0, \quad i = 1, 2, \\ &p_i(u^{(1)})p_j(u^{(1)}) + p_i(u^{(2)})p_j(u^{(2)}) + p_i(u^{(3)})p_j(u^{(3)}) = 3\delta_{ij}, \quad i, j = 1, 2. \end{aligned} \tag{3}$$

Let $T : (\xi, \eta) \mapsto (s, t)$ be a linear transform from $\mathbb{R}^2$ to $\mathbb{R}^2$ where $s = p_1(\xi, \eta)$ and $t = p_2(\xi, \eta)$. Let $v^{(k)} = (v_{k1}, v_{k2}) = T(u^{(k)}), \quad k = 1, 2, 3$, then (3) can be rewritten as

$$\begin{aligned} &v_{11} + v_{21} + v_{31} = 0, \quad v_{12} + v_{22} + v_{32} = 0, \\ &v_{1i}v_{1j} + v_{2i}v_{2j} + v_{3i}v_{3j} = 3\delta_{ij}, \quad i, j = 1, 2. \end{aligned} \tag{4}$$

By defining a matrix

$$A = \begin{bmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ 1 & 1 & 1 \end{bmatrix}$$

we can rewrite (4) as

$$AA^T = 3I$$

where I is the identity matrix. Hence

$$A^T A = 3I$$

is equivalent to the following equations

$$v_{i1}v_{j1} + v_{i2}v_{j2} + 1 = 3\delta_{ij}, \quad i, j = 1, 2, 3.$$

Therefore the points $v^{(k)}$ lie on a sphere of radius $r = \sqrt{2}$ with centroid at the origin. It is also straightforward to show that they are equidistant, i.e.,

$$d^2\left(v^{(k)}, v^{(j)}\right) = v_{k1}^2 + v_{k2}^2 + v_{j1}^2 + v_{j2}^2 - 2\left(v_{k1}v_{j1} + v_{k2}v_{j2}\right) = 6.$$

Therefore $v^{(k)}$ are the vertices of a regular triangle. One can see that the nodes of quadrature formula with equal weights are $u^{(k)} = T^{-1}(v^{(k)})(k = 1, 2, 3)$, where T^{-1} is the inverse mapping of T. Let

$$v^{(k)} = \sqrt{2}\left(\cos\theta_k, \sin\theta_k\right), \quad \theta_k = \theta + 2(k-1)\pi/3, \quad k = 1, 2, 3,$$

where θ is a free parameter, then the quadrature points over $\tilde{Q}$ can be written as $u^{(k)} = (\xi_k, \eta_k)$, $k = 1, 2, 3$ with

$$\begin{aligned}\xi_k &= \sqrt{2C_1}\cos\theta_k + \frac{1}{3}(1+h_1),\\ \eta_k &= \frac{\sqrt{2C_2}\sin\theta_k - \sqrt{2C_1}\,(1+h_1)(1+h_2)\cos\theta_k}{2\left(1-h_1+h_1^2\right)} + \frac{1}{3}\,(1+h_2)\,.\end{aligned} \tag{5}$$

4 Numerical Examples

In this section, two types of quadrilateral meshes are employed: uniformly trapezoid meshes as shown in Fig. 3 and the randomly perturbed quadrilateral meshes depicted in Fig. 4.

4.1 The Elliptic Problem

Consider a second-order elliptic problem:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

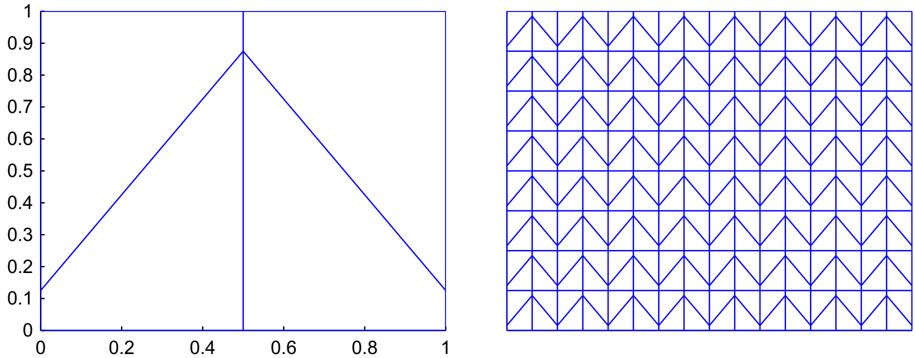

Fig. 3 **a** A partition of the square into four trapezoids. **b** A mesh composed of translated dilates of this partition

