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A new analysis of discontinuous Galerkin methods for a fourth order variational inequality

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Abstract

We study a family of discontinuous Galerkin methods for the displacement obstacle problem of Kirchhoff plates on two and three dimensional convex polyhedral domains, which are characterized as fourth order elliptic variational inequalities of the first kind. We prove that the error in an H^2 -like energy norm is $O(h^{\alpha})$ for the quadratic method, where $\alpha \in (\frac{1}{2}, 1]$ is determined by the geometry of the domain. Under additional assumptions on the contact set such that the solution has improved regularity, we derive the optimal error estimate with $\alpha \in (1, \frac{3}{2})$ for the cubic method. Numerical experiments demonstrate the performance of the methods and confirm the theoretical results.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$ (d = 2 or 3) be a bounded polyhedral domain, $f \in L_2(\Omega)$, $g \in H^4(\Omega)$ and $\psi_1, \psi_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\psi_1 < \psi_2 \quad \text{in } \Omega \qquad \text{and} \qquad \psi_1 < g < \psi_2 \quad \text{on } \partial \Omega.$$
 (1.1)

In this paper, we consider the following displacement obstacle problem: Find $u \in K$ such that

$$u = \underset{v \in K}{\operatorname{argmin}} \left[\frac{1}{2} a(v, v) - (f, v) \right], \tag{1.2}$$

where

$$K = \{ v \in H^2(\Omega) : v - g \in H^2_0(\Omega), \ \psi_1 \le v \le \psi_2 \text{ in } \Omega \},$$

$$(1.3)$$

$$a(w,v) = \int_{\Omega} (\Delta w) (\Delta v) \, dx \quad \text{and} \quad (f,v) = \int_{\Omega} f \, v \, dx. \tag{1.4}$$

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In (1.2), *u* is the vertical displacement of the midsurface of the clamped Kirchhoff plate and *f* denotes the vertical load density divided by the flexural rigidity of the plate. Since *K* is a nonempty closed convex subset of $H^2(\Omega)$, and $a(\cdot, \cdot)$ is symmetric bounded on $H^2(\Omega)$ and coercive on $K - K \subseteq H_0^2(\Omega)$, it follows from the standard theory [1–4] that the problem (1.2)–(1.4) has a unique solution, which is also characterized by the variational inequality of the first kind (cf. [5]):

$$a(u, v - u) \ge (f, v - u) \qquad \forall v \in K.$$

$$(1.5)$$

Finite element methods for second order obstacle problems where $a(w, v) = \int_{\Omega} \nabla w \cdot \nabla v \, dx$ in (1.4) have been investigated in [6–9] and references therein. The key ingredient of the analysis is the use of the strong complementarity form of variational inequality which is valid since $u \in H^2(\Omega)$ under appropriate conditions (cf. [10]). However for the obstacle problem (1.2), it was shown in [4,11–14] that $u \in H^3_{loc}(\Omega) \cap W^{2,\infty}_{loc}(\Omega) \cap C^2(\Omega)$ under our assumptions on data. Because of (1.1), u is unconstrained near $\partial \Omega$ and thus $\Delta^2 u = f$ in a neighborhood of $\partial \Omega$. Therefore the elliptic regularity theory [15–18] implies that u belongs to $H^{2+\alpha}$ near $\partial \Omega$, where $\alpha \in (1/2, 1]$ denotes the index of elliptic regularity depending on the geometry of Ω . In the case of convex domain, we can take $\alpha = 1$. Nevertheless, the lack of full regularity is the main challenge of numerical analysis for the problem (1.2).

In [14], a unified convergence analysis for conforming, nonconforming finite element methods and C^0 interior penalty methods was developed and extensions to more general cases and other finite element methods can also be found in [19–22]. In these works, an intermediate obstacle problem is introduced to connect the continuous and discrete obstacle problems. However, when we extend the analysis to three dimensions, only the suboptimal error estimate can be obtained. Furthermore, in these methods, finite element spaces need to be continuous at the vertices of triangulation which limits the practical applications.

The goal of this paper is to study fully discontinuous Galerkin (DG) methods [23,24] for the problem (1.2)/(1.5). This is motivated by the advantages of DG methods in regard to naturally handling nonhomogeneous boundary conditions, designing adaptive meshes and parallel computing. In particular, we develop a unified approach for a family of DG methods where only the weak complementarity form of the variational inequality is used, without using an intermediate problem as developed in [14,19,21,22]. In [25,26], a similar technique had been applied to elliptic optimal control problems with pointwise state constraints. In this paper, we prove optimal error estimates for both quadratic and cubic fully DG methods on two and three dimensions with nonhomogeneous boundary conditions. To the best of our knowledge, the analysis for the cubic method is new and this is the first paper that provides rigorous error analysis of fully DG methods for fourth order variational inequality of the first kind.

The rest of the paper is organized as follows. In Section 2, we introduce a family of DG methods for (1.5). In Section 3, we derive the convergence analysis with help of the weak complementarity condition (cf. Section 3.1). Numerical examples are presented in Section 4 to illustrate the performance of the DG methods. We end the paper with some concluding remarks in Section 5.

2. Discontinuous Galerkin methods

2.1. Notation

Let \mathcal{T}_h be a shape-regular triangulation of Ω with mesh size h without hanging nodes. We will use the following notation throughout the paper.

- T: a triangle of \mathcal{T}_h in two dimensions and a tetrahedron of \mathcal{T}_h in three dimensions.
- \mathcal{V}_h : the set of the vertices of \mathcal{T}_h .
- \mathcal{V}_h^i : the set of all interior vertices of \mathcal{T}_h .
- \mathcal{V}_T : the set of the vertices of $T \in \mathcal{T}_h$.
- \mathcal{E}_h^i : the set of all interior edges/faces of \mathcal{T}_h .
- \mathcal{E}_h^b : the set of all boundary edges/faces of \mathcal{T}_h .
- $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$: the set of all edges/faces of \mathcal{T}_h .
- $\mathcal{E}_{\mathcal{V}_T}$: the set of edges/faces emanating from the vertices of $T \in \mathcal{T}_h$.
- h_e : the length of the edge $e \in \mathcal{E}_h$.

- $H^{s}(\Omega, \mathcal{T}_{h}) = \{v \in L_{2}(\Omega) : v_{T} = v|_{T} \in H^{s}(T) \quad \forall T \in \mathcal{T}_{h}\}$: piecewise Sobolev space.
- $V_h = \{v \in L_2(\Omega) : v_T \in P_r(T) \quad \forall T \in \mathcal{T}_h\}$: the discontinuous finite element space associated with \mathcal{T}_h , where r = 2 or 3.

We will also denote by *C* a generic positive constant independent of mesh sizes that can take different values at different occurrences. Next, we introduce average and jump operators that are needed in the construction and analysis of DG methods. Let $e \in \mathcal{E}_h^i$, then $e = \partial T^+ \cap \partial T^-$ for some $T^+, T^- \in \mathcal{T}_h$. We define the jump [v] and $[\nabla v]$ on *e* as

$$\llbracket v \rrbracket = v^+ n^+ + v^- n^-$$
 and $\llbracket \nabla v \rrbracket = \nabla v^+ \cdot n^+ + \nabla v^- \cdot n^-,$

for any $v \in H^s(\Omega, \mathcal{T}_h)$ with s > 3/2, where $v^{\pm} = v|_{T^{\pm}}$, and n^+ (resp., n^-) is the unit outward normal on e corresponding to ∂T^+ (resp., ∂T^-). The average $\{\Delta v\}$ and $\{\!\{\nabla \Delta v\}\!\}$ on e are defined by

$$\{\Delta v\} = \frac{1}{2}(\Delta v^+ + \Delta v^-) \quad \text{and} \quad \{\{\nabla \Delta v\}\} = \frac{1}{2}(\nabla \Delta v^+ + \nabla \Delta v^-)$$

for any $v \in H^s(\Omega, \mathcal{T}_h)$ with s > 7/2. For $e \in \mathcal{E}_h^b$, we denote T^+ the triangle that contains e and define

$$\llbracket v \rrbracket = v^+ n^+, \qquad [\nabla v] = \nabla v^+ \cdot n^+, \qquad \{ \Delta v \} = \Delta v^+, \qquad \{ [\nabla \Delta v] \} = \nabla \Delta v^+.$$

