AMA1D01C – Ancient Indian Mathematics

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References

These notes mainly follow material from the following book:

- Katz, V. *A History of Mathematics: an Introduction.* Addison-Wesley, 1998.

and also use material from the following sources:

- Calinger, R. *A Contextual History of Mathematics.* Prentice-Hall, 1999.
- Kline, M. *Mathematical Thought from Ancient to Modern Times.* Oxford University Press, 1972.
- MacTutor History of Mathematics Archive, University of St Andrews. `http://www-history.mcs.st-and.ac.uk/`

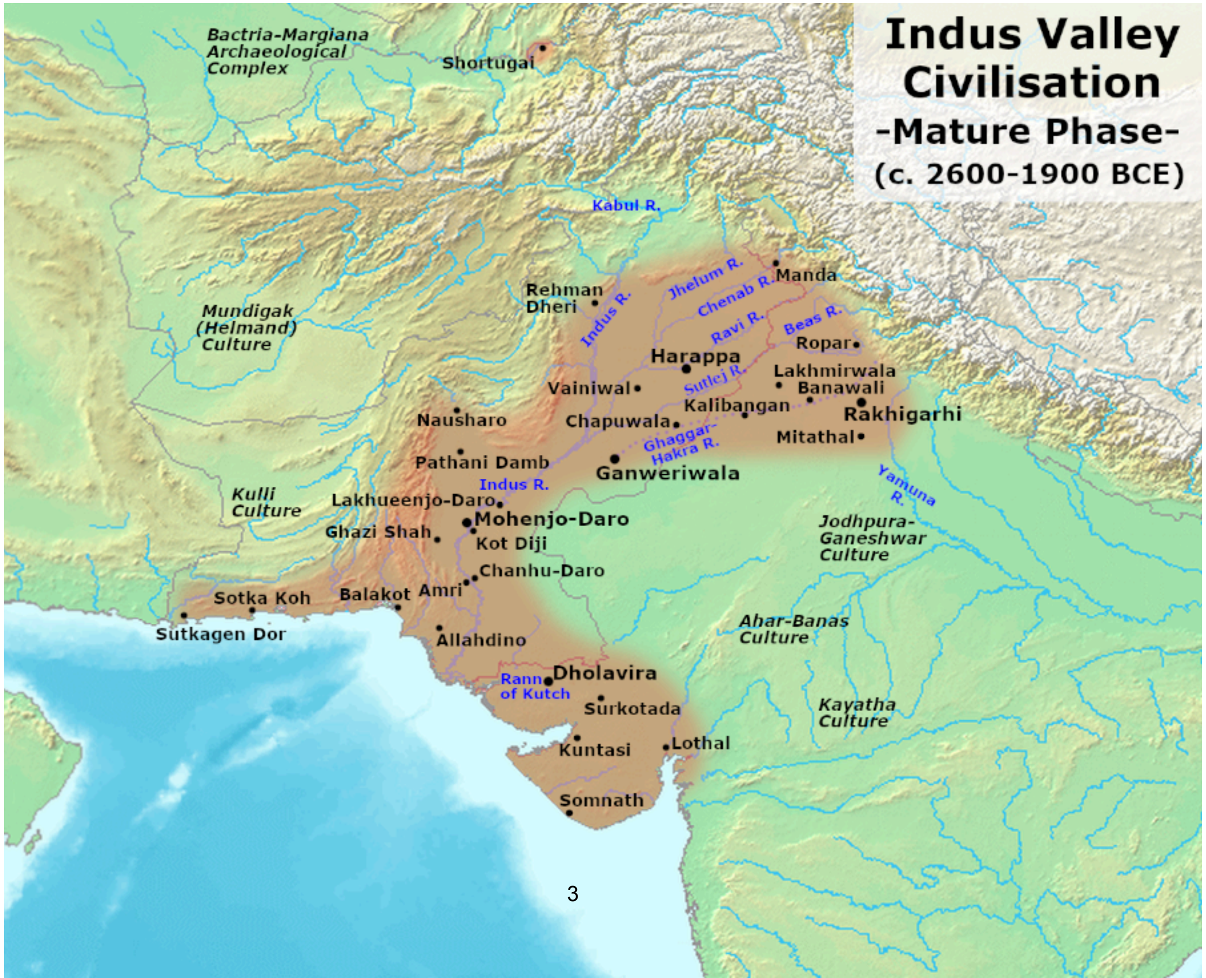
Ancient India

<u>Indus Valley Civilization 印度河流域文明</u>

also known as the Mature Harappan civilization (2600-1900 BCE)

Two famous ancient cities (among others):

- Harappa (哈拉帕) in the Punjab (旁遮普省/巴基斯坦), and
- Mohenjo-daro (摩亨佐-達羅) in the Sindh (信德省/巴基斯坦)



- It is believed that they were Dravidian peoples 達羅毗荼人 (or maybe Munda people 蒙達人) 古印度數學史：羅毗荼時期 (約公元前 3000-前 1400 年)

- Indus Valley Civilization was approximately contemporaneous with Old Kingdom Egypt, as well as, the Early Dynastic Period of the Sumerian Civilization and the Akkadian Empire in Mesopotamia.

