# A new analysis of discontinuous Galerkin methods for a fourth order variational inequality 

Jintao Cui ${ }^{\text {a,* }}$, Yi Zhang ${ }^{\text {b }}$<br>${ }^{\text {a }}$ Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong<br>${ }^{\mathrm{b}}$ The University of North Carolina at Greensboro, Greensboro, NC 27402, USA

Received 6 December 2018; received in revised form 2 April 2019; accepted 3 April 2019
Available online 10 April 2019


#### Abstract

We study a family of discontinuous Galerkin methods for the displacement obstacle problem of Kirchhoff plates on two and three dimensional convex polyhedral domains, which are characterized as fourth order elliptic variational inequalities of the first kind. We prove that the error in an $H^{2}$-like energy norm is $O\left(h^{\alpha}\right)$ for the quadratic method, where $\alpha \in\left(\frac{1}{2}, 1\right]$ is determined by the geometry of the domain. Under additional assumptions on the contact set such that the solution has improved regularity, we derive the optimal error estimate with $\alpha \in\left(1, \frac{3}{2}\right)$ for the cubic method. Numerical experiments demonstrate the performance of the methods and confirm the theoretical results.


(C) 2019 Elsevier B.V. All rights reserved.

MSC: 65N15; 65N30; 35J20
Keywords: Displacement obstacle; Fourth order variational inequality; Discontinuous Galerkin methods; Error estimate

## 1. Introduction

Let $\Omega \subset \mathbb{R}^{d}(d=2$ or 3$)$ be a bounded polyhedral domain, $f \in L_{2}(\Omega), g \in H^{4}(\Omega)$ and $\psi_{1}, \psi_{2} \in C^{2}(\Omega) \cap C(\bar{\Omega})$ such that

$$
\begin{equation*}
\psi_{1}<\psi_{2} \quad \text { in } \Omega \quad \text { and } \quad \psi_{1}<g<\psi_{2} \quad \text { on } \partial \Omega \tag{1.1}
\end{equation*}
$$

In this paper, we consider the following displacement obstacle problem: Find $u \in K$ such that

$$
\begin{equation*}
u=\underset{v \in K}{\operatorname{argmin}}\left[\frac{1}{2} a(v, v)-(f, v)\right] \tag{1.2}
\end{equation*}
$$

where

$$
\begin{align*}
& K=\left\{v \in H^{2}(\Omega): v-g \in H_{0}^{2}(\Omega), \psi_{1} \leq v \leq \psi_{2} \text { in } \Omega\right\}  \tag{1.3}\\
& a(w, v)=\int_{\Omega}(\Delta w)(\Delta v) d x \quad \text { and } \quad(f, v)=\int_{\Omega} f v d x \tag{1.4}
\end{align*}
$$

[^0]In (1.2), $u$ is the vertical displacement of the midsurface of the clamped Kirchhoff plate and $f$ denotes the vertical load density divided by the flexural rigidity of the plate. Since $K$ is a nonempty closed convex subset of $H^{2}(\Omega)$, and $a(\cdot, \cdot)$ is symmetric bounded on $H^{2}(\Omega)$ and coercive on $K-K \subseteq H_{0}^{2}(\Omega)$, it follows from the standard theory [1-4] that the problem (1.2)-(1.4) has a unique solution, which is also characterized by the variational inequality of the first kind (cf. [5]):

$$
\begin{equation*}
a(u, v-u) \geq(f, v-u) \quad \forall v \in K \tag{1.5}
\end{equation*}
$$

Finite element methods for second order obstacle problems where $a(w, v)=\int_{\Omega} \nabla w \cdot \nabla v d x$ in (1.4) have been investigated in [6-9] and references therein. The key ingredient of the analysis is the use of the strong complementarity form of variational inequality which is valid since $u \in H^{2}(\Omega)$ under appropriate conditions (cf. [10]). However for the obstacle problem (1.2), it was shown in [4,11-14] that $u \in H_{l o c}^{3}(\Omega) \cap W_{l o c}^{2, \infty}(\Omega) \cap C^{2}(\Omega)$ under our assumptions on data. Because of (1.1), $u$ is unconstrained near $\partial \Omega$ and thus $\Delta^{2} u=f$ in a neighborhood of $\partial \Omega$. Therefore the elliptic regularity theory [15-18] implies that $u$ belongs to $H^{2+\alpha}$ near $\partial \Omega$, where $\alpha \in(1 / 2,1]$ denotes the index of elliptic regularity depending on the geometry of $\Omega$. In the case of convex domain, we can take $\alpha=1$. Nevertheless, the lack of full regularity is the main challenge of numerical analysis for the problem (1.2).

In [14], a unified convergence analysis for conforming, nonconforming finite element methods and $C^{0}$ interior penalty methods was developed and extensions to more general cases and other finite element methods can also be found in [19-22]. In these works, an intermediate obstacle problem is introduced to connect the continuous and discrete obstacle problems. However, when we extend the analysis to three dimensions, only the suboptimal error estimate can be obtained. Furthermore, in these methods, finite element spaces need to be continuous at the vertices of triangulation which limits the practical applications.

The goal of this paper is to study fully discontinuous Galerkin (DG) methods [23,24] for the problem (1.2)/(1.5). This is motivated by the advantages of DG methods in regard to naturally handling nonhomogeneous boundary conditions, designing adaptive meshes and parallel computing. In particular, we develop a unified approach for a family of DG methods where only the weak complementarity form of the variational inequality is used, without using an intermediate problem as developed in [14,19,21,22]. In [25,26], a similar technique had been applied to elliptic optimal control problems with pointwise state constraints. In this paper, we prove optimal error estimates for both quadratic and cubic fully DG methods on two and three dimensions with nonhomogeneous boundary conditions. To the best of our knowledge, the analysis for the cubic method is new and this is the first paper that provides rigorous error analysis of fully DG methods for fourth order variational inequality of the first kind.

The rest of the paper is organized as follows. In Section 2, we introduce a family of DG methods for (1.5). In Section 3, we derive the convergence analysis with help of the weak complementarity condition (cf. Section 3.1). Numerical examples are presented in Section 4 to illustrate the performance of the DG methods. We end the paper with some concluding remarks in Section 5.

## 2. Discontinuous Galerkin methods

### 2.1. Notation

Let $\mathcal{T}_{h}$ be a shape-regular triangulation of $\Omega$ with mesh size $h$ without hanging nodes. We will use the following notation throughout the paper.

- $T$ : a triangle of $\mathcal{T}_{h}$ in two dimensions and a tetrahedron of $\mathcal{T}_{h}$ in three dimensions.
- $\mathcal{V}_{h}$ : the set of the vertices of $\mathcal{T}_{h}$.
- $\mathcal{V}_{h}^{i}$ : the set of all interior vertices of $\mathcal{T}_{h}$.
- $\mathcal{V}_{T}$ : the set of the vertices of $T \in \mathcal{T}_{h}$.
- $\mathcal{E}_{h}^{i}$ : the set of all interior edges/faces of $\mathcal{T}_{h}$.
- $\mathcal{E}_{h}^{b}$ : the set of all boundary edges/faces of $\mathcal{T}_{h}$.
- $\mathcal{E}_{h}=\mathcal{E}_{h}^{i} \cup \mathcal{E}_{h}^{b}$ : the set of all edges/faces of $\mathcal{T}_{h}$.
- $\mathcal{E}_{\mathcal{V}_{T}}$ : the set of edges/faces emanating from the vertices of $T \in \mathcal{T}_{h}$.
- $h_{e}$ : the length of the edge $e \in \mathcal{E}_{h}$.
- $H^{s}\left(\Omega, \mathcal{T}_{h}\right)=\left\{v \in L_{2}(\Omega): v_{T}=\left.v\right|_{T} \in H^{s}(T) \quad \forall T \in \mathcal{T}_{h}\right\}$ : piecewise Sobolev space.
- $V_{h}=\left\{v \in L_{2}(\Omega): v_{T} \in P_{r}(T) \quad \forall T \in \mathcal{T}_{h}\right\}$ : the discontinuous finite element space associated with $\mathcal{T}_{h}$, where $r=2$ or 3 .
We will also denote by $C$ a generic positive constant independent of mesh sizes that can take different values at different occurrences. Next, we introduce average and jump operators that are needed in the construction and analysis of DG methods. Let $e \in \mathcal{E}_{h}^{i}$, then $e=\partial T^{+} \cap \partial T^{-}$for some $T^{+}, T^{-} \in \mathcal{T}_{h}$. We define the jump $\llbracket v \rrbracket$ and [ $\nabla v$ ] on $e$ as

$$
\llbracket v \rrbracket=v^{+} n^{+}+v^{-} n^{-} \quad \text { and } \quad[\nabla v]=\nabla v^{+} \cdot n^{+}+\nabla v^{-} \cdot n^{-}
$$

for any $v \in H^{s}\left(\Omega, \mathcal{T}_{h}\right)$ with $s>3 / 2$, where $v^{ \pm}=\left.v\right|_{T^{ \pm}}$, and $n^{+}$(resp., $n^{-}$) is the unit outward normal on $e$ corresponding to $\partial T^{+}$(resp., $\partial T^{-}$). The average $\{\Delta v\}$ and $\left.\{\nabla \Delta v\}\right\}$ on $e$ are defined by

$$
\left.\{\Delta v\}=\frac{1}{2}\left(\Delta v^{+}+\Delta v^{-}\right) \quad \text { and } \quad\{\nabla \Delta v\}\right\}=\frac{1}{2}\left(\nabla \Delta v^{+}+\nabla \Delta v^{-}\right)
$$

for any $v \in H^{s}\left(\Omega, \mathcal{T}_{h}\right)$ with $s>7 / 2$. For $e \in \mathcal{E}_{h}^{b}$, we denote $T^{+}$the triangle that contains $e$ and define

$$
\llbracket v \rrbracket=v^{+} n^{+}, \quad[\nabla v]=\nabla v^{+} \cdot n^{+}, \quad\{\Delta v\}=\Delta v^{+}, \quad\{\nabla \Delta v\}=\nabla \Delta v^{+} .
$$

