

AMA1D01C – Mathematics in the Islamic World

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References

These notes mainly follow material from the following book:

- ▶ Katz, V. *A History of Mathematics: an Introduction*. Addison-Wesley, 1998.

and also use material from the following sources:

- ▶ Calinger, R. *A Contextual History of Mathematics*. Prentice-Hall, 1999.
- ▶ Kline, M. *Mathematical Thought from Ancient to Modern Times*. Oxford University Press, 1972.
- ▶ Struik, D. *A Concise History of Mathematics*. G. Bell and Sons, 1954.
- ▶ MacTutor History of Mathematics Archive, University of St Andrews. <http://www-history.mcs.st-and.ac.uk/>

Introduction

- ▶ Major mathematician: al-Khwarizmi (780-850), al-Uqlidisi (920-980), abul-Wafa (940-998), al-Karaji (953-1029), al-Biruni (973-1048), Khayyam (1048-1131), al-Samawal (1130-1180), al-Kashi (1380-1429)
- ▶ Inherited knowledge from the ancient Babylonians, the Greeks and the Indians
- ▶ Major development in algebra, trigonometry and combinatorics
- ▶ Not “Arabic” mathematics since the Persians also made significant contributions

Early Motivations

- ▶ Inheritance (<https://www.journals.uchicago.edu/doi/10.1086/368492>)
- ▶ Prayer times (determined by the position of the sun) and facing direction (to Mecca)
- ▶ Logic (so as not to lose theological arguments with the Christians and the Jews)

- ▶ Set up in Baghdad by Caliph al-Ma'mun
- ▶ Caliph: an Islamic ruler, considered a successor of Muhammad the prophet
- ▶ Comparable to the Museum of Alexandria
- ▶ Intense effort to acquire and translate Greek and Indian manuscripts into Arabic
- ▶ These Arabic scholars were also influenced by Babylonian mathematical traditions

- ▶ Influenced by Greek thought
- ▶ Saw Greek logic as a path to truth complementary to the Islamic faith

Muhammad ibn Musa al-Khwarizmi (780-850)

- ▶ Came from Khwarizm (now part of Uzbekistan and Turkmenistan)
- ▶ One of the first scholars in the House of Wisdom
- ▶ His name was the root of the modern word *algorithm*.
- ▶ Algorithm: a set of well-defined mechanical instructions to carry out a task



Figure: al-Khwarizmi, on a Soviet stamp. Source:
[http://http://www-groups.dcs.st-and.ac.uk/history/
PictDisplay/Al-Khwarizmi.html](http://http://www-groups.dcs.st-and.ac.uk/history/PictDisplay/Al-Khwarizmi.html)

Kitab al-jam'wal tafriq bi hisab al-Hind (“Book on Addition and Subtraction after the Method of the Indians”)

- ▶ No surviving Arabic manuscripts
- ▶ Only Latin versions remain
- ▶ Introduced nine characters to represent 1 to 9, and a circle to represent zero
- ▶ Demonstrated how to use these characters to write down any number
- ▶ Also demonstrated *algorithms* of addition, subtraction, multiplication, division, halving, doubling and finding square roots
- ▶ Such algorithms assume the use of a dustboard, where some figures are erased at each step
- ▶ Most of the time fractions were written in the Egyptian way
- ▶ Decimal fractions still missing

Al-kitab al-muhtasar fi hisab al-jabr wa-l-muqabala (“The Condensed Book on the Calculation of al-Jabr and al-Muqabala”)

- ▶ al-jabr – “restoring” – moving a subtracted (negative) quantity to the other side of an equation to make it an added (positive) quantity
- ▶ Root of the modern word *algebra*
- ▶ Example: $3x + 2 = 4 - 2x$ to $5x + 2 = 4$
- ▶ al-muqabala – “comparing” – reduction of a positive term by subtracting the same quantity from both sides of an equation
- ▶ Example: $5x + 2 = 4$ to $5x = 2$

Six types of equations:

- ▶ $ax^2 = bx$
- ▶ $ax^2 = c$
- ▶ $bx = c$
- ▶ $ax^2 + bx = c$
- ▶ $ax^2 + c = bx$
- ▶ $bx + c = ax^2$

The last three types were considered distinct because Islamic mathematicians, unlike the Indians, did not have negative numbers.

“What must be the square which, when increased by ten of its own roots, amounts to thirty-nine? The solution is this: You halve the number of roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this which is eight, and subtract from it half the number of the roots, which is five; the remainder is three. This is the root of the square which you sought for.”