- Urban, and built around planned cities, which also had drainage and sanitation.

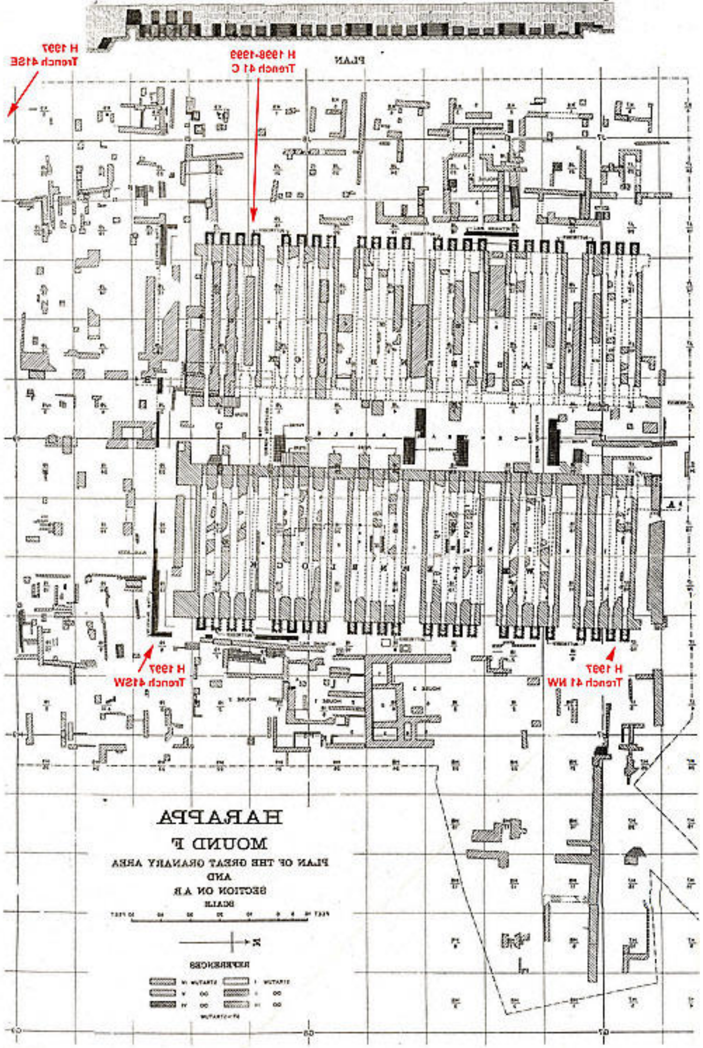
- Archaeological evidence suggests craft specialization and long-distance trade between Indus Valley and Mesopotamia, and with Central Asia.

- Maritime trade along the shores of Africa and Asia started several millennia ago (sea level was much lower at that time).

- Several Indus seals with Harappan script have also been found in Mesopotamia.

- Akkadian Empire records mention timber, carnelian and ivory as being imported from **Meluhha** by Meluhhan ships, Meluhha being generally considered as the Mesopotamian name for the Indus Valley.

- Recent genetic analysis of ancient Mesopotamian skeletal DNA tends to confirm a significant association.