Let $T \in \mathcal{T}_{h}, v \in P_{r}(T)$ and $w \in H^{2}(T)$, we have an integration by parts formula

$$
\begin{aligned}
\int_{T}(\Delta v)(\Delta w) d x & =\int_{T}\left(\Delta^{2} v\right) w d x-\int_{\partial T} \nabla(\Delta v) \cdot w n d s+\int_{\partial T} \Delta v \nabla w \cdot n d s \\
& =-\int_{\partial T} \nabla(\Delta v) \cdot w n d s+\int_{\partial T} \Delta v \nabla w \cdot n d s
\end{aligned}
$$

where $n$ is the unit normal vector to $\partial T$. Summing up over all $T \in \mathcal{T}_{h}$, we obtain

$$
\begin{align*}
& \sum_{T \in \mathcal{T}_{h}} \int_{T}(\Delta v)(\Delta w) d x=\sum_{T \in \mathcal{T}_{h}} \int_{T}\left(\Delta^{2} v\right) w d x-\sum_{e \in \mathcal{E}_{h}} \int_{e}\{\| \nabla(\Delta v)\} \llbracket \llbracket w \rrbracket d s \\
&-\sum_{e \in \mathcal{E}_{h}^{i}} \int_{e}\{w\}[\nabla(\Delta v)] d s+\sum_{e \in \mathcal{E}_{h}} \int_{e}\{\Delta v\}[\nabla w] d s \\
&+\sum_{e \in \mathcal{E}_{h}^{i}} \int_{e}\{\nabla w\} \llbracket \llbracket \Delta v \rrbracket d s, \tag{2.1}
\end{align*}
$$

for all $v \in H^{4}\left(\Omega, \mathcal{T}_{h}\right)$ and $w \in H^{2}\left(\Omega, \mathcal{T}_{h}\right)$.

### 2.2. Discontinuous Galerkin methods

Let

$$
\begin{equation*}
K_{h}=\left\{v \in V_{h}: \psi_{1}(p) \leq v_{T}(p) \leq \psi_{2}(p) \quad \forall p \in \mathcal{V}_{T}, T \in \mathcal{T}_{h}\right\} . \tag{2.2}
\end{equation*}
$$

The discrete variational inequality for (1.5) is: Find $u_{h} \in K_{h}$ such that

$$
\begin{equation*}
a_{h}\left(u_{h}, v-u_{h}\right) \geq F\left(v-u_{h}\right) \quad \forall v \in K_{h}, \tag{2.3}
\end{equation*}
$$

where

$$
\begin{align*}
a_{h}(w, v)= & \sum_{T \in \mathcal{T}_{h}} \int_{T}(\Delta w)(\Delta v) d x+\sum_{e \in \mathcal{E}_{h}} \int_{e}\left\{\llbracket \Delta w \rrbracket \llbracket v \rrbracket d s-\sum_{e \in \mathcal{E}_{h}} \int_{e}\{\Delta w\}[\nabla v] d s\right. \\
& +\lambda_{1} \sum_{e \in \mathcal{E}_{h}} \int_{e}\{\nabla \Delta v\} \rrbracket \llbracket w d s-\lambda_{2} \sum_{e \in \mathcal{E}_{h}} \int_{e}\{\Delta v\}[\nabla w] d s  \tag{2.4}\\
& +\sum_{e \in \mathcal{E}_{h}} \frac{\sigma_{1}}{h_{e}^{3}} \int_{e} \llbracket w \rrbracket \llbracket v \rrbracket d s+\sum_{e \in \mathcal{E}_{h}} \frac{\sigma_{2}}{h_{e}} \int_{e}[\nabla w][\nabla v] d s,
\end{align*}
$$

Table 1
DG methods.

| Method | $\lambda_{1}$ | $\lambda_{2}$ | $\sigma_{1}$ | $\sigma_{2}$ |
| :--- | :--- | :--- | :--- | :--- |
| SIPG | 1 | 1 | $\sigma_{1}>\sigma^{*}$ | $\sigma_{2}>\sigma^{*}$ |
| NIPG | -1 | -1 | $\sigma_{1}>0$ | $\sigma_{2}>0$ |
| SSIPG1 | -1 | 1 | $\sigma_{1}>0$ | $\sigma_{2}>\sigma^{*}$ |
| SSIPG2 | 1 | -1 | $\sigma_{1}>\sigma^{*}$ | $\sigma_{2}>0$ |

$$
\begin{align*}
& F(v)=(f, v)+\tilde{F}(v),  \tag{2.5}\\
& \tilde{F}(v)=\sum_{e \in \mathcal{E}_{h}^{b}} \int_{e}\left(\lambda_{1}\{\nabla \Delta v\}+\frac{\sigma_{1}}{h_{e}^{3}} \llbracket v \rrbracket\right) \llbracket g \rrbracket d s \\
&  \tag{2.6}\\
& \quad+\sum_{e \in \mathcal{E}_{h}^{b}} \int_{e}\left(\frac{\sigma_{2}}{h_{e}}[\nabla v]-\lambda_{2}\{\Delta v\}\right)[\nabla g] d s .
\end{align*}
$$

In view of (1.3), (2.2), (2.4)-(2.6), the boundary conditions are weakly imposed in the DG schemes.
The different choices for $\lambda_{1}, \lambda_{2}, \sigma_{1}$ and $\sigma_{2}$ lead to four different DG methods [23], which are SIPG method, NIPG methods and their combinations: the semisymmetric methods SSIPG1 and SSIPG2 (cf. Table 1). Here $\sigma^{*}$ is a sufficiently large positive number that depend on the constants in the inverse inequalities, the degree of polynomial and the shape regularity of $\mathcal{T}_{h}$ [23].

To measure the error, we define a mesh-dependent energy norm by

$$
\begin{equation*}
\|v\|_{h}^{2}=\sum_{T \in \mathcal{T}_{h}}\|\Delta v\|_{L_{2}(T)}^{2}+\sum_{e \in \mathcal{E}}\left(\frac{\sigma_{1}}{h_{e}^{3}}\|\llbracket v \rrbracket\|_{L_{2}(e)}^{2}+\frac{\sigma_{2}}{h_{e}}\|[\nabla v]\|_{L_{2}(e)}^{2}\right) \quad \forall v \in V_{h} \tag{2.7}
\end{equation*}
$$

By the standard inverse estimate $[8,27]$, the bilinear form $a_{h}(\cdot, \cdot)$ is bounded and coercive with respect to $\|\cdot\|_{h}$ provided that $\sigma_{1}$ and $\sigma_{2}$ are chosen according to Table 1 for different DG methods, i.e.,

$$
\begin{array}{ll}
a_{h}(w, v) \leq C\|w\|_{h}\|v\|_{h} & \forall w, v \in V_{h} \\
a_{h}(v, v) \geq C\|v\|_{h}^{2} & \forall v \in V_{h} \tag{2.9}
\end{array}
$$

In particular for the NIPG method, the coercivity estimate (2.9) becomes

$$
a_{h}(v, v)=\|v\|_{h}^{2} \quad \forall v \in V_{h}
$$

Let $\Pi_{h}: H^{2}(\Omega) \longrightarrow V_{h} \cap H^{1}(\Omega)$ be the nodal interpolation operator for the $P_{r}(r=2$ or 3$)$ element such that

$$
\begin{equation*}
\left.\left(\Pi_{h} \zeta\right)\right|_{T}(p)=\zeta(p) \quad \forall p \in \mathcal{V}_{T}, T \in \mathcal{T}_{h} \tag{2.10}
\end{equation*}
$$

We have the following standard interpolation error estimate [8,27]

$$
\begin{align*}
& \left\|\zeta-\Pi_{h} \zeta\right\|_{L_{2}(\Omega)}+h\left\|\zeta-\Pi_{h} \zeta\right\|_{H^{1}(\Omega)}+h^{2}\left\|\zeta-\Pi_{h} \zeta\right\|_{h} \\
& \quad \leq C h^{\min (r+1, s)}\|\zeta\|_{H^{s}(\Omega)} \quad \forall \zeta \in H^{s}(\Omega), s \geq 2 \tag{2.11}
\end{align*}
$$

For $r=3$, we also have

$$
\begin{equation*}
\left|\zeta-\Pi_{h} \zeta\right|_{H^{3}\left(\mathcal{T}_{h}\right)} \leq C h_{T}^{\min (1, s-3)}\|\zeta\|_{H^{s}(T)} \quad \forall \zeta \in H^{s}(T), \quad T \in \mathcal{T}_{h}, s \geq 3 \tag{2.12}
\end{equation*}
$$

where $h_{T}=\operatorname{diam}(T)$.
Since $\Pi_{h} u \in K_{h}$, the set $K_{h}$ is nonempty closed convex in $V_{h}$. It then follows from (2.9) that the discrete problem (2.3) has a unique solution.

### 2.3. Enriching operator

Since the DG space $V_{h} \not \subset H^{2}(\Omega)$, we need an enriching operator [14,19,21,22] to measure the difference between the finite element space $V_{h}$ and the Sobolev space $H^{2}(\Omega)$. For simplicity, we consider two dimensional case.


Fig. 1. Degrees of freedom for the Hsieh-Clough-Tocher macro element.

Let $\mathcal{T}_{h}$ be simplicial triangulation of a polygonal domain $\Omega \subset \mathbb{R}^{2}$. We consider a linear operator

$$
E_{h}: V_{h} \longrightarrow W_{h} \cap H_{0}^{2}(\Omega),
$$

where $W_{h}$ is the Hsieh-Clough-Tocher macro finite element space [8,27]. For $v \in V_{h}$, we define $E_{h} v$ by specifying its degrees of freedom (dofs), which are the values of the derivatives up to order 1 at the vertices, and the values of the normal derivative at the midpoints of the edges (cf. Fig. 1). The dofs of $E_{h} v$ at any interior nodal point of $V_{h}$ are defined to be the average of the corresponding dofs of $v$ from the triangles of $\mathcal{T}_{h}$ sharing the nodal point. For the boundary nodal point, we set the dofs of $E_{h} v$ to be 0 .