Fig. 4 A nonuniform randomly perturbed quadrilateral triangulation

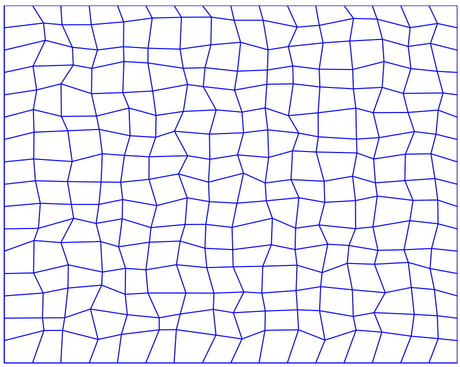

Table 1 Computational results on the uniformly trapezoid meshes for the elliptic problem

	h	1/4	1/8	1/16	1/32	1/64	1/128
2×2 Gauss formula	$\|u - u_h\|_1$	1.37E-0	8.66E-1	4.66E-1	2.42E-1	1.24E-1	6.24E-2
	Ratio		0.66	0.89	0.94	0.97	0.99
	$\\|u - u_h\\|_0$	7.38E-2	2.55E-2	7.68E-3	2.18E-3	5.83E-4	1.51E-4
	Ratio		1.53	1.73	1.82	1.90	1.95
Our formula	$\|u - u_h\|_1$	1.65E-0	8.86E-1	4.68E-1	2.43E-1	1.24E-1	6.24E-2
	Ratio		0.90	0.92	0.95	0.97	0.99
	$\\|u - u_h\\|_0$	5.96E-2	2.19E-2	6.62E-3	1.87E-3	5.00E-4	1.30E-4
	Ratio		1.44	1.73	1.83	1.90	1.95

Table 2 Computational results on the randomly perturbed meshes for the elliptic problem

	h	1/4	1/8	1/16	1/32	1/64	1/128
2×2 Gauss formula	$\|u - u_h\|_1$	1.28E-0	6.67E-1	3.63E-1	1.88E-1	9.58E-2	4.85E-2
	Ratio		0.94	0.88	0.95	0.98	0.98
	$\\|u - u_h\\|_0$	5.07E-2	1.24E-2	3.76E-3	9.85E-4	2.56E-4	6.58E-5
	Ratio		2.04	1.72	1.93	1.94	1.96
Our formula	$\|u - u_h\|_1$	1.39E-0	6.92E-1	3.75E-1	1.89E-1	9.65E-2	4.85E-2
	Ratio		1.01	0.89	0.99	0.97	0.99
	$\\|u - u_h\\|_0$	3.91E-2	1.02E-2	3.00E-3	8.00E-4	2.13E-4	5.35E-5
	Ratio		1.94	1.74	1.92	1.91	1.99

where $\Omega = (0, 1)^2$. The source term f is generated from the exact solution

$$u(x, y) = \sin(2\pi x)\sin(2\pi y)\left(x^3 - y^4 + x^2 y^3\right).$$

The entries of the stiffness matrix are computed by the 2×2-Gauss rule and our formula, respectively. Here we take $\theta = 0$ in Eq. (5). Table 1 presents the results on a uniform trapezoidal meshes. Similarly, Table 2 shows the numerical results on the randomly perturbed meshes. We observe the optimal convergence rates of $O(h)$ and $O(h^2)$ in energy norm and L^2 norm, respectively. In view of errors, there is no big difference in the energy norm between our formula and 2×2-Gauss rule. But the errors obtained by our formula is smaller than that by 2×2-Gauss rule in the L^2 norm. In view of computational cost, our formula is more efficient when the proposed element is employed, since the Jacobian determinant is constant and less quadrature points are used.