Let $T \in \mathcal{T}_h$, $v \in P_r(T)$ and $w \in H^2(T)$, we have an integration by parts formula

$$\int_{T} (\Delta v) (\Delta w) \, dx = \int_{T} (\Delta^2 v) w \, dx - \int_{\partial T} \nabla (\Delta v) \cdot wn \, ds + \int_{\partial T} \Delta v \, \nabla w \cdot n \, ds,$$
$$= -\int_{\partial T} \nabla (\Delta v) \cdot wn \, ds + \int_{\partial T} \Delta v \, \nabla w \cdot n \, ds,$$

where *n* is the unit normal vector to ∂T . Summing up over all $T \in \mathcal{T}_h$, we obtain

$$\sum_{T \in \mathcal{T}_{h}} \int_{T} (\Delta v) (\Delta w) \, dx = \sum_{T \in \mathcal{T}_{h}} \int_{T} (\Delta^{2} v) w \, dx - \sum_{e \in \mathcal{E}_{h}} \int_{e} \{ \nabla (\Delta v) \} \, [\![w]\!] \, ds$$
$$- \sum_{e \in \mathcal{E}_{h}^{i}} \int_{e} \{ w \} [\nabla (\Delta v)] \, ds + \sum_{e \in \mathcal{E}_{h}} \int_{e} \{ \Delta v \} [\nabla w] \, ds$$
$$+ \sum_{e \in \mathcal{E}_{h}^{i}} \int_{e} \{ \nabla w \} \, [\![\Delta v]\!] \, ds, \tag{2.1}$$

for all $v \in H^4(\Omega, \mathcal{T}_h)$ and $w \in H^2(\Omega, \mathcal{T}_h)$.

2.2. Discontinuous Galerkin methods

Let

$$K_h = \{ v \in V_h : \psi_1(p) \le v_T(p) \le \psi_2(p) \qquad \forall \ p \in \mathcal{V}_T, \ T \in \mathcal{T}_h \}.$$

$$(2.2)$$

The discrete variational inequality for (1.5) is: Find $u_h \in K_h$ such that

$$a_h(u_h, v - u_h) \ge F(v - u_h) \qquad \forall v \in K_h,$$
(2.3)

where

$$a_{h}(w, v) = \sum_{T \in \mathcal{T}_{h}} \int_{T} (\Delta w) (\Delta v) \, dx + \sum_{e \in \mathcal{E}_{h}} \int_{e} \{\!\!\{\nabla \Delta w\}\!\} [\![v]\!] \, ds - \sum_{e \in \mathcal{E}_{h}} \int_{e} \{\!\!\{\Delta w\}\!\} [\![\nabla v]\!] \, ds + \lambda_{1} \sum_{e \in \mathcal{E}_{h}} \int_{e} \{\!\!\{\nabla \Delta v\}\!\} [\![w]\!] \, ds - \lambda_{2} \sum_{e \in \mathcal{E}_{h}} \int_{e} \{\!\!\{\Delta v\}\!\} [\![\nabla w]\!] \, ds + \sum_{e \in \mathcal{E}_{h}} \frac{\sigma_{1}}{h_{e}^{3}} \int_{e} [\![w]\!] [\![v]\!] \, ds + \sum_{e \in \mathcal{E}_{h}} \frac{\sigma_{2}}{h_{e}} \int_{e} [\![\nabla w]\!] [\![\nabla v]\!] \, ds,$$

$$(2.4)$$

Table 1DG methods.				
Method	λ_1	λ_2	σ_1	σ_2
SIPG	1	1	$\sigma_1 > \sigma^*$	$\sigma_2 > \sigma^*$
NIPG	-1	-1	$\sigma_1 > 0$	$\sigma_2 > 0$
SSIPG1	-1	1	$\sigma_1 > 0$	$\sigma_2 > \sigma^*$
SSIPG2	1	-1	$\sigma_1 > \sigma^*$	$\sigma_2 > 0$

$$F(v) = (f, v) + F(v),$$

$$\tilde{F}(v) = \sum_{e \in \mathcal{E}_h^b} \int_e \left(\lambda_1 \{\!\!\{ \nabla \Delta v \}\!\!\} + \frac{\sigma_1}{h_e^3} [\![v]\!] \right) [\![g]\!] ds$$

$$+ \sum_{e \in \mathcal{E}_h^b} \int_e \left(\frac{\sigma_2}{h_e} [\nabla v] - \lambda_2 \{ \Delta v \} \right) [\nabla g] ds.$$

$$(2.6)$$

In view of (1.3), (2.2), (2.4)–(2.6), the boundary conditions are weakly imposed in the DG schemes.

The different choices for λ_1 , λ_2 , σ_1 and σ_2 lead to four different DG methods [23], which are SIPG method, NIPG methods and their combinations: the semisymmetric methods SSIPG1 and SSIPG2 (cf. Table 1). Here σ^* is a sufficiently large positive number that depend on the constants in the inverse inequalities, the degree of polynomial and the shape regularity of T_h [23].

To measure the error, we define a mesh-dependent energy norm by

$$\|v\|_{h}^{2} = \sum_{T \in \mathcal{T}_{h}} \|\Delta v\|_{L_{2}(T)}^{2} + \sum_{e \in \mathcal{E}} \left(\frac{\sigma_{1}}{h_{e}^{3}} \| [v] \|_{L_{2}(e)}^{2} + \frac{\sigma_{2}}{h_{e}} \| [\nabla v] \|_{L_{2}(e)}^{2}\right) \qquad \forall v \in V_{h}.$$

$$(2.7)$$

By the standard inverse estimate [8,27], the bilinear form $a_h(\cdot, \cdot)$ is bounded and coercive with respect to $\|\cdot\|_h$ provided that σ_1 and σ_2 are chosen according to Table 1 for different DG methods, i.e.,

$$a_h(w,v) \le C \|w\|_h \|v\|_h \qquad \forall w,v \in V_h,$$

$$(2.8)$$

$$a_h(v,v) \ge C \|v\|_h^2 \qquad \forall v \in V_h.$$

$$(2.9)$$

In particular for the NIPG method, the coercivity estimate (2.9) becomes

$$a_h(v, v) = \|v\|_h^2 \qquad \forall v \in V_h$$

Let $\Pi_h: H^2(\Omega) \longrightarrow V_h \cap H^1(\Omega)$ be the nodal interpolation operator for the P_r (r = 2 or 3) element such that

$$(\Pi_h \zeta)|_T(p) = \zeta(p) \qquad \forall \ p \in \mathcal{V}_T, \ T \in \mathcal{T}_h.$$

$$(2.10)$$

We have the following standard interpolation error estimate [8,27]

$$\begin{aligned} \|\zeta - \Pi_h \zeta\|_{L_2(\Omega)} + h \|\zeta - \Pi_h \zeta\|_{H^1(\Omega)} + h^2 \|\zeta - \Pi_h \zeta\|_h \\ &\leq C h^{\min(r+1,s)} \|\zeta\|_{H^s(\Omega)} \quad \forall \zeta \in H^s(\Omega), \ s \ge 2. \end{aligned}$$

$$(2.11)$$

For r = 3, we also have

$$|\zeta - \Pi_h \zeta|_{H^3(\mathcal{T}_h)} \le Ch_T^{\min(1,s-3)} \|\zeta\|_{H^s(T)} \qquad \forall \zeta \in H^s(T), \ T \in \mathcal{T}_h, \ s \ge 3,$$

$$(2.12)$$

where $h_T = \operatorname{diam}(T)$.