For any $v \in K_{h}$, we have by (2.2) and the definition of $E_{h}$ that

$$
\begin{equation*}
\psi_{1}(p) \leq E_{h} v(p) \leq \psi_{2}(p) \quad \forall p \in \mathcal{V}_{h}^{i} \tag{2.13}
\end{equation*}
$$

where $\mathcal{V}_{h}^{i}=\mathcal{V}_{h} \cap \Omega$. In particular,

$$
E_{h} v(p)=v(p) \quad \forall p \in \mathcal{V}_{h}^{i}, v \in V_{h} \cap H^{1}(\Omega) .
$$

For any $v \in V_{h}$ and $T \in \mathcal{T}_{h}$, the following local approximation property can be proved similar to Lemma 4.1 in [28]:

$$
\begin{equation*}
\sum_{m=0}^{2} h_{T}^{2 m}\left|v-E_{h} v\right|_{H^{m}(T)}^{2} \leq C h_{T}^{4} \sum_{e \in \mathcal{E}_{\mathcal{V}(T)}}\left(\frac{1}{h_{e}^{3}}\|\llbracket v\|\left\|_{L_{2}(e)}^{2}+\frac{1}{h_{e}}\right\|[\nabla v] \|_{L_{2}(e)}^{2}\right) . \tag{2.14}
\end{equation*}
$$

From (2.14) and the standard inverse inequality [8,27], we have the global estimates

$$
\begin{align*}
& \left\|v-E_{h} v\right\|_{L_{2}(\Omega)}^{2}+h^{2} \sum_{T \in \mathcal{T}_{h}}\left|v-E_{h} v\right|_{H^{1}(T)}^{2}+h^{4}\left\|v-E_{h} v\right\|_{h}^{2} \\
& \leq C h^{4} \sum_{e \in \mathcal{E}_{h}}\left(h_{e}^{-3}\|\llbracket v \rrbracket\|_{L_{2}(e)}^{2}+h_{e}^{-1}\|[\nabla v]\|_{L_{2}(e)}^{2}\right) \leq C h^{4}\|v\|_{h},  \tag{2.15}\\
& \left.\sum_{e \in \mathcal{E}_{h}}\left(h_{e}^{-1} \|\left\{\nabla\left(v-E_{h} v\right)\right\}\right\}\left\|_{L_{2}(e)}^{2}+h_{e}^{-3}\right\|\left\{v-E_{h} v\right\} \|_{L_{2}(e)}^{2}\right) \\
& \leq C \sum_{e \in \mathcal{E}_{h}}\left(h_{e}^{-3}\|\llbracket v \rrbracket\|_{L_{2}(e)}^{2}+h_{e}^{-1}\|[\nabla v]\|_{L_{2}(e)}^{2}\right) \leq C\|v\|_{h}, \tag{2.16}
\end{align*}
$$

for all $v \in V_{h}$.
Furthermore, (2.11), (2.15) and the standard inverse inequality imply that

$$
\begin{align*}
& \left\|\zeta-E_{h} \Pi_{h} \zeta\right\|_{L_{2}(\Omega)}+h^{2-\beta}\left\|\zeta-E_{h} \Pi_{h} \zeta\right\|_{H^{2-\beta}(\Omega)} \\
& \quad+h\left|\zeta-E_{h} \Pi_{h} \zeta\right|_{H^{1}(\Omega)}+h^{2}\left|\zeta-E_{h} \Pi_{h} \zeta\right|_{H^{2}(\Omega)} \\
& \leq C h^{\min (r+1, s)}\|\zeta\|_{H^{s}(\Omega)} \quad \forall \zeta \in H^{s}(\Omega) \cap H_{0}^{2}(\Omega), \tag{2.17}
\end{align*}
$$

for any $\beta \in(1,2)$.

Remark 2.1. The enriching operator can also be constructed in three dimensions where the Hsieh-Clough-Tocher macro finite element space is replaced by the Ženíšek finite element space [29].

In order to deal with the nonhomogeneous boundary conditions, we consider an affine operator $T_{h}: V_{h} \longrightarrow$ $H^{2}(\Omega)$ defined by

$$
\begin{equation*}
T_{h} v=g+E_{h}\left(v-\Pi_{h} g\right) \quad \forall v \in V_{h} \tag{2.18}
\end{equation*}
$$

It follows from (2.10) and (2.18) that

$$
\begin{equation*}
u-T_{h} \Pi_{h} u=(u-g)-E_{h} \Pi_{h}(u-g) . \tag{2.19}
\end{equation*}
$$

Therefore we can take $\zeta=u-g$ in (2.17) since $u-g \in H^{s}(\Omega) \cap H_{0}^{2}(\Omega)$. Furthermore,

$$
\begin{equation*}
T_{h} \Pi_{h} u-T_{h} v=E_{h}\left(\Pi_{h} u-v\right) \tag{2.20}
\end{equation*}
$$

Since for all $v \in V_{h}$,

$$
\begin{align*}
& \left(T_{h} v\right)(p)=g(p)+E_{h}\left(v-\Pi_{h} g\right)(p)=\left(E_{h} v\right)(p) \quad \forall p \in \mathcal{V}_{h}^{i},  \tag{2.21}\\
& \left(T_{h} v\right)(p)=g(p) \quad \forall p \in \mathcal{V}_{h} \cap \partial \Omega, \tag{2.22}
\end{align*}
$$

we have by (2.13),

$$
\begin{equation*}
\psi_{1}(p) \leq T_{h} v(p) \leq \psi_{2}(p) \quad \forall p \in \mathcal{V}_{h} \tag{2.23}
\end{equation*}
$$

for all $v \in K_{h}$.

## 3. Convergence analysis

In this section, we will prove the optimal error estimate for $\left\|u-u_{h}\right\|_{h}$ in both two and three dimensions. First of all, we introduce the complementarity form of the variational inequality (1.5) that will be crucial for the error analysis.

### 3.1. Complementarity form of the variational inequality

Let the contact set defined by

$$
\mathcal{A}_{i}=\left\{x \in \Omega: u(x)=\psi_{i}(x)\right\} \quad i=1,2 .
$$

Due to (1.1), we know $\mathcal{A}_{1} \cap \mathcal{A}_{2}=\emptyset$ and they are both compact and disjoint from $\partial \Omega$. By the Riesz representation theorem [30,31], there exists a Borel measure $\mu$ such that

$$
\begin{equation*}
a(u, v)-(f, v)=\int_{\Omega} v d \mu \quad \forall v \in H_{0}^{2}(\Omega), \tag{3.1}
\end{equation*}
$$

where $\mu=\mu_{1}-\mu_{2}$ is the Jordan decomposition of $\mu$. In particular, $\mu_{1}$ and $\mu_{2}$ are nonnegative Borel measures that concentrate on $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$, respectively, i.e.,

$$
\begin{equation*}
\mu_{1}\left(\Omega \backslash \mathcal{A}_{1}\right)=\mu_{2}\left(\Omega \backslash \mathcal{A}_{2}\right)=0 \tag{3.2}
\end{equation*}
$$

Furthermore, $\mu_{i}(\Omega)=\mu_{i}\left(\mathcal{A}_{i}\right)<\infty(i=1,2)$ and the solution $u$ to (1.5) satisfies the following weak complementarity conditions:

$$
\begin{equation*}
\int_{\Omega}\left(u-\psi_{i}\right) d \mu_{i}=0 \quad i=1,2 \tag{3.3}
\end{equation*}
$$

We denote by $|\mu|$ the total variation measure of $\mu$, then

$$
\begin{equation*}
|\mu|(\Omega)=\mu_{1}(\Omega)+\mu_{2}(\Omega)<\infty \tag{3.4}
\end{equation*}
$$

Let $\mathcal{G}$ be an open neighborhood of $\mathcal{A}:=\mathcal{A}_{1} \cup \mathcal{A}_{2}$ with a smooth boundary such that $\overline{\mathcal{G}}$ is a compact subset of $\Omega$, and let $\phi \in C_{c}^{\infty}(\Omega)$ with compact support in $\mathcal{G}$ and $\phi=1$ in $\mathcal{A}$. Since $u \in H_{l o c}^{3}(\Omega)$, we have by (3.1) and integration by parts that

$$
\begin{align*}
\int_{\Omega} v d \mu=\int_{\Omega}(\phi v) d \mu & =-\int_{\Omega} \nabla(\Delta u) \cdot \nabla(v \phi) d x-(f, v \phi) \\
& =B(u, v)-(f, v \phi) \quad \forall v \in H_{0}^{2}(\Omega), \tag{3.5}
\end{align*}
$$

where $B(\zeta, w)=-\int_{\Omega} \nabla(\Delta \zeta) \cdot \nabla(w \phi) d x$. We also have

$$
\begin{align*}
|B(\zeta, w)| & \leq\|\nabla(\Delta \zeta)\|_{L_{2}(\Omega)}\|\nabla(w \phi)\|_{L_{2}(\Omega)}  \tag{3.6}\\
& \leq C_{\mathcal{G}}\|\zeta\|_{H^{3}(\mathcal{G})}\|w\|_{H^{1}(\mathcal{G})} \quad \forall \zeta \in H^{3}(\mathcal{G}), w \in H^{1}(\mathcal{G}),
\end{align*}
$$

by the Cauchy-Schwarz inequality. Now we combine (3.5) and (3.6) to get

$$
\begin{equation*}
\left|\int_{\Omega} v d \mu\right| \leq\left(C_{\mathcal{G}}\|u\|_{H^{3}(\mathcal{G})}+\|f\|_{L_{2}(\mathcal{G})}\right)\|v\|_{H^{1}(\mathcal{G})} \quad \forall v \in H_{0}^{2}(\Omega), \tag{3.7}
\end{equation*}
$$

which together with a density argument implies $\mu \in H^{-1}(\Omega)=\left[H_{0}^{1}(\Omega)\right]^{\prime}$.