- ▶ Trying to solve $x^2 + 10x = 39$
- ▶ The solution given was $\sqrt{\left(\frac{10}{2}\right)^2 + 39} - \frac{10}{2} = 3$
- ▶ In general to solve $x^2 + bx = c$ we have $x = \sqrt{\left(\frac{b}{2}\right)^2 + c} - \frac{b}{2}$
- ▶ Half of the quadratic formula

- ▶ Although al-Khwarizmi gave a geometric argument which looked “Babylonian”, he was going in a more abstract direction
- ▶ At some point in his text he added two polynomials, and said “this does not admit of any figure”, admitting that there was no geometric interpretation to what he did
- ▶ For equations of type 5, he was comfortable with an equation having more than one solutions

al-Uqlidisi (920-980)

- ▶ The name, which means “The Euclidean”, may indicate he was well-trained in geometry, but may also indicate he made copies of Euclid's *Elements* for sale.
- ▶ Nothing is known about his life

Kitab al-Fusul fi-l-Hisab al-Hindi (“The Book of Chapters on Hindu Arithmetic”)

- ▶ Written in Damascus in 952
- ▶ Not a theorem-proof book in Greek style, but a practical manual on how to do calculations with Indian numbers
- ▶ Methods were designed for doing calculations on paper
- ▶ Decimal fractions: “half of 1 in any place is 5 before it”
- ▶ The earliest record of such fractions outside of China
- ▶ Provided an example where 19 was halved 5 times
- ▶ However he didn't use decimal fractions to full strength. All of his divisions were by 2 or 10

al-Karaji (953-1029)

- ▶ Major work was called *al-Fakhri* (“*The Marvelous*”)
- ▶ Used the method of induction to prove
$$1^3 + 2^3 + \dots + 10^3 = (1 + 2 + \dots + 10)^2$$
- ▶ Note he did not use a general n , but his proof can be easily generalized (Euclid’s proof that there are infinitely many primes was similar in this regard.)
- ▶ Earliest surviving proof of a formula for sums of cubes of consecutive integers

Khayyam (1048-1131)

- ▶ *Risala fi-l-barahin ala masa'il al-jabr wa'l-muqabala* (“Treatise on Demonstrations of Problems of al-jabr and al-muqabala”)
- ▶ Treatise: a piece of writing which is a formal and systematic study of a subject
- ▶ Devoted to the solution of cubic equations
- ▶ Made it clear in the preface that the reader must be familiar with Euclid's *Elements* and Apollonius' *Conics*
- ▶ This shows those works were well known in the Islamic world.



Figure: Khayyam. Source: <http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Khayyam.html>

- ▶ Having no negative coefficients, Khayyam divided his equations into 14 different types
- ▶ $x^3 = d$
- ▶ $x^3 + cx = d$, $x^3 + d = cx$, $x^3 = cx + d$, $x^3 + bx^2 = d$,
 $x^3 + d = bx^2$, $x^3 = bx^2 + d$.
- ▶ $x^3 + bx^2 + cx = d$, $x^3 + bx^2 + d = c$, $x^3 + cx + d = bx^2$,
 $x^3 = bx^2 + cx + d$, $x^3 + bx^2 = cx + d$, $x^3 + cx = bx^2 + d$,
 $x^3 + d = bx^2 + cs$.
- ▶ Such equations arose from intersections of conic sections.
Example: the intersection of $x(\frac{d}{c} - x) = y^2$ (a circle) and $\sqrt{c}y = x^2$ (a parabola)

- ▶ Such equations arose from intersections of conic sections.
Example: Solving for the intersection of $x(\frac{d}{c} - x) = y^2$ (a circle) and $\sqrt{c}y = x^2$ (a parabola) gives $x(\frac{d}{c} - x) = (\frac{x}{\sqrt{c}})^2$.
Simplifying gives us $x^3 + cx = d$.
- ▶ $x(\frac{d}{c} - x) = y^2$ is the circle $(x - \frac{d}{2c})^2 + y^2 = \frac{d^2}{4c^2}$.

- ▶ Today, outside of the circle of mathematicians, Omar Khayyam is better known as a poet.
- ▶ The collection of his four-lined poems is known as the *Rubaiyat of Omar Khayyam* in the West.
- ▶ His poems were mostly about the transience of life and uncertainties about the nature or existence of an afterlife.
- ▶ Orthodox Muslims thought he was one of “those unfortunate philosophers and materialists who detached from divine blessings, wander in stupefaction (not thinking or feeling properly) and error.” (J. A. Boyle. *The Cambridge History of Iran.*)

Here are two poems from the collection (English rendition by Edward FitzGerald):

No. 63

Oh, threats of Hell and Hopes of Paradise!
One thing at least is certain—This Life flies;
One thing is certain and the rest is Lies;
The Flower that once has blown for ever dies.