Since $\Pi_h u \in K_h$, the set K_h is nonempty closed convex in V_h . It then follows from (2.9) that the discrete problem (2.3) has a unique solution.

2.3. Enriching operator

Since the DG space $V_h \not\subset H^2(\Omega)$, we need an enriching operator [14,19,21,22] to measure the difference between the finite element space V_h and the Sobolev space $H^2(\Omega)$. For simplicity, we consider two dimensional case.



Fig. 1. Degrees of freedom for the Hsieh-Clough-Tocher macro element.

Let \mathcal{T}_h be simplicial triangulation of a polygonal domain $\Omega \subset \mathbb{R}^2$. We consider a linear operator

$$E_h: V_h \longrightarrow W_h \cap H^2_0(\Omega),$$

where W_h is the Hsieh–Clough–Tocher macro finite element space [8,27]. For $v \in V_h$, we define $E_h v$ by specifying its degrees of freedom (dofs), which are the values of the derivatives up to order 1 at the vertices, and the values of the normal derivative at the midpoints of the edges (cf. Fig. 1). The dofs of $E_h v$ at any interior nodal point of V_h are defined to be the average of the corresponding dofs of v from the triangles of \mathcal{T}_h sharing the nodal point. For the boundary nodal point, we set the dofs of $E_h v$ to be 0.

For any $v \in K_h$, we have by (2.2) and the definition of E_h that

$$\psi_1(p) \le E_h v(p) \le \psi_2(p) \qquad \forall \, p \in \mathcal{V}_h^i, \tag{2.13}$$

where $\mathcal{V}_h^i = \mathcal{V}_h \cap \Omega$. In particular,

$$E_h v(p) = v(p)$$
 $\forall p \in \mathcal{V}_h^i, v \in V_h \cap H^1(\Omega).$

For any $v \in V_h$ and $T \in \mathcal{T}_h$, the following local approximation property can be proved similar to Lemma 4.1 in [28]:

$$\sum_{m=0}^{2} h_{T}^{2m} |v - E_{h}v|_{H^{m}(T)}^{2} \leq Ch_{T}^{4} \sum_{e \in \mathcal{E}_{\mathcal{V}(T)}} \left(\frac{1}{h_{e}^{3}} \| \left[\!\left[v\right]\!\right]\!\right]_{L_{2}(e)}^{2} + \frac{1}{h_{e}} \| \left[\!\left[\nabla v\right]\!\right]\!\right]_{L_{2}(e)}^{2} \right).$$

$$(2.14)$$

From (2.14) and the standard inverse inequality [8,27], we have the global estimates

$$\|v - E_{h}v\|_{L_{2}(\Omega)}^{2} + h^{2} \sum_{T \in \mathcal{T}_{h}} |v - E_{h}v|_{H^{1}(T)}^{2} + h^{4} \|v - E_{h}v\|_{h}^{2}$$

$$\leq Ch^{4} \sum_{e \in \mathcal{E}_{h}} \left(h_{e}^{-3}\|[v]\|_{L_{2}(e)}^{2} + h_{e}^{-1}\|[\nabla v]\|_{L_{2}(e)}^{2}\right) \leq Ch^{4}\|v\|_{h}, \qquad (2.15)$$

$$\sum_{e \in \mathcal{E}_{h}} \left(h_{e}^{-1} \| \left\{ \nabla (v - E_{h}v) \right\} \|_{L_{2}(e)}^{2} + h_{e}^{-3} \| \left\{ v - E_{h}v \right\} \|_{L_{2}(e)}^{2} \right)$$

$$\leq C \sum_{e \in \mathcal{E}_{h}} \left(h_{e}^{-3} \| \left[v \right] \|_{L_{2}(e)}^{2} + h_{e}^{-1} \| \left[\nabla v \right] \|_{L_{2}(e)}^{2} \right) \leq C \| v \|_{h}, \qquad (2.16)$$

for all $v \in V_h$.

Furthermore, (2.11), (2.15) and the standard inverse inequality imply that

$$\begin{aligned} \|\zeta - E_h \Pi_h \zeta\|_{L_2(\Omega)} + h^{2-\beta} \|\zeta - E_h \Pi_h \zeta\|_{H^{2-\beta}(\Omega)} \\ &+ h |\zeta - E_h \Pi_h \zeta|_{H^1(\Omega)} + h^2 |\zeta - E_h \Pi_h \zeta|_{H^2(\Omega)} \\ &\leq C h^{\min(r+1,s)} \|\zeta\|_{H^s(\Omega)} \qquad \forall \zeta \in H^s(\Omega) \cap H_0^2(\Omega), \end{aligned}$$

$$(2.17)$$

for any $\beta \in (1, 2)$.

Remark 2.1. The enriching operator can also be constructed in three dimensions where the Hsieh–Clough–Tocher macro finite element space is replaced by the Ženíšek finite element space [29].

In order to deal with the nonhomogeneous boundary conditions, we consider an affine operator $T_h: V_h \longrightarrow H^2(\Omega)$ defined by

$$T_h v = g + E_h (v - \Pi_h g) \qquad \forall v \in V_h.$$
(2.18)

It follows from (2.10) and (2.18) that

$$u - T_h \Pi_h u = (u - g) - E_h \Pi_h (u - g).$$
(2.19)

Therefore we can take $\zeta = u - g$ in (2.17) since $u - g \in H^s(\Omega) \cap H^2_0(\Omega)$. Furthermore,

$$T_h \Pi_h u - T_h v = E_h (\Pi_h u - v). \tag{2.20}$$

Since for all $v \in V_h$,

$$(T_h v)(p) = g(p) + E_h(v - \Pi_h g)(p) = (E_h v)(p) \quad \forall p \in \mathcal{V}_h^i,$$
(2.21)

$$(T_h v)(p) = g(p) \qquad \forall \, p \in \mathcal{V}_h \cap \partial \Omega, \tag{2.22}$$

we have by (2.13),

$$\psi_1(p) \le T_h v(p) \le \psi_2(p) \qquad \forall \, p \in \mathcal{V}_h, \tag{2.23}$$

for all $v \in K_h$.

3. Convergence analysis

In this section, we will prove the optimal error estimate for $||u - u_h||_h$ in both two and three dimensions. First of all, we introduce the complementarity form of the variational inequality (1.5) that will be crucial for the error analysis.

3.1. Complementarity form of the variational inequality

Let the contact set defined by

$$\mathcal{A}_i = \{ x \in \Omega : u(x) = \psi_i(x) \} \qquad i = 1, 2.$$

Due to (1.1), we know $A_1 \cap A_2 = \emptyset$ and they are both compact and disjoint from $\partial \Omega$. By the Riesz representation theorem [30,31], there exists a Borel measure μ such that

$$a(u,v) - (f,v) = \int_{\Omega} v \, d\mu \qquad \forall v \in H_0^2(\Omega),$$
(3.1)

where $\mu = \mu_1 - \mu_2$ is the Jordan decomposition of μ . In particular, μ_1 and μ_2 are nonnegative Borel measures that concentrate on A_1 and A_2 , respectively, i.e.,

$$\mu_1(\Omega \setminus \mathcal{A}_1) = \mu_2(\Omega \setminus \mathcal{A}_2) = 0. \tag{3.2}$$

Furthermore, $\mu_i(\Omega) = \mu_i(A_i) < \infty$ (i = 1, 2) and the solution u to (1.5) satisfies the following weak complementarity conditions:

$$\int_{\Omega} (u - \psi_i) d\mu_i = 0 \qquad i = 1, 2.$$
(3.3)

We denote by $|\mu|$ the total variation measure of μ , then

$$|\mu|(\Omega) = \mu_1(\Omega) + \mu_2(\Omega) < \infty.$$
(3.4)