### 3.2. Error estimate for $\left\|u-u_{h}\right\|_{h}$

We begin by a useful technical lemma that holds for all four DG methods proposed in Section 2.2.
Lemma 3.1. There exists a positive constant $C$, depending on the shape regularity of $\mathcal{T}_{h}$, such that

$$
\begin{equation*}
a_{h}\left(\Pi_{h} \zeta, w\right)-\tilde{F}(w)-a\left(\zeta, E_{h} w\right) \leq C h^{\min (r+1, s)-2}\|\zeta\|_{H^{s}(\Omega)}\|w\|_{h}, \tag{3.8}
\end{equation*}
$$

for all $\zeta \in H^{s}(\Omega), s \in[2,4]$ with $\zeta-g \in H_{0}^{2}(\Omega)$ and all $w \in V_{h}$.
Proof. Since $E_{h} w \in H_{0}^{2}(\Omega)$ for any $w \in V_{h}$, by (2.1), (2.4) and (2.6) we have

$$
\begin{align*}
& a_{h}\left(\Pi_{h} \zeta, w\right)-\tilde{F}(w)-a\left(\zeta, E_{h} w\right)=\sum_{T \in \mathcal{T}_{h}} \int_{T} \Delta\left(\Pi_{h} \zeta-\zeta\right) \Delta\left(E_{h} w\right) d x \\
& \quad+\lambda_{1} \sum_{e \in \mathcal{E}_{h}} \int_{e} \llbracket \nabla \Delta w \rrbracket \llbracket \Pi_{h} \zeta-\zeta \rrbracket d s-\lambda_{2} \sum_{e \in \mathcal{E}_{h}} \int_{e}\{\Delta w\}\left[\nabla\left(\Pi_{h} \zeta-\zeta\right)\right] d s \\
& \quad+\sum_{e \in \mathcal{E}_{h}} \frac{\sigma_{1}}{h_{e}^{3}} \int_{e} \llbracket \Pi_{h} \zeta-\zeta \rrbracket \llbracket w \rrbracket d s+\sum_{e \in \mathcal{E}_{h}} \frac{\sigma_{2}}{h_{e}} \int_{e}\left[\nabla\left(\Pi_{h} \zeta-\zeta\right)\right][\nabla w] d s  \tag{3.9}\\
& \quad-\sum_{e \in \mathcal{E}_{h}^{i}} \int_{e}\left\{w-E_{h} w\right\}\left[\nabla \Delta\left(\Pi_{h} \zeta\right)\right] d s+\sum_{e \in \mathcal{E}_{h}^{i}} \int_{e}\left\{\nabla\left(w-E_{h} w\right)\right\} \llbracket \Delta\left(\Pi_{h} \zeta\right) \rrbracket d s \\
& :=S_{1}+S_{2}+S_{3}+S_{4}+S_{5}+S_{6}+S_{7},
\end{align*}
$$

where $\lambda_{1}$ and $\lambda_{2}$ are taken to be 1 or -1 for different DG methods (cf. Table 1).
By using a standard inverse estimate, trace theorem and (2.11), the terms $S_{i}(i=1,2, \ldots, 5)$ on the right-hand side of (3.9) can be estimated as follows:

$$
\begin{align*}
\left|S_{1}\right| & \leq C\left(\sum_{T \in \mathcal{T}_{h}}\left|\Pi_{h} \zeta-\zeta\right|_{H^{2}(T)}^{2}\right)^{\frac{1}{2}}\|w\|_{h} \leq C h^{\min (r+1, s)-2}|\zeta|_{H^{s}(\Omega)}\|w\|_{h},  \tag{3.10}\\
\left|S_{2}\right| & \leq\left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{3}\|\{\nabla \Delta w\}\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}}\left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{-3}\left\|\llbracket \Pi_{h} \zeta-\zeta \rrbracket\right\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}}  \tag{3.11}\\
& \leq C\left(\sum_{T \in \mathcal{T}_{h}}\|\Delta w\|_{L_{2}(T)}^{2}\right)^{\frac{1}{2}} h^{\min (r+1, s)-2}|\zeta|_{H^{s}(\Omega)} \leq C h^{\min (r+1, s)-2}|\zeta|_{H^{s}(\Omega)}\|w\|_{h} . \\
\left|S_{3}\right| & \leq\left(\sum_{e \in \mathcal{E}_{\mathcal{G}}} h_{e}\|\{\Delta w\}\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}}\left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{-1}\left\|\left[\nabla\left(\Pi_{h} \zeta-\zeta\right)\right]\right\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}} \\
& \leq C h^{\min (r+1, s)-2}|\zeta|_{H^{s}(\Omega)}\|w\|_{h} .  \tag{3.12}\\
\left|S_{4}\right| & \leq\left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{-3}\left\|\llbracket \Pi_{h} \zeta-\zeta \rrbracket\right\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}}\left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{-3}\|\llbracket w \rrbracket\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}}
\end{align*}
$$

$$
\begin{align*}
& \leq C h^{\min (r+1, s)-2}|\zeta|_{H^{s}(\Omega)}\|w\|_{h} .  \tag{3.13}\\
\left|S_{5}\right| & \leq\left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{-1}\left\|\left[\nabla\left(\Pi_{h} \zeta-\zeta\right)\right]\right\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}}\left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{-1}\|[\nabla w]\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}} \\
& \leq C h^{\min (r+1, s)-2}|\zeta|_{H^{s}(\Omega)}\|w\|_{h} . \tag{3.14}
\end{align*}
$$

Now we estimate $S_{6}$ on the right-hand side of (3.9). For $r=2$, we have $S_{6}=0$. In order to estimate $S_{6}$ in the case of $r=3$, we denote

$$
\begin{equation*}
G_{1}(\xi)=-\sum_{e \in \mathcal{E}_{h}^{i}} \int_{e}\left\{w-E_{h} w\right\}\left[\nabla \Delta\left(\Pi_{h} \xi\right)\right] d s \tag{3.15}
\end{equation*}
$$

Whenever $\xi \in H^{2}(\Omega)$, by a standard inverse estimate and (2.16), we have

$$
\begin{align*}
\left|G_{1}(\xi)\right| & \leq C\left(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e}^{-3}\left\|\left\{w-E_{h} w\right\}\right\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}}\left(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e}^{3}\left\|\left[\nabla \Delta\left(\Pi_{h} \xi\right)\right]\right\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}} \\
& \leq C\|\xi\|_{H^{2}(\Omega)}\|w\|_{h} . \tag{3.16}
\end{align*}
$$

When $\xi \in H^{4}(\Omega)$, we apply (2.12), (2.16) and a standard inverse estimate to obtain

$$
\begin{align*}
\left|G_{1}(\xi)\right| & =\left|\sum_{e \in \mathcal{E}_{h}^{i}} \int_{e}\left\{w-E_{h} w\right\}\left[\nabla \Delta\left(\Pi_{h} \xi-\xi\right)\right] d s\right| \\
& \leq C\left(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e}^{-3}\left\|\left\{w-E_{h} w\right\}\right\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}}\left(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e}^{3}\left\|\left[\nabla \Delta\left(\Pi_{h} \xi-\xi\right)\right]\right\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}}  \tag{3.17}\\
& \leq C h^{2}\|\xi\|_{H^{4}(\Omega)}\|w\|_{h} .
\end{align*}
$$

Using interpolation between Sobolev spaces $H^{2}(\Omega)$ and $H^{4}(\Omega)$ (cf. [8,27]), we have by (3.16) and (3.17) that

$$
\begin{equation*}
\left|G_{1}(\xi)\right| \leq C h^{s-2}\|\xi\|_{H^{s}(\Omega)}\|w\|_{h} \tag{3.18}
\end{equation*}
$$

for all $\xi \in H^{s}(\Omega)$ with $s \in[2,4]$ and $w \in V_{h}$. In particular, we take $\xi=\zeta$ in (3.18) to obtain the estimate of $S_{6}$ in the case of $r=3$ :

$$
\begin{equation*}
\left|S_{6}\right| \leq C h^{s-2}\|\zeta\|_{H^{s}(\Omega)}\|w\|_{h} . \tag{3.19}
\end{equation*}
$$

To estimate $S_{7}$, we define

$$
\begin{equation*}
G_{2}(\xi)=\sum_{e \in \mathcal{E}_{h}^{i}} \int_{e}\left\{\nabla\left(w-E_{h} w\right)\right\} \llbracket \llbracket \Delta\left(\Pi_{h} \xi\right) \rrbracket d s . \tag{3.20}
\end{equation*}
$$

By (2.11), (2.16) and a standard inverse estimate, we have

$$
\begin{align*}
\left|G_{2}(\xi)\right| & \left.\leq C\left(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e}^{-1} \|\left\{\|\left(w-E_{h} w\right)\right\}\right\} \|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}}\left(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e}\left\|\llbracket \Delta\left(\Pi_{h} \xi\right) \rrbracket\right\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}} \\
& \leq C\|\xi\|_{H^{2}(\Omega)}\|w\|_{h} \quad \forall \xi \in H^{2}(\Omega), \tag{3.21}
\end{align*}
$$

and

$$
\begin{align*}
\left|G_{2}(\xi)\right| & =\mid \sum_{e \in \mathcal{E}_{h}^{i}} \int_{e}\left\{\left\|\nabla\left(w-E_{h} w\right)\right\| \llbracket \llbracket \Delta\left(\Pi_{h} \xi-\xi\right) \rrbracket d s \mid\right. \\
& \leq C\left(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e}^{-1}\left\|\left\{\nabla\left(w-E_{h} w\right)\right\}\right\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}}\left(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e}\left\|\llbracket \Delta\left(\Pi_{h} \xi-\xi\right) \rrbracket\right\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}} \\
& \leq C h\|\xi\|_{H^{3}(\Omega)}\|w\|_{h}, \quad \forall \xi \in H^{3}(\Omega) . \tag{3.22}
\end{align*}
$$

Moreover, in the case of $r=3$,

$$
\begin{equation*}
\left|G_{2}(\xi)\right| \leq C h^{2}\|\xi\|_{H^{4}(\Omega)}\|w\|_{h} \quad \forall \xi \in H^{4}(\Omega), \tag{3.23}
\end{equation*}
$$

where we used (2.12), (2.16) and a standard inverse estimate. Taking $\xi=\zeta$, combining (3.21)-(3.23) and applying interpolation between Sobolev spaces [8,27], we get

$$
\begin{equation*}
\left|S_{7}\right| \leq C h^{\min (r+1, s)-2}\|\zeta\|_{H^{s}(\Omega)}\|w\|_{h} \tag{3.24}
\end{equation*}
$$

Finally, the estimate (3.8) follows from (3.9)-(3.14), (3.19) and (3.24).
Due to the nonconformity of the DG space and the discrete constraint set, i.e., $V_{h} \not \subset H_{0}^{2}(\Omega)$ and $K_{h} \not \subset K$, it is important to establish the connection between these spaces. For this aim, we reduce the error between $u$ and $u_{h}$ in the energy norm to an estimate that only involves the continuous bilinear form $a(\cdot, \cdot)$. In next two lemmas, we assume $u \in H^{s}(\Omega) \cap W_{l o c}^{2, \infty}(\Omega)$ for some $s \in[2,4]$.