No. 74

Yesterday This Day's Madness did prepare;
To-morrow's Silence, Triumph, or Despair:
Drink! for you know not whence you came, nor why:
Drink! for you know not why you go, nor where.

63.

碧落黃泉皆妄語
三生因果盡荒唐
濁醪以外無真理
一謝花魂再不香

74.

風雲一一安排好
世界如茶入眼來
痛飲不聞狂客問
明朝何處化塵埃

Figure: A Chinese rendition of the two poems on the previous slide, by the late Dr Kerson Huang, Professor Emeritus of Physics at the Massachusetts Institute of Technology

al-Bahir fi-l jabr (“The Shining Treatise on Algebra”)

- ▶ Used decimal fractions for approximation
- ▶ For example he expressed $\frac{210}{13}$ and the square root of 10 using decimal fractions.
- ▶ $\frac{210}{13}$: 16 plus 1 part of 10 plus 5 parts of 100 plus 3 parts of 1000 plus 8 parts of 10000 plus 4 parts of 100000
- ▶ Still used words instead of symbols.
- ▶ Gave (in words) the square root of 10 as (in modern notation) 3.162277
- ▶ Also, gave (in words) the Binomial Theorem

$$(a + b)^n = \sum_{k=0}^n C_k^n a^{n-k} b^k$$

- ▶ Gave the coefficients in what was equivalent to Pascal's triangle
- ▶ Did what was equivalent to long division of polynomials

abul-Wafa (940-998)

- ▶ Worked on spherical trigonometry
- ▶ A spherical triangle is a triangle on a sphere where all three sides are sections of great circles (i.e., circles on the sphere whose centre is the centre of the sphere)
- ▶ Great circles give the shortest paths between two points on a sphere
- ▶ Spherical trigonometry was important for religious reasons: determining the direction of Mecca (people needed to know which direction to face when praying), and setting prayer times which were determined by sunrise and sunset times
- ▶ Theorem: In any spherical triangle ABC , $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$.

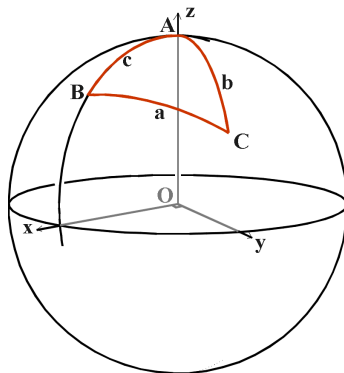


Figure: Spherical triangles. Source:

<http://star-www.st-and.ac.uk/~fv/webnotes/chapter2.htm>

al-Biruni (973-1048)

- ▶ Used abul-Wafa's work to find *qibla*, the direction of Mecca given one's position
- ▶ Calculated a sine table at intervals of 15 minutes

Nasir al-Din al-Tusi (1201-1274)

- ▶ 1256: Fall of Baghdad to the Mongols
- ▶ Hulagu, the Mongol ruler, built a centre of learning for al-Tusi
- ▶ https://en.wikipedia.org/wiki/Hulagu_Khan
- ▶ Treated trigonometry as a separate science from astronomy
- ▶ Attempted to prove Euclid's fifth postulate
- ▶ Many mathematicians to come would attempt the same thing, leading to the development of non-Euclidean geometries.

al-Kashi (1380-1429)

- ▶ Major work was Miftah al-hisab (“The Calculator’s Key”)
- ▶ Developed a method to calculate $\sin 1^\circ$
- ▶ Used $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$
- ▶ Let $x = \sin 1^\circ$, then $3x - 4x^3 = \sin 3^\circ$, where $\sin 3^\circ$ can be found using the difference-of-angle and half-angle formulae
- ▶ In fact the value he found was $y = 60 \sin 1^\circ$, which satisfied the equation $3y - \frac{4y^3}{60^2} = 60 \sin 3^\circ$
- ▶ He found a way to approximate y one (base-60) place at a time

- ▶ Invasion by the Crusades, the Turks and the Mongols created political and institutional instability
- ▶ The rise of religious elites who believed free scientific research restricted their influence <https://scholar.harvard.edu/files/chaney/files/paper.pdf>