Let \mathcal{G} be an open neighborhood of $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$ with a smooth boundary such that $\overline{\mathcal{G}}$ is a compact subset of Ω , and let $\phi \in C_c^{\infty}(\Omega)$ with compact support in \mathcal{G} and $\phi = 1$ in \mathcal{A} . Since $u \in H^3_{loc}(\Omega)$, we have by (3.1) and integration by parts that

$$\int_{\Omega} v \, d\mu = \int_{\Omega} (\phi v) \, d\mu = -\int_{\Omega} \nabla(\Delta u) \cdot \nabla(v\phi) \, dx - (f, v\phi)$$
$$= B(u, v) - (f, v\phi) \quad \forall v \in H_0^2(\Omega),$$
(3.5)

where $B(\zeta, w) = -\int_{\Omega} \nabla(\Delta \zeta) \cdot \nabla(w\phi) dx$. We also have

$$|B(\zeta, w)| \le \|\nabla(\Delta\zeta)\|_{L_2(\Omega)} \|\nabla(w\phi)\|_{L_2(\Omega)}$$

$$\le C_{\mathcal{G}} \|\zeta\|_{H^3(\mathcal{G})} \|w\|_{H^1(\mathcal{G})} \quad \forall \zeta \in H^3(\mathcal{G}), \ w \in H^1(\mathcal{G}),$$
(3.6)

by the Cauchy-Schwarz inequality. Now we combine (3.5) and (3.6) to get

$$\left| \int_{\Omega} v \, d\mu \right| \le (C_{\mathcal{G}} \|u\|_{H^{3}(\mathcal{G})} + \|f\|_{L_{2}(\mathcal{G})}) \|v\|_{H^{1}(\mathcal{G})} \qquad \forall v \in H_{0}^{2}(\Omega),$$
(3.7)

which together with a density argument implies $\mu \in H^{-1}(\Omega) = [H_0^1(\Omega)]'$.

3.2. Error estimate for $||u - u_h||_h$

. .

We begin by a useful technical lemma that holds for all four DG methods proposed in Section 2.2.

Lemma 3.1. There exists a positive constant C, depending on the shape regularity of T_h , such that

$$a_h(\Pi_h\zeta, w) - F(w) - a(\zeta, E_hw) \le Ch^{\min(r+1,s)-2} \|\zeta\|_{H^s(\Omega)} \|w\|_h,$$
for all $\zeta \in H^s(\Omega)$, $s \in [2, 4]$ with $\zeta - g \in H^2_0(\Omega)$ and all $w \in V_h$.
$$(3.8)$$

Proof. Since $E_h w \in H_0^2(\Omega)$ for any $w \in V_h$, by (2.1), (2.4) and (2.6) we have

$$a_{h}(\Pi_{h}\zeta, w) - \tilde{F}(w) - a(\zeta, E_{h}w) = \sum_{T \in \mathcal{T}_{h}} \int_{T} \Delta(\Pi_{h}\zeta - \zeta)\Delta(E_{h}w) dx$$

$$+ \lambda_{1} \sum_{e \in \mathcal{E}_{h}} \int_{e} \{\!\{\nabla \Delta w\}\!\} [\![\Pi_{h}\zeta - \zeta]\!] ds - \lambda_{2} \sum_{e \in \mathcal{E}_{h}} \int_{e} \{\!\Delta w\}\![\nabla(\Pi_{h}\zeta - \zeta)] ds$$

$$+ \sum_{e \in \mathcal{E}_{h}} \frac{\sigma_{1}}{h_{e}^{3}} \int_{e} [\![\Pi_{h}\zeta - \zeta]\!] [\![w]\!] ds + \sum_{e \in \mathcal{E}_{h}} \frac{\sigma_{2}}{h_{e}} \int_{e} [\![\nabla(\Pi_{h}\zeta - \zeta)]\![\nabla w] ds$$

$$- \sum_{e \in \mathcal{E}_{h}^{i}} \int_{e} \{\!w - E_{h}w\}\![\nabla\Delta(\Pi_{h}\zeta)] ds + \sum_{e \in \mathcal{E}_{h}^{i}} \int_{e} \{\!\{\nabla(w - E_{h}w)\}\!\} [\![\Delta(\Pi_{h}\zeta)]\!] ds$$

$$:= S_{1} + S_{2} + S_{3} + S_{4} + S_{5} + S_{6} + S_{7},$$
(3.9)

where λ_1 and λ_2 are taken to be 1 or -1 for different DG methods (cf. Table 1).

By using a standard inverse estimate, trace theorem and (2.11), the terms S_i (i = 1, 2, ..., 5) on the right-hand side of (3.9) can be estimated as follows:

$$|S_1| \le C \left(\sum_{T \in \mathcal{T}_h} |\Pi_h \zeta - \zeta|^2_{H^2(T)} \right)^{\frac{1}{2}} \|w\|_h \le C h^{\min(r+1,s)-2} |\zeta|_{H^s(\Omega)} \|w\|_h,$$
(3.10)

$$\begin{aligned} |S_{2}| &\leq \left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{3} \| \left\{\!\!\{\nabla \Delta w\}\!\}\right\} \|_{L_{2}(e)}^{2} \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{-3} \| \left[\![\Pi_{h} \zeta - \zeta]\!]\right] \|_{L_{2}(e)}^{2} \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{T \in \mathcal{T}_{h}} \|\Delta w\|_{L_{2}(T)}^{2} \right)^{\frac{1}{2}} h^{\min(r+1,s)-2} |\zeta|_{H^{s}(\Omega)} \leq C h^{\min(r+1,s)-2} |\zeta|_{H^{s}(\Omega)} \|w\|_{h}. \end{aligned}$$
(3.11)

$$|S_{3}| \leq \left(\sum_{e \in \mathcal{E}_{h}} h_{e} \|\{\Delta w\}\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{-1} \|[\nabla(\Pi_{h}\zeta - \zeta)]\|_{L_{2}(e)}^{2}\right)^{\frac{1}{2}}$$

$$\leq Ch^{\min(r+1,s)-2} |\zeta|_{H^{s}(\Omega)} \|w\|_{h}.$$
(3.12)

$$|S_4| \le \left(\sum_{e \in \mathcal{E}_h} h_e^{-3} \| [\![\Pi_h \zeta - \zeta]\!] \|_{L_2(e)}^2 \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}_h} h_e^{-3} \| [\![w]\!] \|_{L_2(e)}^2 \right)^{\frac{1}{2}}$$

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$$\leq Ch^{\min(r+1,s)-2} |\zeta|_{H^{s}(\Omega)} ||w||_{h}.$$

$$|S_{5}| \leq \left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{-1} ||[\nabla(\Pi_{h}\zeta - \zeta)]||_{L_{2}(e)}^{2}\right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{-1} ||[\nabla w]||_{L_{2}(e)}^{2}\right)^{\frac{1}{2}}$$

$$\leq Ch^{\min(r+1,s)-2} |\zeta|_{H^{s}(\Omega)} ||w||_{h}.$$
(3.14)

Now we estimate S_6 on the right-hand side of (3.9). For r = 2, we have $S_6 = 0$. In order to estimate S_6 in the case of r = 3, we denote

$$G_1(\xi) = -\sum_{e \in \mathcal{E}_h^i} \int_e \{w - E_h w\} [\nabla \Delta(\Pi_h \xi)] \, ds.$$
(3.15)

Whenever $\xi \in H^2(\Omega)$, by a standard inverse estimate and (2.16), we have

$$|G_{1}(\xi)| \leq C \left(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e}^{-3} \| \{\!\!\{w - E_{h}w\}\!\} \|_{L_{2}(e)}^{2} \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e}^{3} \| [\nabla \Delta(\Pi_{h}\xi)] \|_{L_{2}(e)}^{2} \right)^{\frac{1}{2}} \\ \leq C \| \xi \|_{H^{2}(\Omega)} \| w \|_{h}.$$
(3.16)