Lemma 3.2. There exists a positive constant $C$, depending on the shape regularity of $\mathcal{T}_{h}$, such that

$$
\begin{align*}
\left\|u-u_{h}\right\|_{h}^{2} \leq C & \left(h^{2 \min (r+1, s)-4}+h^{\min (r+1, s)-2}\left\|\Pi_{h} u-u_{h}\right\|_{h}\right. \\
& \left.+a\left(u, E_{h}\left(\Pi_{h} u-u_{h}\right)\right)-f, E_{h}\left(\Pi_{h} u-u_{h}\right)\right) . \tag{3.25}
\end{align*}
$$

Proof. Since $\Pi_{h} u \in K_{h}$, it follows from (2.3) and (2.9) that

$$
\begin{align*}
\left\|u-u_{h}\right\|_{h}^{2} & \leq 2\left\|u-\Pi_{h} u\right\|_{h}^{2}+2\left\|\Pi_{h} u-u_{h}\right\|_{h}^{2} \\
& \leq 2\left\|u-\Pi_{h} u\right\|_{h}^{2}+C a_{h}\left(\Pi_{h} u-u_{h}, \Pi_{h} u-u_{h}\right)  \tag{3.26}\\
& \leq 2\left\|u-\Pi_{h} u\right\|_{h}^{2}+C\left(a_{h}\left(\Pi_{h} u, \Pi_{h} u-u_{h}\right)-F\left(\Pi_{h} u-u_{h}\right)\right) .
\end{align*}
$$

Taking $\zeta=u$ and $w=\Pi_{h} u-u_{h}$ in Lemma 3.1, we have

$$
\begin{align*}
& a_{h}\left(\Pi_{h} u, \Pi_{h} u-u_{h}\right)-\tilde{F}\left(\Pi_{h} u-u_{h}\right) \\
& \quad \leq C h^{\min (r+1, s)-2}\left\|\Pi_{h} u-u_{h}\right\|_{h}+a\left(u, E_{h}\left(\Pi_{h} u-u_{h}\right)\right) \tag{3.27}
\end{align*}
$$

Furthermore using (2.16), we find

$$
\begin{equation*}
-\left(f, \Pi_{h} u-u_{h}\right) \leq-\left(f, E_{h}\left(\Pi_{h} u-u_{h}\right)\right)+C h^{2}\left\|\Pi_{h} u-u_{h}\right\|_{h} . \tag{3.28}
\end{equation*}
$$

Now the estimate (3.25) follows from (2.11) and (3.26)-(3.28).
Lemma 3.3. There exists a positive constant $C$, depending on the shape regularity of $\mathcal{T}_{h}$, such that

$$
\begin{align*}
& a\left(u, E_{h}\left(\Pi_{h} u-u_{h}\right)\right)-\left(f, E_{h}\left(\Pi_{h} u-u_{h}\right)\right) \\
& \leq C\left(h^{\min (r+1, s)-1}+h^{2}+h\left\|\Pi_{h} u-u_{h}\right\|_{h}\right) . \tag{3.29}
\end{align*}
$$

Proof. By (2.20) and (3.1) and the fact that $E_{h}\left(\Pi_{h} u-u_{h}\right) \in H_{0}^{2}(\Omega)$, we have

$$
\begin{align*}
& a\left(u, E_{h}\left(\Pi_{h} u-u_{h}\right)\right)-\left(f, E_{h}\left(\Pi_{h} u-u_{h}\right)\right)=\int_{\Omega} E_{h}\left(\Pi_{h} u-u_{h}\right) d \mu \\
&= \int_{\Omega}\left(T_{h} \Pi_{h} u-T_{h} u_{h}\right) d \mu \\
&= \int_{\Omega}\left(T_{h} \Pi_{h} u-u\right) d \mu+\left[\int_{\Omega}\left(u-\psi_{1}\right) d \mu_{1}-\int_{\Omega}\left(u-\psi_{2}\right) d \mu_{2}\right]  \tag{3.30}\\
&+\left[\int_{\Omega}\left(I_{h}\left(u-\psi_{1}\right)-\left(u-\psi_{1}\right)\right) d \mu_{1}-\int_{\Omega}\left(I_{h}\left(u-\psi_{2}\right)-\left(u-\psi_{2}\right)\right) d \mu_{2}\right] \\
& \quad+\left[\int_{\Omega}\left(I_{h} \psi_{1}-I_{h} T_{h} u_{h}\right) d \mu_{1}-\int_{\Omega}\left(I_{h} \psi_{2}-I_{h} T_{h} u_{h}\right) d \mu_{2}\right] \\
& \quad+\int_{\Omega}\left(I_{h}\left(T_{h} u_{h}-u\right)-\left(T_{h} u_{h}-u\right)\right) d \mu \\
&= R_{1}+R_{2}+R_{3}+R_{4}+R_{5},
\end{align*}
$$

where $I_{h}$ is the standard nodal interpolation operator for the conforming linear finite element space associated with $\mathcal{T}_{h}$.

It directly follows from (3.3) that

$$
\begin{equation*}
R_{2}=0 \tag{3.31}
\end{equation*}
$$

In view of (1.3) and (2.23) and the facts that $\mu_{i} \geq 0(i=1,2)$, we can estimate $R_{4}$ by

$$
\begin{equation*}
R_{4} \leq 0 \tag{3.32}
\end{equation*}
$$

For $R_{1}$, by (2.17), (2.19) and (3.1) we have

$$
\begin{align*}
\left|R_{1}\right| & \leq\|\mu\|_{H^{-1}(\Omega)}\left\|T_{h} \Pi_{h} u-u\right\|_{H^{1}(\Omega)} \\
& \leq C\left\|E_{h} \Pi_{h}(u-g)-(u-g)\right\|_{H^{1}(\Omega)}  \tag{3.33}\\
& \leq C h^{\min (r+1, s)-1} .
\end{align*}
$$

Next we introduce notation

$$
\mathcal{A}_{i, h}=\cup_{T \in \mathcal{T}_{h}}\left\{T \cap \mathcal{A}_{i} \neq \emptyset\right\} \quad i=1,2 .
$$

Without loss of generality, we assume $h$ is small enough such that the distance between $\mathcal{A}_{i, h}$ and $\partial \Omega$ is positive. Then by standard interpolation error estimate for $I_{h}[8,27]$, we can bound $R_{3}$ as

$$
\begin{align*}
\left|R_{3}\right| \leq & \mu_{1}\left(\mathcal{A}_{1}\right)\left\|I_{h}\left(u-\psi_{1}\right)-\left(u-\psi_{1}\right)\right\|_{L_{\infty}\left(\mathcal{A}_{1, h}\right)} \\
& \quad+\mu_{2}\left(\mathcal{A}_{2}\right)\left\|I_{h}\left(u-\psi_{2}\right)-\left(u-\psi_{2}\right)\right\|_{L_{\infty}\left(\mathcal{A}_{2, h}\right)} \\
\leq & C h^{2}\left(\left\|u-\psi_{1}\right\|_{W^{2, \infty}\left(\mathcal{A}_{1, h}\right)}+\left\|u-\psi_{2}\right\|_{W^{2, \infty}\left(\mathcal{A}_{2, h}\right)}\right) \\
\leq & C h^{2} . \tag{3.34}
\end{align*}
$$

By using (2.17), we can estimate $R_{5}$ as follows:

$$
\begin{align*}
\left|R_{5}\right| & \leq\|\mu\|_{H^{-1}(\Omega)}\left\|I_{h}\left(T_{h} u_{h}-u\right)-\left(T_{h} u_{h}-u\right)\right\|_{H^{1}(\Omega)} \\
& \leq C h\left|T_{h} u_{h}-u\right|_{H^{2}(\Omega)} \\
& \leq C h\left(\left|E_{h}\left(u_{h}-\Pi_{h} u\right)\right|_{H^{2}(\Omega)}+\left|T_{h} \Pi_{h} u-u\right|_{H^{2}(\Omega)}\right)  \tag{3.35}\\
& \leq C\left(h\left\|u_{h}-\Pi_{h} u\right\|_{h}+h^{\min (r+1, s)-1}\right) .
\end{align*}
$$

From (3.30)-(3.35), we obtain the estimate (3.29).
Now we combine Lemma 3.2 and Lemma 3.3 to obtain the optimal error estimate for the quadratic method ( $r=2$ ).

Theorem 3.1. Suppose the regularity result $u \in H^{2+\alpha}(\Omega)$ holds for some $\alpha \in(1 / 2,1]$. There exists a positive constant $C$, depending on the shape regularity of $\mathcal{T}_{h}$, such that

$$
\begin{equation*}
\left\|u-u_{h}\right\|_{h} \leq C h^{\alpha}, \tag{3.36}
\end{equation*}
$$

for the quadratic method $(r=2)$.
Proof. By taking $s=2+\alpha$ in (3.25) and (3.29), we obtain

$$
\begin{equation*}
\left\|u-u_{h}\right\|_{h}^{2} \leq C\left(h^{2 \alpha}+h^{1+\alpha}+h^{2}\right)+\frac{1}{2}\left\|u-u_{h}\right\|_{h}^{2} . \tag{3.37}
\end{equation*}
$$

Since $\alpha \in(1 / 2,1]$, we prove (3.36).
In the case where the contact sets $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ are smooth and do not degenerate to lower dimensional surfaces in $\mathbb{R}^{d}$, we may improve the regularity of $u$. Indeed, under the following assumption:

The free boundary $\partial \mathcal{A}:=\partial \mathcal{A}_{1} \cup \partial \mathcal{A}_{2}$ is smooth and $u \in H^{4}(\mathcal{G} \backslash \mathcal{A}) \cap H^{4}(\mathcal{A})$,
where $\mathcal{A}$ is the interior of $\mathcal{A}$ and $\mathcal{G}$ is described as in Section 3.1. From (3.38), we have $u \in H_{l o c}^{2+\alpha}(\Omega)$ for any $\alpha<1.5$. In this case, it holds that

$$
\begin{equation*}
\left|u(x)-\psi_{i}(x)\right| \leq C_{\alpha} h^{2 \alpha} \quad i=1,2 \tag{3.39}
\end{equation*}
$$

for any $x \in \Omega$ whose distance to $\mathcal{A}_{i}(i=1,2)$ is $\leq h$ and $\alpha \in(1,1.5)$. Note that the proof of (3.39) can be found in Lemma 5.5 of [26].