4.2 Numerical Examples for the Stokes Problem

In this subsection, we apply our element to approximate each component of the velocity fields in solving the incompressible Stokes equations in two dimensions, while the piecewise constant element is employed to approximate the pressure.

Table 3 The Stokes problem: The apparent L^2 and broken energy norm errors and their reduction ratios on the uniformly trapezoid meshes

h	$\lvert \boldsymbol{u}-\boldsymbol{u}_h\rvert_1$	Ratio	$\lVert \boldsymbol{u}-\boldsymbol{u}_h\rVert_0$	Ratio	$\lVert p-p_h\rVert$	Ratio
1/2	4.94E-1	–	2.58E-2	–	4.46E-1	–
1/4	2.82E-1	0.81	1.55E-2	0.74	4.14E-1	0.11
1/8	2.17E-1	0.38	8.89E-3	0.80	2.08E-1	0.99
1/16	1.20E-1	0.85	2.77E-3	1.68	9.90E-2	1.07
1/32	6.24E-2	0.95	7.52E-4	1.88	5.05E-2	0.97
1/64	3.16E-2	0.98	1.93E-4	1.96	2.50E-2	1.02
1/128	1.59E-2	0.99	4.88E-5	1.99	1.23E-2	1.02

Table 4 The Stokes problem: The apparent L^2 and broken energy norm errors and their reduction ratios on the randomly perturbed meshes

h	$\lvert \boldsymbol{u}-\boldsymbol{u}_h\rvert_1$	Ratio	$\lVert \boldsymbol{u}-\boldsymbol{u}_h\rVert_0$	Ratio	$\lVert p-p_h\rVert$	Ratio
1/2	4.06E-1	–	3.49E-2	–	3.70E-1	–
1/4	3.00E-1	0.43	1.83E-2	0.93	3.40E-1	0.13
1/8	1.56E-1	0.95	4.44E-3	2.04	2.14E-1	0.66
1/16	8.25E-2	0.92	1.21E-3	1.87	9.35E-2	1.20
1/32	4.21E-2	0.97	3.09E-4	1.97	4.38E-2	1.09
1/64	2.15E-2	0.97	7.99E-5	1.95	2.12E-2	1.05
1/128	1.08E-2	0.99	2.02E-5	1.99	1.06E-2	1.00

Set $\Omega = (0,1)^2$ and the Stokes equations are given by

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

The exact solution for $\boldsymbol{u}$ is given by $\nabla \times \psi$, where

$$\psi(x,y) = e^{(x+2y)} x^2 (x-1)^2 y^2 (y-1)^2.$$

The exact solution for p is given by

$$p(x,y) = \sin(2\pi x)\sin(2\pi y).$$

Then the body force term $\boldsymbol{f}$ can be generated by $-\Delta\boldsymbol{u} + \nabla p$. Table 3 presents the results on a uniform trapezoidal meshes. Similarly, Table 4 shows the numerical results on the randomly perturbed meshes.

5 Conclusion

We developed a new nonparametric nonconforming finite element method that can be used on general quadrilateral meshes. The new element has at least two advantages compared to other nonparametric elements in the literature. Firstly, for other nonparametric elements, one must solve at least four linear systems to get the basis functions. However the basis functions of our element can be expressed explicitly. Secondly, the integration can be evaluated like the parametric element and only linear maps are required, which implies that the Jacobi determinants are constants. Therefore, we can construct an efficient quadrature formula with less quadrature points for the computation. Moreover, the shape functions are just second-degree polynomials and hence quadrature rule with second degree exactness is enough to achieve optimal convergence rates, whereas for most other other nonconforming elements, at least 2×2-Gauss rule is required.

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