When $\xi \in H^4(\Omega)$, we apply (2.12), (2.16) and a standard inverse estimate to obtain

$$|G_{1}(\xi)| = \left| \sum_{e \in \mathcal{E}_{h}^{i}} \int_{e}^{\xi} \{w - E_{h}w\} [\nabla \Delta(\Pi_{h}\xi - \xi)] ds \right|$$

$$\leq C \left(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e}^{-3} \|\{w - E_{h}w\}\|_{L_{2}(e)}^{2} \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e}^{3} \|[\nabla \Delta(\Pi_{h}\xi - \xi)]\|_{L_{2}(e)}^{2} \right)^{\frac{1}{2}}$$

$$\leq Ch^{2} \|\xi\|_{H^{4}(\Omega)} \|w\|_{h}.$$
(3.17)

Using interpolation between Sobolev spaces $H^2(\Omega)$ and $H^4(\Omega)$ (cf. [8,27]), we have by (3.16) and (3.17) that

$$|G_1(\xi)| \le Ch^{s-2} \|\xi\|_{H^s(\Omega)} \|w\|_h \tag{3.18}$$

for all $\xi \in H^s(\Omega)$ with $s \in [2, 4]$ and $w \in V_h$. In particular, we take $\xi = \zeta$ in (3.18) to obtain the estimate of S_6 in the case of r = 3:

$$|S_6| \le Ch^{s-2} \|\zeta\|_{H^s(\Omega)} \|w\|_h.$$
(3.19)

To estimate S_7 , we define

$$G_2(\xi) = \sum_{e \in \mathcal{E}_h^i} \int_e \{\!\!\{\nabla(w - E_h w)\}\!\!\} [\![\Delta(\Pi_h \xi)]\!] ds.$$
(3.20)

By (2.11), (2.16) and a standard inverse estimate, we have

$$|G_{2}(\xi)| \leq C \Big(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e}^{-1} \| \{\!\!\{ \nabla(w - E_{h}w) \}\!\!\} \|_{L_{2}(e)}^{2} \Big)^{\frac{1}{2}} \Big(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e} \| [\![\Delta(\Pi_{h}\xi)]\!] \|_{L_{2}(e)}^{2} \Big)^{\frac{1}{2}} \\ \leq C \| \xi \|_{H^{2}(\Omega)} \| w \|_{h} \qquad \forall \xi \in H^{2}(\Omega),$$
(3.21)

and

$$\begin{aligned} |G_{2}(\xi)| &= \left| \sum_{e \in \mathcal{E}_{h}^{i}} \int_{e} \{\!\!\{\nabla(w - E_{h}w)\}\!\} [\![\Delta(\Pi_{h}\xi - \xi)]\!] ds \right| \\ &\leq C \Big(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e}^{-1} \| \{\!\!\{\nabla(w - E_{h}w)\}\!\} \|_{L_{2}(e)}^{2} \Big)^{\frac{1}{2}} \Big(\sum_{e \in \mathcal{E}_{h}^{i}} h_{e} \| [\![\Delta(\Pi_{h}\xi - \xi)]\!] \|_{L_{2}(e)}^{2} \Big)^{\frac{1}{2}} \\ &\leq Ch \|\xi\|_{H^{3}(\Omega)} \|w\|_{h}, \quad \forall \xi \in H^{3}(\Omega). \end{aligned}$$
(3.22)

Moreover, in the case of r = 3,

$$|G_2(\xi)| \le Ch^2 \|\xi\|_{H^4(\Omega)} \|w\|_h \qquad \forall \xi \in H^4(\Omega),$$
(3.23)

where we used (2.12), (2.16) and a standard inverse estimate. Taking $\xi = \zeta$, combining (3.21)–(3.23) and applying interpolation between Sobolev spaces [8,27], we get

$$|S_7| \le Ch^{\min(r+1,s)-2} \|\zeta\|_{H^s(\Omega)} \|w\|_h.$$
(3.24)

Finally, the estimate (3.8) follows from (3.9)–(3.14), (3.19) and (3.24).

Due to the nonconformity of the DG space and the discrete constraint set, i.e., $V_h \not\subset H_0^2(\Omega)$ and $K_h \not\subset K$, it is important to establish the connection between these spaces. For this aim, we reduce the error between u and u_h in the energy norm to an estimate that only involves the continuous bilinear form $a(\cdot, \cdot)$. In next two lemmas, we assume $u \in H^s(\Omega) \cap W_{loc}^{2,\infty}(\Omega)$ for some $s \in [2, 4]$.

Lemma 3.2. There exists a positive constant C, depending on the shape regularity of T_h , such that

$$\|u - u_h\|_h^2 \le C \left(h^{2\min(r+1,s)-4} + h^{\min(r+1,s)-2} \|\Pi_h u - u_h\|_h + a(u, E_h(\Pi_h u - u_h)) - f, E_h(\Pi_h u - u_h) \right).$$
(3.25)

Proof. Since $\Pi_h u \in K_h$, it follows from (2.3) and (2.9) that

$$\begin{aligned} \|u - u_h\|_h^2 &\leq 2\|u - \Pi_h u\|_h^2 + 2\|\Pi_h u - u_h\|_h^2 \\ &\leq 2\|u - \Pi_h u\|_h^2 + Ca_h(\Pi_h u - u_h, \Pi_h u - u_h) \\ &\leq 2\|u - \Pi_h u\|_h^2 + C(a_h(\Pi_h u, \Pi_h u - u_h) - F(\Pi_h u - u_h)). \end{aligned}$$
(3.26)

Taking $\zeta = u$ and $w = \prod_h u - u_h$ in Lemma 3.1, we have

$$a_{h}(\Pi_{h}u, \Pi_{h}u - u_{h}) - \tilde{F}(\Pi_{h}u - u_{h}) \\ \leq Ch^{\min(r+1,s)-2} \|\Pi_{h}u - u_{h}\|_{h} + a(u, E_{h}(\Pi_{h}u - u_{h})).$$
(3.27)

Furthermore using (2.16), we find

$$-(f, \Pi_h u - u_h) \le -(f, E_h(\Pi_h u - u_h)) + Ch^2 \|\Pi_h u - u_h\|_h.$$
(3.28)

Now the estimate (3.25) follows from (2.11) and (3.26)–(3.28).

Lemma 3.3. There exists a positive constant C, depending on the shape regularity of \mathcal{T}_h , such that

$$a(u, E_h(\Pi_h u - u_h)) - (f, E_h(\Pi_h u - u_h)) \leq C(h^{\min(r+1,s)-1} + h^2 + h \|\Pi_h u - u_h\|_h).$$
(3.29)

Proof. By (2.20) and (3.1) and the fact that $E_h(\Pi_h u - u_h) \in H_0^2(\Omega)$, we have

$$\begin{aligned} a(u, E_{h}(\Pi_{h}u - u_{h})) &- (f, E_{h}(\Pi_{h}u - u_{h})) = \int_{\Omega} E_{h}(\Pi_{h}u - u_{h}) d\mu \\ &= \int_{\Omega} (T_{h}\Pi_{h}u - T_{h}u_{h}) d\mu \\ &= \int_{\Omega} (T_{h}\Pi_{h}u - u) d\mu + \left[\int_{\Omega} (u - \psi_{1}) d\mu_{1} - \int_{\Omega} (u - \psi_{2}) d\mu_{2} \right] \\ &+ \left[\int_{\Omega} (I_{h}(u - \psi_{1}) - (u - \psi_{1})) d\mu_{1} - \int_{\Omega} (I_{h}(u - \psi_{2}) - (u - \psi_{2})) d\mu_{2} \right] \\ &+ \left[\int_{\Omega} (I_{h}\psi_{1} - I_{h}T_{h}u_{h}) d\mu_{1} - \int_{\Omega} (I_{h}\psi_{2} - I_{h}T_{h}u_{h}) d\mu_{2} \right] \\ &+ \int_{\Omega} (I_{h}(T_{h}u_{h} - u) - (T_{h}u_{h} - u)) d\mu \\ &:= R_{1} + R_{2} + R_{3} + R_{4} + R_{5}, \end{aligned}$$
(3.30)

where I_h is the standard nodal interpolation operator for the conforming linear finite element space associated with T_h .