Next, we show that the regularity of $\mu$ can also be improved.
Lemma 3.4. There exists a positive constant $C$, depending on the shape regularity of $\mathcal{T}_{h}$, such that

$$
\begin{equation*}
\int_{\Omega} v d \mu \leq C\left(\|u\|_{H_{l o c}^{2+\alpha}(\Omega)}+\|f\|_{L_{2}(\Omega)}\right)\|v\|_{H^{2-\alpha}(\Omega)} \quad \forall v \in H_{0}^{2-\alpha}(\Omega), \alpha \in(1,1.5), \tag{3.40}
\end{equation*}
$$

i.e., $\mu \in H^{\alpha-2}(\Omega)=\left[H_{0}^{2-\alpha}(\Omega)\right]^{\prime}$. Here $H_{0}^{2-\alpha}(\Omega):=\left\{v \in H^{2-\alpha}(\Omega): v=0\right.$ on $\left.\partial \Omega\right\}$.

Proof. Let $B(\cdot, \cdot)$ be defined in Section 3.1. Through integration by parts, we have

$$
\begin{equation*}
|B(\zeta, w)| \leq C_{\mathcal{G}}\|\zeta\|_{H^{4}(\mathcal{G})}\|w\|_{L_{2}(\mathcal{G})} \quad \forall \zeta \in H^{4}(\mathcal{G}), w \in H^{1}(\mathcal{G}) . \tag{3.41}
\end{equation*}
$$

Combining (3.6) and (3.41), we can apply the interpolation of bilinear forms on Sobolev spaces (see Section 4.4 in [32]) to extend $B$ to $H^{2+\alpha}(\mathcal{G}) \times H^{2-\alpha}(\mathcal{G})$ such that

$$
\begin{equation*}
|B(\zeta, w)| \leq C_{\mathcal{G}, \alpha}\|\zeta\|_{H^{2+\alpha}(\mathcal{G})}\|w\|_{H^{2-\alpha}(\mathcal{G})} \quad \forall \zeta \in H^{2+\alpha}(\mathcal{G}), w \in H^{2-\alpha}(\mathcal{G}) . \tag{3.42}
\end{equation*}
$$

This together with (3.5) implies

$$
\begin{equation*}
\left|\int_{\Omega} v d \mu\right| \leq C_{\alpha}\left(\|u\|_{H_{l o c}^{2+\alpha}(\Omega)}+\|f\|_{L_{2}(\Omega)}\right)\|v\|_{H^{2-\alpha}(\Omega)} \quad \forall v \in H_{0}^{2}(\Omega) . \tag{3.43}
\end{equation*}
$$

Finally, the estimate (3.40) follows from the density argument.
In the rest of the section, we aim to extend the optimal error estimate (3.36) to the cubic method $(r=3)$ under the regularity $u \in H^{2+\alpha}(\Omega)$ for some $\alpha \in(1,1.5)$.

Theorem 3.2. Suppose the assumption (3.38) holds and u belongs to $H^{2+\alpha}(\Omega)$ for some $\alpha \in(1,1.5)$. There exists a positive constant $C$, depending on the shape regularity of $\mathcal{T}_{h}$, such that

$$
\begin{equation*}
\left\|u-u_{h}\right\|_{h} \leq C h^{\alpha} \tag{3.44}
\end{equation*}
$$

for the cubic method $(r=3)$.
Proof. In order to extend the error estimate in Theorem 3.1 to the cubic element, we need to improve the estimates of $R_{1}, R_{3}$ and $R_{5}$ that appeared in the proof of Lemma 3.3.

For $R_{1}$, by combining (2.17), (2.19), (3.1) and Lemma 3.4, we have

$$
\begin{align*}
\left|R_{1}\right| & \leq\|\mu\|_{H^{\alpha-2}(\Omega)}\left\|T_{h} \Pi_{h} u-u\right\|_{H^{2-\alpha}(\Omega)} \\
& \leq C\left\|E_{h} \Pi_{h}(u-g)-(u-g)\right\|_{H^{2-\alpha}(\Omega)} \\
& \leq C h^{2 \alpha} \tag{3.45}
\end{align*}
$$

Similarly, for $R_{5}$ we have

$$
\begin{align*}
\left|R_{5}\right| & \leq\|\mu\|_{H^{\alpha-2}(\Omega)}\left\|I_{h}\left(T_{h} u_{h}-u\right)-\left(T_{h} u_{h}-u\right)\right\|_{H^{2-\alpha}(\Omega)} \\
& \leq C h^{\alpha}\left|T_{h} u_{h}-u\right|_{H^{2}(\Omega)} \\
& \leq C h^{\alpha}\left(\left|E_{h}\left(u_{h}-\Pi_{h} u\right)\right|_{H^{2}(\Omega)}+\left|T_{h} \Pi_{h} u-u\right|_{H^{2}(\Omega)}\right)  \tag{3.46}\\
& \leq C\left(h^{\alpha}\left\|u_{h}-\Pi_{h} u\right\|_{h}+h^{2 \alpha}\right) .
\end{align*}
$$

Table 2
Quadratic method $(r=2)$ for Example 4.1 with exact solution.

| $h$ | $\left\\|e_{h}\right\\|_{h}$ | Order | $\left\\|e_{h}\right\\|_{H^{1}}$ | Order | $\left\\|e_{h}\right\\|_{L_{\infty}}$ | Order |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $1 / 4$ | $6.8886 \mathrm{E}-04$ | - | $3.3545 \mathrm{E}-02$ | - | $9.9629 \mathrm{E}-03$ | - |
| $1 / 8$ | $2.9617 \mathrm{E}-04$ | 1.2178 | $1.1980 \mathrm{E}-02$ | 1.4855 | $1.7108 \mathrm{E}-03$ | 2.5419 |
| $1 / 16$ | $1.2106 \mathrm{E}-04$ | 1.2907 | $3.4353 \mathrm{E}-03$ | 1.8021 | $3.1192 \mathrm{E}-04$ | 2.4554 |
| $1 / 32$ | $5.4987 \mathrm{E}-05$ | 1.1386 | $9.1452 \mathrm{E}-04$ | 1.9094 | $6.2420 \mathrm{E}-05$ | 2.3211 |
| $1 / 64$ | $2.6346 \mathrm{E}-05$ | 1.0615 | $2.3506 \mathrm{E}-04$ | 1.9600 | $1.3422 \mathrm{E}-05$ | 2.2174 |

By (3.3) and (3.39), we can estimate $R_{3}$ as follows.

$$
\begin{align*}
\left|R_{3}\right| & \leq\left|\int_{\Omega}\left(I_{h}\left(u-\psi_{1}\right)-\left(u-\psi_{1}\right)\right) d \mu_{1}+\int_{\Omega}\left(I_{h}\left(u-\psi_{2}\right)-\left(u-\psi_{2}\right)\right) d \mu_{2}\right| \\
& \leq\left|\int_{\Omega} I_{h}\left(u-\psi_{1}\right) d \mu_{1}\right|+\left|\int_{\Omega} I_{h}\left(u-\psi_{2}\right) d \mu_{2}\right|  \tag{3.47}\\
& \leq C\left(\left\|I_{h}\left(u-\psi_{1}\right)\right\|_{L_{\infty}\left(\mathcal{A}_{1, h}\right)}+\left\|I_{h}\left(u-\psi_{2}\right)\right\|_{L_{\infty}\left(\mathcal{A}_{2, h}\right)}\right) \leq C_{\alpha} h^{2 \alpha} .
\end{align*}
$$

Finally, by combining with other estimates in Lemma 3.3, we obtain

$$
\begin{equation*}
\left\|u-u_{h}\right\|_{h}^{2} \leq C h^{2 \alpha}+\frac{1}{2}\left\|u-u_{h}\right\|_{h}^{2} \tag{3.48}
\end{equation*}
$$

for $\alpha \in(1,1.5)$, and thus the estimate (3.44) is proved.

## 4. Numerical results

In this section, we report the numerical results of several examples obtained by SIPG method for two-dimensional obstacle problem. We consider one-obstacle problems in Examples 4.1 and 4.2. In the first example, an exact solution with nonhomogeneous boundary conditions is constructed on the square domain. The second example is constructed on a $L$-shaped domain so that the index of elliptic regularity $\alpha<1$. In the third example, we consider a two-obstacle problem. We will investigate numerical errors and rates of convergence in various norms and also plot the discrete contact sets. In all numerical tests, we take $\sigma_{1}=30, \sigma_{2}=15$ for the quadratic method, and $\sigma_{1}=650, \sigma_{2}=50$ for the cubic method. To solve the discrete problems, we have used the tools developed via the FEniCS project [33,34]. We denote the lower and upper obstacle functions by $\psi(x)$ and $\widetilde{\psi}(x)$ respectively, and solve the discrete obstacle problems on uniform triangulations with the mesh size $h$.

Example 4.1. In this example, we consider the obstacle problem on the disc $\{x:|x|<2\}$ for the data $\psi(x)=1-|x|^{2}, f(x)=0$ and $g(x)=0$. We can find the exact solution $u(x)$ of this obstacle problem as follows:

$$
u(r)=\left\{\begin{array}{ll}
C_{1}|x|^{2} \ln |x|+C_{2}|x|^{2}+C_{3} \ln |x|+C_{4} & |x|>r_{0}  \tag{4.1}\\
1-|x|^{2} & |x| \leq r_{0}
\end{array},\right.
$$

where $r_{0} \approx 0.181345, C_{1} \approx 0.525041, C_{2} \approx 0.628609, C_{3} \approx 0.017266, C_{4} \approx 1.046746$. Then we restrict the problem to $\Omega=(-0.5,0.5)^{2}$ with the same $\psi$ so that the exact solution is the restriction of $u$ on $\Omega$ with nonhomogeneous Dirichlet boundary conditions determined by (4.1). Note that $u$ belongs to $H^{2+\alpha}(\Omega)$ for any $\alpha<1.5$. We consider the error $e_{h}=\Pi_{h} u-u_{h}$ and evaluate the errors in the energy norm, $H^{1}$ norm and $L_{\infty}$ norm, where

$$
\left\|e_{h}\right\|_{L_{\infty}}=\max _{T \in \mathcal{T}_{h}, p \in \mathcal{V}_{T}}\left|e_{h, T}(p)\right| \quad \forall e \in V_{h},
$$

and $e_{h, T}=\left.e_{h}\right|_{T}$.
The numerical results for the quadratic method (resp. cubic method) are given in Table 2 (resp. Table 3). The asymptotic convergence rate is 1 for $r=2$ and 1.5 for $r=3$ in the energy norm as predicted by Theorems 3.1 and 3.2. We also observe that the $H^{1}$ norm and $L_{\infty}$ norm errors are of order $O\left(h^{2}\right)$ for the quadratic method. For the cubic method, the average rates of convergence in lower order norms are close to 2.5 .

Table 3
Cubic method $(r=3)$ for Example 4.1 with exact solution.

| $h$ | $\left\\|e_{h}\right\\|_{h}$ | Order | $\left\\|e_{h}\right\\|_{H^{1}}$ | Order | $\left\\|e_{h}\right\\|_{L_{\infty}}$ | Order |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $1 / 4$ | $1.5527 \mathrm{E}-04$ | - | $2.5613 \mathrm{E}-03$ | - | $6.2322 \mathrm{E}-04$ | - |
| $1 / 8$ | $5.8237 \mathrm{E}-05$ | 1.4147 | $6.4097 \mathrm{E}-04$ | 1.9986 | $1.7554 \mathrm{E}-04$ | 1.8279 |
| $1 / 16$ | $2.2578 \mathrm{E}-05$ | 1.3670 | $1.2552 \mathrm{E}-04$ | 2.3523 | $2.9250 \mathrm{E}-05$ | 2.5853 |
| $1 / 32$ | $7.8761 \mathrm{E}-06$ | 1.5194 | $1.9308 \mathrm{E}-05$ | 2.7007 | $2.4223 \mathrm{E}-06$ | 3.5940 |
| $1 / 64$ | $2.4573 \mathrm{E}-06$ | 1.6804 | $2.9657 \mathrm{E}-06$ | 2.7027 | $2.2796 \mathrm{E}-07$ | 3.4095 |



Fig. 2. Discrete contact sets of Example 4.1 obtained by the quadratic method $(r=2)$. Left: $h=1 / 64$. Right: $h=1 / 128$.


Fig. 3. Discrete contact sets of Example 4.1 obtained by the cubic method $(r=3)$. Left: $h=1 / 64$. Right: $h=1 / 128$.

We plot the discrete contact set $\mathcal{A}_{h}$ for levels 7-8 in Figs. 2 and 3 for quadratic and cubic methods respectively, where

$$
\begin{equation*}
\mathcal{A}_{h}=\cup_{T \in \mathcal{T}_{h}}\left\{p \in \mathcal{V}_{T}:\left|u_{h, T}(p)-\psi(p)\right| \leq\left\|e_{h}\right\|_{L_{\infty}}\right\}, \tag{4.2}
\end{equation*}
$$

and $u_{h, T}=\left.u_{h}\right|_{T}$.
Example 4.2. In this example we consider an $L$-shaped domain $\Omega=(-0.5,0.5)^{2} \backslash[0,0.5]^{2}$ with $f=g=0$ and

$$
\psi(x)=1-\left(\frac{\left(x_{1}+0.25\right)^{2}}{0.2^{2}}+\frac{x_{2}^{2}}{0.35^{2}}\right) .
$$

Since the exact solution is unknown, we take $e_{h}=u_{H}-u_{h}$, where $u_{H}$ and $u_{h}$ are the discrete solutions obtained by DG method on two consecutive levels.

The numerical results are given in Table 4 (resp. Table 5) for the quadratic method (resp. cubic method). Since $\Omega$ is nonconvex in this example, we have $\alpha \approx 0.5445<1$, which are observed in the energy norm error convergence rates for both $r=2$ and $r=3$. Note that the energy norm errors have not reached the asymptotic region, but the rate for the cubic method is closer to 0.5445 than that of the quadratic method. Furthermore the convergence rates for errors in the $H^{1}$ and $L_{\infty}$ norms are of higher orders.

Since $\Delta^{2} \psi=0$ in this example, the non-contact set is connected (cf. [13]). This is confirmed by Figs. 4 and 5, where we plot the discrete contact sets obtained by quadratic and cubic methods respectively for levels 7-8.

Table 4
Quadratic method $(r=2)$ for Example 4.2 without exact solution.

| $h$ | $\left\\|e_{h}\right\\|_{h}$ | Order | $\left\\|e_{h}\right\\|_{H^{1}}$ | Order | $\left\\|e_{h}\right\\|_{L_{\infty}}$ | Order |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $1 / 4$ | $3.6503 \mathrm{E}-01$ | - | $6.0233 \mathrm{E}-01$ | - | $6.8806 \mathrm{E}-01$ | - |
| $1 / 8$ | $3.8320 \mathrm{E}-01$ | -0.0701 | $3.4651 \mathrm{E}-01$ | 0.7977 | $2.7195 \mathrm{E}-01$ | 1.3392 |
| $1 / 16$ | $2.1479 \mathrm{E}-01$ | 0.8352 | $1.2569 \mathrm{E}-01$ | 1.4630 | $9.1738 \mathrm{E}-02$ | 1.5677 |
| $1 / 32$ | $1.0921 \mathrm{E}-01$ | 0.9759 | $3.8351 \mathrm{E}-02$ | 1.7125 | $2.8423 \mathrm{E}-02$ | 1.6905 |
| $1 / 64$ | $5.7240 \mathrm{E}-02$ | 0.9320 | $1.0638 \mathrm{E}-02$ | 1.8500 | $7.2566 \mathrm{E}-03$ | 1.9697 |

Table 5
Cubic method $(r=3)$ for Example 4.2 without exact solution.

| $h$ | $\left\\|e_{h}\right\\|_{h}$ | Order | $\left\\|e_{h}\right\\|_{H^{1}}$ | Order | $\left\\|e_{h}\right\\|_{L_{\infty}}$ | Order |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $1 / 4$ | $3.2645 \mathrm{E}-01$ | - | $1.7378 \mathrm{E}-01$ | - | $1.7076 \mathrm{E}-01$ | - |
| $1 / 8$ | $1.1954 \mathrm{E}-01$ | 1.4493 | $4.0127 \mathrm{E}-02$ | 2.1147 | $4.9318 \mathrm{E}-02$ | 1.7918 |
| $1 / 16$ | $6.3338 \mathrm{E}-02$ | 0.9164 | $1.2338 \mathrm{E}-02$ | 1.7015 | $1.4988 \mathrm{E}-02$ | 1.7183 |
| $1 / 32$ | $3.4444 \mathrm{E}-02$ | 0.8788 | $4.2430 \mathrm{E}-03$ | 1.5399 | $4.6227 \mathrm{E}-03$ | 1.6970 |
| $1 / 64$ | $2.0463 \mathrm{E}-02$ | 0.7512 | $1.2971 \mathrm{E}-03$ | 1.7098 | $1.2159 \mathrm{E}-03$ | 1.9267 |



Fig. 4. Discrete contact sets of Example 4.2 obtained by the quadratic method ( $r=2$ ). Left: $h=1 / 64$. Right: $h=1 / 128$.


Fig. 5. Discrete contact sets of Example 4.2 obtained by the cubic method ( $r=3$ ). Left: $h=1 / 64$. Right: $h=1 / 128$.

Example 4.3. In this example we consider a two-obstacle problem on $\Omega=(-0.5,0.5)^{2}$ with $f=g=0$, $\psi(x)=1-36|x|^{4}$ and $\tilde{\psi}(x)=1.07$. Similar to Example 4.2, we take $e_{h}=u_{H}-u_{h}$, where $u_{H}$ and $u_{h}$ are the discrete solutions on two consecutive levels.

From Tables 6-7, we observe $O\left(h^{\tau}\right)(\tau=1$ for $r=2$ and $\tau=1.5$ for $r=3)$ convergence in the energy norm error, which agrees with the estimates in Theorems 3.1 and 3.2. We also observe that the $H^{1}$ norm errors are of order $O\left(h^{2}\right)$, and the $L_{\infty}$ norm errors are of higher order for the quadratic method. However for the cubic method, both the $H^{1}$ norm and $L_{\infty}$ norm errors are of order $O\left(h^{2.5}\right)$ in average.

In Fig. 6-7, we also observe that two contact sets are disjoint. In particular, the lower contact set (blue color) has no interior point since $\Delta^{2} \psi<0$ (cf. [13]).