It directly follows from (3.3) that

$$R_2 = 0.$$
 (3.31)

In view of (1.3) and (2.23) and the facts that $\mu_i \ge 0$ (i = 1, 2), we can estimate R_4 by

$$R_4 \le 0. \tag{3.32}$$

For R_1 , by (2.17), (2.19) and (3.1) we have

$$|R_{1}| \leq \|\mu\|_{H^{-1}(\Omega)} \|T_{h}\Pi_{h}u - u\|_{H^{1}(\Omega)}$$

$$\leq C \|E_{h}\Pi_{h}(u - g) - (u - g)\|_{H^{1}(\Omega)}$$

$$< Ch^{\min(r+1,s)-1}.$$
(3.33)

Next we introduce notation

$$\mathcal{A}_{i,h} = \bigcup_{T \in \mathcal{T}_h} \{T \cap \mathcal{A}_i \neq \emptyset\} \qquad i = 1, 2$$

Without loss of generality, we assume h is small enough such that the distance between $A_{i,h}$ and $\partial \Omega$ is positive. Then by standard interpolation error estimate for I_h [8,27], we can bound R_3 as

$$|R_{3}| \leq \mu_{1}(\mathcal{A}_{1}) \| I_{h}(u - \psi_{1}) - (u - \psi_{1}) \|_{L_{\infty}(\mathcal{A}_{1,h})} + \mu_{2}(\mathcal{A}_{2}) \| I_{h}(u - \psi_{2}) - (u - \psi_{2}) \|_{L_{\infty}(\mathcal{A}_{2,h})} \leq Ch^{2} (\| u - \psi_{1} \|_{W^{2,\infty}(\mathcal{A}_{1,h})} + \| u - \psi_{2} \|_{W^{2,\infty}(\mathcal{A}_{2,h})}) \leq Ch^{2}.$$

$$(3.34)$$

By using (2.17), we can estimate R_5 as follows:

$$\begin{aligned} |R_{5}| &\leq \|\mu\|_{H^{-1}(\Omega)} \|I_{h}(T_{h}u_{h} - u) - (T_{h}u_{h} - u)\|_{H^{1}(\Omega)} \\ &\leq Ch|T_{h}u_{h} - u|_{H^{2}(\Omega)} \\ &\leq Ch(|E_{h}(u_{h} - \Pi_{h}u)|_{H^{2}(\Omega)} + |T_{h}\Pi_{h}u - u|_{H^{2}(\Omega)}) \\ &\leq C(h\|u_{h} - \Pi_{h}u\|_{h} + h^{\min(r+1,s)-1}). \end{aligned}$$

$$(3.35)$$

From (3.30)–(3.35), we obtain the estimate (3.29).

Now we combine Lemma 3.2 and Lemma 3.3 to obtain the optimal error estimate for the quadratic method (r = 2).

Theorem 3.1. Suppose the regularity result $u \in H^{2+\alpha}(\Omega)$ holds for some $\alpha \in (1/2, 1]$. There exists a positive constant C, depending on the shape regularity of \mathcal{T}_h , such that

$$\|u - u_h\|_h \le Ch^{\alpha},\tag{3.36}$$

for the quadratic method (r = 2).

Proof. By taking $s = 2 + \alpha$ in (3.25) and (3.29), we obtain

$$\|u - u_h\|_h^2 \le C(h^{2\alpha} + h^{1+\alpha} + h^2) + \frac{1}{2}\|u - u_h\|_h^2.$$
(3.37)

Since $\alpha \in (1/2, 1]$, we prove (3.36). \Box

In the case where the contact sets A_1 and A_2 are smooth and do not degenerate to lower dimensional surfaces in \mathbb{R}^d , we may improve the regularity of *u*. Indeed, under the following assumption:

The free boundary
$$\partial \mathcal{A} := \partial \mathcal{A}_1 \cup \partial \mathcal{A}_2$$
 is smooth and $u \in H^4(\mathcal{G} \setminus \mathcal{A}) \cap H^4(\mathcal{A})$, (3.38)

where \mathring{A} is the interior of A and G is described as in Section 3.1. From (3.38), we have $u \in H^{2+\alpha}_{loc}(\Omega)$ for any $\alpha < 1.5$. In this case, it holds that

$$|u(x) - \psi_i(x)| \le C_{\alpha} h^{2\alpha} \qquad i = 1, 2, \tag{3.39}$$

for any $x \in \Omega$ whose distance to A_i (i = 1, 2) is $\leq h$ and $\alpha \in (1, 1.5)$. Note that the proof of (3.39) can be found in Lemma 5.5 of [26].

Next, we show that the regularity of μ can also be improved.

Lemma 3.4. There exists a positive constant C, depending on the shape regularity of \mathcal{T}_h , such that

$$\int_{\Omega} v \, d\mu \le C(\|u\|_{H^{2+\alpha}_{loc}(\Omega)} + \|f\|_{L_{2}(\Omega)}) \|v\|_{H^{2-\alpha}(\Omega)} \qquad \forall v \in H^{2-\alpha}_{0}(\Omega), \ \alpha \in (1, 1.5),$$
(3.40)

i.e., $\mu \in H^{\alpha-2}(\Omega) = [H_0^{2-\alpha}(\Omega)]'$. Here $H_0^{2-\alpha}(\Omega) := \{v \in H^{2-\alpha}(\Omega) : v = 0 \text{ on } \partial \Omega\}.$

Proof. Let $B(\cdot, \cdot)$ be defined in Section 3.1. Through integration by parts, we have

$$|B(\zeta, w)| \le C_{\mathcal{G}} \|\zeta\|_{H^4(\mathcal{G})} \|w\|_{L_2(\mathcal{G})} \qquad \forall \zeta \in H^4(\mathcal{G}), \ w \in H^1(\mathcal{G}).$$

$$(3.41)$$

Combining (3.6) and (3.41), we can apply the interpolation of bilinear forms on Sobolev spaces (see Section 4.4 in [32]) to extend B to $H^{2+\alpha}(\mathcal{G}) \times H^{2-\alpha}(\mathcal{G})$ such that

$$|B(\zeta, w)| \le C_{\mathcal{G},\alpha} \|\zeta\|_{H^{2+\alpha}(\mathcal{G})} \|w\|_{H^{2-\alpha}(\mathcal{G})} \qquad \forall \zeta \in H^{2+\alpha}(\mathcal{G}), \ w \in H^{2-\alpha}(\mathcal{G}).$$
(3.42)

This together with (3.5) implies

$$\left| \int_{\Omega} v \, d\mu \right| \le C_{\alpha}(\|u\|_{H^{2+\alpha}_{loc}(\Omega)} + \|f\|_{L_{2}(\Omega)}) \|v\|_{H^{2-\alpha}(\Omega)} \qquad \forall v \in H^{2}_{0}(\Omega).$$
(3.43)

Finally, the estimate (3.40) follows from the density argument. \Box

In the rest of the section, we aim to extend the optimal error estimate (3.36) to the cubic method (r = 3) under the regularity $u \in H^{2+\alpha}(\Omega)$ for some $\alpha \in (1, 1.5)$.