Table 6
Quadratic method $(r=2)$ for Example 4.3 without exact solution.

| $h$ | $\left\\|e_{h}\right\\|_{h}$ | Order | $\left\\|e_{h}\right\\|_{H^{1}}$ | Order | $\left\\|e_{h}\right\\|_{L_{\infty}}$ | Order |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $1 / 4$ | $3.6966 \mathrm{E}-01$ | - | $4.3391 \mathrm{E}-01$ | - | $3.3750 \mathrm{E}-01$ | - |
| $1 / 8$ | $3.3397 \mathrm{E}-01$ | 0.1465 | $2.1768 \mathrm{E}-01$ | 0.9952 | $8.6803 \mathrm{E}-02$ | 1.9591 |
| $1 / 16$ | $1.4973 \mathrm{E}-01$ | 1.1574 | $6.9404 \mathrm{E}-02$ | 1.6491 | $2.3841 \mathrm{E}-02$ | 1.8643 |
| $1 / 32$ | $6.8184 \mathrm{E}-02$ | 1.1348 | $1.8572 \mathrm{E}-02$ | 1.9019 | $4.0470 \mathrm{E}-03$ | 2.5585 |
| $1 / 64$ | $3.2921 \mathrm{E}-02$ | 1.0504 | $4.6473 \mathrm{E}-03$ | 1.9986 | $6.4597 \mathrm{E}-04$ | 2.6473 |

Table 7
Cubic method $(r=3)$ for Example 4.3 without exact solution.

| $h$ | $\left\\|e_{h}\right\\|_{h}$ | Order | $\left\\|e_{h}\right\\|_{H^{1}}$ | Order | $\left\\|e_{h}\right\\|_{L_{\infty}}$ | Order |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $1 / 4$ | $2.4108 \mathrm{E}-01$ | - | $8.7294 \mathrm{E}-02$ | - | $3.1551 \mathrm{E}-02$ | - |
| $1 / 8$ | $8.4913 \mathrm{E}-02$ | 1.5054 | $1.7229 \mathrm{E}-02$ | 2.3410 | $5.5377 \mathrm{E}-03$ | 2.5104 |
| $1 / 16$ | $3.0659 \mathrm{E}-02$ | 1.4697 | $2.9819 \mathrm{E}-03$ | 2.5306 | $8.0149 \mathrm{E}-04$ | 2.7885 |
| $1 / 32$ | $1.0782 \mathrm{E}-02$ | 1.5077 | $7.5179 \mathrm{E}-04$ | 1.9878 | $2.4638 \mathrm{E}-04$ | 1.7018 |
| $1 / 64$ | $3.8562 \mathrm{E}-03$ | 1.4834 | $1.2559 \mathrm{E}-04$ | 2.5817 | $2.3063 \mathrm{E}-05$ | 3.4172 |



Fig. 6. Discrete contact sets of Example 4.3 obtained by the quadratic method ( $r=2$ ). Left: $h=1 / 64$. Right: $h=1 / 128$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)


Fig. 7. Discrete contact sets of Example 4.3 obtained by the cubic method ( $r=3$ ). Left: $h=1 / 64$. Right: $h=1 / 128$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

## 5. Conclusion

In this work, we studied a family of DG methods (SIPG, NIPG, SSIPG1 and SSIPG2) for a fourth order variational inequality of the first kind. We unified the convergence analysis and derived optimal error estimates for both quadratic and cubic DG methods. Note the SIPG scheme has a symmetric formulation, which results in a symmetric positive definite stiffness matrix. On the other hand, though NIPG, SSIPG1 and SSIPG2 schemes are not symmetric, in theory they have less restrictions on penalty parameters compared with SIPG method. However, it is shown that there is no essential difference in their error analysis. We also tested NIPG and SSIPG methods
for the same numerical examples in Section 4. They all have similar performance as the SIPG method. We only presented the results of SIPG method for illustration.

Our previous work also includes $C^{0} \mathrm{IP}$ method for the fourth order variational inequality problems (see [21]). Note that the $C^{0} \mathrm{IP}$ method requires $C^{0}$ weak continuity in the design of discrete basis functions. In comparison, the fully DG methods studied in this paper have more flexibilities with general meshes, such as the ones with hanging nodes. Moreover, the fully DG methods are more ideal to be used with $h p$-adaptive strategy, due to the fact that their elements are totally discontinuous in nature. Our future work includes the a posteriori error estimates and developing adaptive algorithms for the fully DG methods.

## Acknowledgments

The first author's research is supported in part by the Hong Kong RGC, General Research Fund (GRF) Grant No. 15302518 and the National Natural Science Foundation of China Grant No. 11771367. The authors would like to thank Professor Susanne C. Brenner and Professor Li-yeng Sung of Louisiana State University for their helpful suggestions and comments for this work.

## References

[1] J.-F. Rodrigues, Obstacle Problems in Mathematical Physics, in: North-Holland Mathematics Studies, vol. 134, North-Holland Publishing Co., Amsterdam, 1987, p. 114, Notas de Matemática [Mathematical Notes].
[2] J.-L. Lions, G. Stampacchia, Variational inequalities, Comm. Pure Appl. Math. 20 (3) (1967) 493-519.
[3] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, in: Classics in Applied Mathematics, vol. 31, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.
[4] A. Friedman, Variational Principles and Free-boundary Problems, second ed., Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1988.
[5] G. Duvaut, J.-L. Lions, Inequalities in Mechanics and Physics, Springer-Verlag, Berlin-New York, 1976, p. 219, Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften.
[6] R.S. Falk, Error estimates for the approximation of a class of variational inequalities, Math. Comp. 28 (128) (1974) $963-971$.
[7] F. Brezzi, W.W. Hager, P.-A. Raviart, Error estimates for the finite element solution of variational inequalities, Numer. Math. 28 (4) (1977) 431-443.
[8] P.G. Ciarlet, The finite element method for elliptic problems, in: Classics in Applied Mathematics, vol. 40, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.
[9] F. Wang, W. Han, X.-L. Cheng, Discontinuous Galerkin methods for solving elliptic variational inequalities, SIAM J. Numer. Anal. 48 (2) (2010) 708-733.
[10] H. Brézis, G. Stampacchia, Sur la régularité de la solution d'inéquations elliptiques, Bulletin de la Société Mathématique de France 96 (1968) 153-180.
[11] J. Frehse, Zum differenzierbarkeitsproblem bei variationsungleichungen höherer ordnung, in: Abhandlungen Aus Dem Mathematischen Seminar Der UniversitäT Hamburg, vol. 36, Springer, 1971, pp. 140-149.
[12] J. Frehse, On the regularity of the solution of the biharmonic variational inequality, Manuscripta Math. 9 (1) (1973) $91-103$.
[13] L.A. Caffarelli, A. Friedman, The obstacle problem for the biharmonic operator, Ann. Sc. Norm. Super Pisa Cl. Sci. 6 (1) (1979) 151-184.
[14] S.C. Brenner, L.-Y. Sung, Y. Zhang, Finite element methods for the displacement obstacle problem of clamped plates, Math. Comp. 81 (279) (2012) 1247-1262.
[15] H. Blum, R. Rannacher, On the boundary value problem of the biharmonic operator on domains with angular corners, Math. Methods Appl. Sci. 2 (4) (1980) 556-581.
[16] P. Grisvard, Elliptic problems in nonsmooth domains, in: Monographs and Studies in Mathematics, vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.
[17] M. Dauge, Elliptic boundary value problems on corner domains, in: Lecture Notes in Mathematics, vol. 1341, Springer-Verlag, Berlin, 1988.
[18] S.A. Nazarov, B.A. Plamenevsky, Elliptic Problems in Domains with Piecewise Smooth Boundaries, in: De Gruyter Expositions in Mathematics, vol. 13, Walter de Gruyter \& Co., Berlin, 1994.
[19] S.C. Brenner, C.B. Davis, L.-Y. Sung, A partition of unity method for the displacement obstacle problem of clamped Kirchhoff plates, J. Comput. Appl. Math. 265 (2014) 3-16.
[20] S.C. Brenner, C.B. Davis, L.-y. Sung, A partition of unity method for a class of fourth order elliptic variational inequalities, Comput. Methods Appl. Mech. Engrg. 276 (2014) 612-626.
[21] S.C. Brenner, L.-Y. Sung, H. Zhang, Y. Zhang, A quadratic $C^{0}$ interior penalty method for the displacement obstacle problem of clamped Kirchhoff plates, SIAM J. Numer. Anal. 50 (6) (2012) 3329-3350.
[22] S.C. Brenner, L.-Y. Sung, H. Zhang, Y. Zhang, A Morley finite element method for the displacement obstacle problem of clamped kirchhoff plates, J. Comput. Appl. Math. 254 (2013) 31-42.
[23] E. Süli, I. Mozolevski, hp-version interior penalty DGFEMs for the biharmonic equation, Comput. Methods Appl. Mech. Engrg. 196 (13) (2007) 1851-1863.
[24] E.H. Georgoulis, P. Houston, Discontinuous Galerkin methods for the biharmonic problem, IMA J. Numer. Anal. 29 (3) (2009) 573-594.
[25] S.C. Brenner, L.-y. Sung, A new convergence analysis of finite element methods for elliptic distributed optimal control problems with pointwise state constraints, SIAM J. Control Optim. 55 (4) (2017) 2289-2304.
[26] S.C. Brenner, J. Gedicke, L.-Y. Sung, $C^{0}$ Interior penalty methods for an elliptic distributed optimal control problem on nonconvex polygonal domains with pointwise state constraints, SIAM J. Numer. Anal. 56 (3) (2018) 1758-1785.
[27] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, third ed., Texts in Applied Mathematics, vol. 15, Springer, New York, 2008.
[28] S.C. Brenner, $C^{0}$ Interior penalty methods, in: Frontiers in Numerical Analysis-Durham 2010, Springer, 2011, pp. 79-147.
[29] A. Ženíšek, Tetrahedral finite $C^{m}$ elements, Acta Univ. Carolin. Math. Phys. 15 (1) (1974) 189-193.
[30] L. Schwartz, Théorie des distributions, in: Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle ed., entiérement corrigée, refondue et augmentée, Hermann, Paris, 1966.
[31] W. Rudin, Real and Complex Analysis, third ed., McGraw-Hill Book Co., New York, 1987.
[32] J. Bergh, J. Löfström, Interpolation Spaces, in: Grundlehren der Mathematischen Wissenschaften, vol. 223, Springer-Verlag, Berlin-New York, 1976.
[33] A. Logg, K.-A. Mardal, G. Wells, Automated Solution of Differential Equations by the Finite Element Method: The FEniCS Book, vol. 84, Springer Science \& Business Media, 2012.
[34] M. Alnæs, J. Blechta, J. Hake, A. Johansson, B. Kehlet, A. Logg, C. Richardson, J. Ring, M. Rognes, G. Wells, The FEniCS Project Version 1.5, Archive of Numerical Software, 3, 2015.


[^0]:    * Corresponding author.

    E-mail addresses: jintao.cui@polyu.edu.hk (J. Cui), y_zhang7@uncg.edu (Y. Zhang).