Theorem 3.2. Suppose the assumption (3.38) holds and u belongs to $H^{2+\alpha}(\Omega)$ for some $\alpha \in (1, 1.5)$. There exists a positive constant C, depending on the shape regularity of \mathcal{T}_h , such that

$$\|u - u_h\|_h \le Ch^{\alpha},\tag{3.44}$$

for the cubic method (r = 3).

Proof. In order to extend the error estimate in Theorem 3.1 to the cubic element, we need to improve the estimates of R_1 , R_3 and R_5 that appeared in the proof of Lemma 3.3.

For R_1 , by combining (2.17), (2.19), (3.1) and Lemma 3.4, we have

$$\begin{aligned} |R_{1}| &\leq \|\mu\|_{H^{\alpha-2}(\Omega)} \|T_{h} \Pi_{h} u - u\|_{H^{2-\alpha}(\Omega)} \\ &\leq C \|E_{h} \Pi_{h} (u - g) - (u - g)\|_{H^{2-\alpha}(\Omega)} \\ &\leq C h^{2\alpha} \end{aligned}$$
(3.45)

Similarly, for R_5 we have

$$\begin{aligned} |R_{5}| &\leq \|\mu\|_{H^{\alpha-2}(\Omega)} \|I_{h}(T_{h}u_{h}-u) - (T_{h}u_{h}-u)\|_{H^{2-\alpha}(\Omega)} \\ &\leq Ch^{\alpha}|T_{h}u_{h}-u|_{H^{2}(\Omega)} \\ &\leq Ch^{\alpha}(|E_{h}(u_{h}-\Pi_{h}u)|_{H^{2}(\Omega)} + |T_{h}\Pi_{h}u-u|_{H^{2}(\Omega)}) \\ &\leq C(h^{\alpha}\|u_{h}-\Pi_{h}u\|_{h} + h^{2\alpha}). \end{aligned}$$
(3.46)

	· · · · · · · · ·				
$\ e_h\ _h$	Order	$\ e_h\ _{H^1}$	Order	$\ e_h\ _{L_{\infty}}$	Order
6.8886E-04	_	3.3545E-02	_	9.9629E-03	-
2.9617E-04	1.2178	1.1980E-02	1.4855	1.7108E-03	2.5419
1.2106E-04	1.2907	3.4353E-03	1.8021	3.1192E-04	2.4554
5.4987E-05	1.1386	9.1452E-04	1.9094	6.2420E-05	2.3211
2.6346E-05	1.0615	2.3506E-04	1.9600	1.3422E-05	2.2174
		$\ e_h\ _h$ Order 6.8886E-04 - 2.9617E-04 1.2178 1.2106E-04 1.2907 5.4987E-05 1.1386 2.6346E-05 1.0615	$\ e_h\ _h$ Order $\ e_h\ _{H^1}$ 6.8886E-04 - 3.3545E-02 2.9617E-04 1.2178 1.1980E-02 1.2106E-04 1.2907 3.4353E-03 5.4987E-05 1.1386 9.1452E-04 2.6346E-05 1.0615 2.3506E-04	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$ \begin{array}{c c c c c c c c c c c c c c c c c c c $

Table 2 Ouadratic method (r = 2) for Example 4.1 with exact solution.

By (3.3) and (3.39), we can estimate R_3 as follows.

$$R_{3}| \leq \left| \int_{\Omega} \left(I_{h}(u - \psi_{1}) - (u - \psi_{1}) \right) d\mu_{1} + \int_{\Omega} \left(I_{h}(u - \psi_{2}) - (u - \psi_{2}) \right) d\mu_{2} \right|$$

$$\leq \left| \int_{\Omega} I_{h}(u - \psi_{1}) d\mu_{1} \right| + \left| \int_{\Omega} I_{h}(u - \psi_{2}) d\mu_{2} \right|$$

$$\leq C(\|I_{h}(u - \psi_{1})\|_{L_{\infty}(\mathcal{A}_{1,h})} + \|I_{h}(u - \psi_{2})\|_{L_{\infty}(\mathcal{A}_{2,h})}) \leq C_{\alpha} h^{2\alpha}.$$
(3.47)

Finally, by combining with other estimates in Lemma 3.3, we obtain

$$\|u - u_h\|_h^2 \le Ch^{2\alpha} + \frac{1}{2}\|u - u_h\|_h^2$$
(3.48)

for $\alpha \in (1, 1.5)$, and thus the estimate (3.44) is proved. \Box

4. Numerical results

In this section, we report the numerical results of several examples obtained by SIPG method for two-dimensional obstacle problem. We consider one-obstacle problems in Examples 4.1 and 4.2. In the first example, an exact solution with nonhomogeneous boundary conditions is constructed on the square domain. The second example is constructed on a *L*-shaped domain so that the index of elliptic regularity $\alpha < 1$. In the third example, we consider a two-obstacle problem. We will investigate numerical errors and rates of convergence in various norms and also plot the discrete contact sets. In all numerical tests, we take $\sigma_1 = 30$, $\sigma_2 = 15$ for the quadratic method, and $\sigma_1 = 650$, $\sigma_2 = 50$ for the cubic method. To solve the discrete problems, we have used the tools developed via the FEniCS project [33,34]. We denote the lower and upper obstacle functions by $\psi(x)$ and $\tilde{\psi}(x)$ respectively, and solve the discrete obstacle problems on uniform triangulations with the mesh size *h*.

Example 4.1. In this example, we consider the obstacle problem on the disc $\{x : |x| < 2\}$ for the data $\psi(x) = 1 - |x|^2$, f(x) = 0 and g(x) = 0. We can find the exact solution u(x) of this obstacle problem as follows:

$$u(r) = \begin{cases} C_1 |x|^2 ln|x| + C_2 |x|^2 + C_3 ln|x| + C_4 & |x| > r_0 \\ 1 - |x|^2 & |x| \le r_0 \end{cases},$$
(4.1)

where $r_0 \approx 0.181345$, $C_1 \approx 0.525041$, $C_2 \approx 0.628609$, $C_3 \approx 0.017266$, $C_4 \approx 1.046746$. Then we restrict the problem to $\Omega = (-0.5, 0.5)^2$ with the same ψ so that the exact solution is the restriction of u on Ω with nonhomogeneous Dirichlet boundary conditions determined by (4.1). Note that u belongs to $H^{2+\alpha}(\Omega)$ for any $\alpha < 1.5$. We consider the error $e_h = \Pi_h u - u_h$ and evaluate the errors in the energy norm, H^1 norm and L_∞ norm, where

$$\|e_h\|_{L_{\infty}} = \max_{T \in \mathcal{T}_h, p \in \mathcal{V}_T} |e_{h,T}(p)| \qquad \forall e \in V_h,$$

and $e_{h,T} = e_h|_T$.

The numerical results for the quadratic method (resp. cubic method) are given in Table 2 (resp. Table 3). The asymptotic convergence rate is 1 for r = 2 and 1.5 for r = 3 in the energy norm as predicted by Theorems 3.1 and 3.2. We also observe that the H^1 norm and L_{∞} norm errors are of order $O(h^2)$ for the quadratic method. For the cubic method, the average rates of convergence in lower order norms are close to 2.5.

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h	$\ e_h\ _h$	Order	$\ e_h\ _{H^1}$	Order	$\ e_h\ _{L_{\infty}}$	Order
1/4	1.5527E-04	_	2.5613E-03	_	6.2322E-04	-
1/8	5.8237E-05	1.4147	6.4097E-04	1.9986	1.7554E-04	1.8279
1/16	2.2578E-05	1.3670	1.2552E-04	2.3523	2.9250E-05	2.5853
1/32	7.8761E-06	1.5194	1.9308E-05	2.7007	2.4223E-06	3.5940
1/64	2.4573E-06	1.6804	2.9657E-06	2.7027	2.2796E-07	3.4095
1/64	2.4573E-06	1.6804	2.9657E-06	2.7027	2.2796E-07	3.4095

Table 3 Cubic method (r = 3) for Example 4.1 with exact solution.



Fig. 2. Discrete contact sets of Example 4.1 obtained by the quadratic method (r = 2). Left: h = 1/64. Right: h = 1/128.



Fig. 3. Discrete contact sets of Example 4.1 obtained by the cubic method (r = 3). Left: h = 1/64. Right: h = 1/128.

We plot the discrete contact set A_h for levels 7–8 in Figs. 2 and 3 for quadratic and cubic methods respectively, where

$$\mathcal{A}_h = \bigcup_{T \in \mathcal{T}_h} \{ p \in \mathcal{V}_T : |u_{h,T}(p) - \psi(p)| \le \|e_h\|_{L_\infty} \}, \tag{4.2}$$

and $u_{h,T} = u_h|_T$.

Example 4.2. In this example we consider an L-shaped domain $\Omega = (-0.5, 0.5)^2 \setminus [0, 0.5]^2$ with f = g = 0 and

$$\psi(x) = 1 - \left(\frac{(x_1 + 0.25)^2}{0.2^2} + \frac{x_2^2}{0.35^2}\right).$$

Since the exact solution is unknown, we take $e_h = u_H - u_h$, where u_H and u_h are the discrete solutions obtained by DG method on two consecutive levels.

The numerical results are given in Table 4 (resp. Table 5) for the quadratic method (resp. cubic method). Since Ω is nonconvex in this example, we have $\alpha \approx 0.5445 < 1$, which are observed in the energy norm error convergence rates for both r = 2 and r = 3. Note that the energy norm errors have not reached the asymptotic region, but the rate for the cubic method is closer to 0.5445 than that of the quadratic method. Furthermore the convergence rates for errors in the H^1 and L_{∞} norms are of higher orders.

Since $\Delta^2 \psi = 0$ in this example, the non-contact set is connected (cf. [13]). This is confirmed by Figs. 4 and 5, where we plot the discrete contact sets obtained by quadratic and cubic methods respectively for levels 7–8.

		··· I ··· ·				
h	$\ e_h\ _h$	Order	$\ e_h\ _{H^1}$	Order	$\ e_h\ _{L_{\infty}}$	Order
1/4	3.6503E-01	_	6.0233E-01	_	6.8806E-01	-
1/8	3.8320E-01	-0.0701	3.4651E-01	0.7977	2.7195E-01	1.3392
1/16	2.1479E-01	0.8352	1.2569E-01	1.4630	9.1738E-02	1.5677
1/32	1.0921E-01	0.9759	3.8351E-02	1.7125	2.8423E-02	1.6905
1/64	5.7240E-02	0.9320	1.0638E-02	1.8500	7.2566E-03	1.9697

Table 4 Quadratic method (r = 2) for Example 4.2 without exact solution.

Table 5 Cubic method (r = 3) for Example 4.2 without exact solution.

h	$\ e_h\ _h$	Order	$\ e_h\ _{H^1}$	Order	$\ e_h\ _{L_\infty}$	Order
1/4	3.2645E-01	_	1.7378E-01	_	1.7076E-01	-
1/8	1.1954E-01	1.4493	4.0127E-02	2.1147	4.9318E-02	1.7918
1/16	6.3338E-02	0.9164	1.2338E-02	1.7015	1.4988E-02	1.7183
1/32	3.4444E-02	0.8788	4.2430E-03	1.5399	4.6227E-03	1.6970
1/64	2.0463E-02	0.7512	1.2971E-03	1.7098	1.2159E-03	1.9267



Fig. 4. Discrete contact sets of Example 4.2 obtained by the quadratic method (r = 2). Left: h = 1/64. Right: h = 1/128.



Fig. 5. Discrete contact sets of Example 4.2 obtained by the cubic method (r = 3). Left: h = 1/64. Right: h = 1/128.

Example 4.3. In this example we consider a two-obstacle problem on $\Omega = (-0.5, 0.5)^2$ with f = g = 0, $\psi(x) = 1 - 36|x|^4$ and $\tilde{\psi}(x) = 1.07$. Similar to Example 4.2, we take $e_h = u_H - u_h$, where u_H and u_h are the discrete solutions on two consecutive levels.

From Tables 6–7, we observe $O(h^{\tau})$ ($\tau = 1$ for r = 2 and $\tau = 1.5$ for r = 3) convergence in the energy norm error, which agrees with the estimates in Theorems 3.1 and 3.2. We also observe that the H^1 norm errors are of order $O(h^2)$, and the L_{∞} norm errors are of higher order for the quadratic method. However for the cubic method, both the H^1 norm and L_{∞} norm errors are of order $O(h^{2.5})$ in average.

In Fig. 6–7, we also observe that two contact sets are disjoint. In particular, the lower contact set (blue color) has no interior point since $\Delta^2 \psi < 0$ (cf. [13]).

Quadratic method $(r = 2)$ for Example 4.3 without exact solution.								
h	$\ e_h\ _h$	Order	$\ e_h\ _{H^1}$	Order	$\ e_h\ _{L_\infty}$	Order		
1/4	3.6966E-01	_	4.3391E-01	_	3.3750E-01	-		
1/8	3.3397E-01	0.1465	2.1768E-01	0.9952	8.6803E-02	1.9591		
1/16	1.4973E-01	1.1574	6.9404E-02	1.6491	2.3841E-02	1.8643		
1/32	6.8184E-02	1.1348	1.8572E - 02	1.9019	4.0470E-03	2.5585		

1.9986

6.4597E-04

2.6473

4.6473E-03

Table 6 Oua

1.0504

h

1/64

Table 7 Cubic method (r = 3) for Example 4.3 without exact solution.

3.2921E-02

h	$\ e_h\ _h$	Order	$\ e_h\ _{H^1}$	Order	$\ e_h\ _{L_{\infty}}$	Order
1/4	2.4108E-01	_	8.7294E-02	_	3.1551E-02	-
1/8	8.4913E-02	1.5054	1.7229E-02	2.3410	5.5377E-03	2.5104
1/16	3.0659E-02	1.4697	2.9819E-03	2.5306	8.0149E-04	2.7885
1/32	1.0782E-02	1.5077	7.5179E-04	1.9878	2.4638E-04	1.7018
1/64	3.8562E-03	1.4834	1.2559E-04	2.5817	2.3063E-05	3.4172



Fig. 6. Discrete contact sets of Example 4.3 obtained by the quadratic method (r = 2). Left: h = 1/64. Right: h = 1/128. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



Fig. 7. Discrete contact sets of Example 4.3 obtained by the cubic method (r = 3). Left: h = 1/64. Right: h = 1/128. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

5. Conclusion

In this work, we studied a family of DG methods (SIPG, NIPG, SSIPG1 and SSIPG2) for a fourth order variational inequality of the first kind. We unified the convergence analysis and derived optimal error estimates for both quadratic and cubic DG methods. Note the SIPG scheme has a symmetric formulation, which results in a symmetric positive definite stiffness matrix. On the other hand, though NIPG, SSIPG1 and SSIPG2 schemes are not symmetric, in theory they have less restrictions on penalty parameters compared with SIPG method. However, it is shown that there is no essential difference in their error analysis. We also tested NIPG and SSIPG methods for the same numerical examples in Section 4. They all have similar performance as the SIPG method. We only presented the results of SIPG method for illustration.

Our previous work also includes C^{0} IP method for the fourth order variational inequality problems (see [21]). Note that the C^{0} IP method requires C^{0} weak continuity in the design of discrete basis functions. In comparison, the fully DG methods studied in this paper have more flexibilities with general meshes, such as the ones with hanging nodes. Moreover, the fully DG methods are more ideal to be used with *hp*-adaptive strategy, due to the fact that their elements are totally discontinuous in nature. Our future work includes the *a posteriori* error estimates and developing adaptive algorithms for the fully DG methods.

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