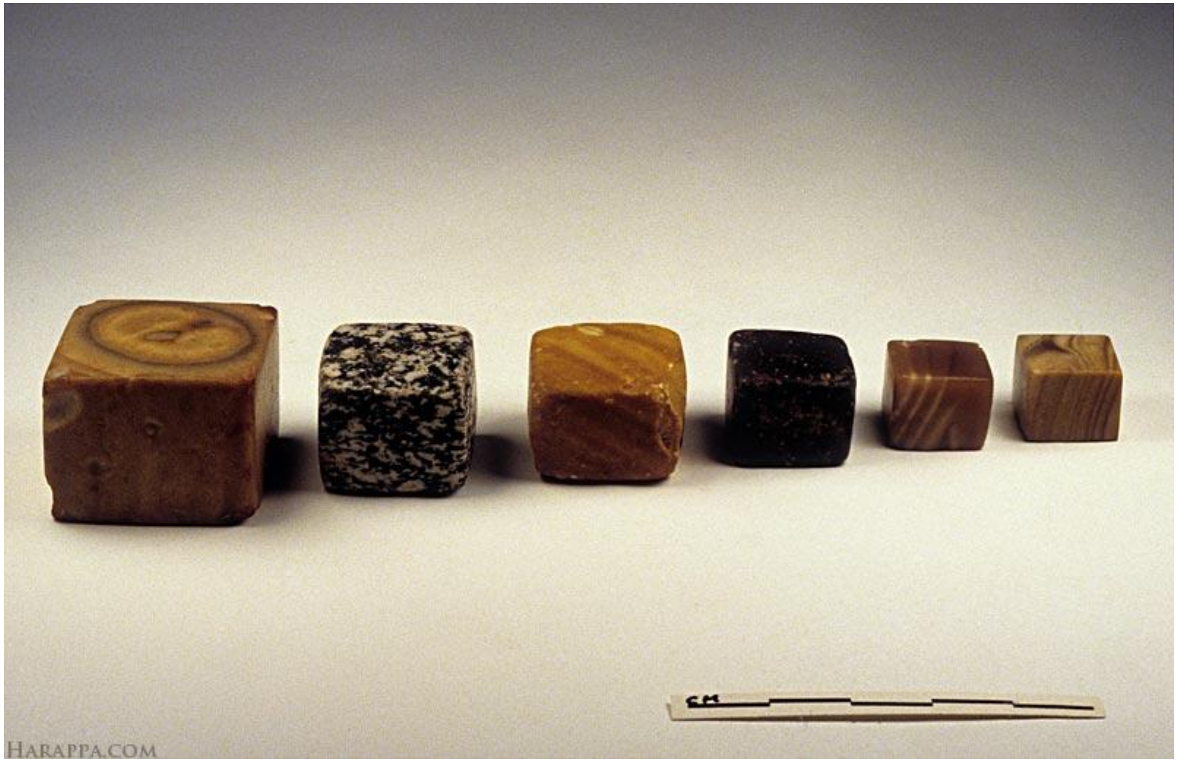


Images from https://www.harappa.com/


Harappa (哈拉帕)

Mohenjo-daro (摩亨佐-達羅）, Great Bath

The mathematics used by the early Harappan civilization were quite application oriented, and was concerned with weights and measuring scales.



Weights ratios: 1/20=0.05, 1/10=0.1, 1/5=0.2, ½=0.5, 1, 2, 5, 10, 20, 50, 100, 200, and 500

Unit Weight approx 28g

Evidence of rulers calibrated to lengths of 1.6 milimetres and 0.13 milimetres, respectively, has been found at the archaeological sites of Lothal and Mohenjo-Daro, two major Harappan towns.

Ivory rulers from Mohenjo-Daro

Unit length approx 3.4cm

Brick Ratio 4:2:1

Indus Valley Civilization ended abruptly.

Several theories suggested :

- Desert encroachment
- Drying up of the Hakra River and changes in the course of the Indus River.
- Floods / Inundation
- Earthquakes and Epidemics
- Tectonic uplift of the coastline
- Aryan invaders (yet to find some archeological evidence to support)
    - Aryan invaders killed people and destroyed the Indus Valley Civilization.
    - The Harappan people were peace loving. They did not have weapons to defend themselves against the invaders. They may only had implements for hunting or farming.
    - The Aryans lived in villages (pastoralists) and knew nothing of urban life. Thus it took hundred of years again for India to have beautiful cities like Mohenjo-daro and Harappa.

Who were they? Aryan tribes (雅利安人) in the Ganges river (恆河) in 2000 BC (they are a branch of the Indo-Iranians, who originated in present-day northern Afghanistan).

Kurgan hypothesis
墳塚假説

The Indo-Aryan Migration (1800-1500 BCE)

Vedic (吠陀) period (c. 1500 – c. 500 BCE)

- 古印度數學史 : 吠陀時期 (約公元前 10 世紀 -- 前 3 世紀)
- Aryan tribes (雅利安人) from the North of Indian subcontinent replaced (invaded? destroyed?) the Harappan culture around 2000 BC.
- Pastoralists 放牧, moving around with their cattle (the word Aryavarta means the 'turning of the Arya', in a reference to a large herd being moved around)
- They founded the Vedic religion.
- The word Vedic comes from the collections of sacred texts known as Vedas, Rigveda 梨俱吠陀, and 3 others….
- Mathematics and astronomy first appear in Vedic works during the 2nd millennium BC.
- **Ganita** (Sanskrit (梵文): "computation", is derived from the root 'gana', which means to count or to enumerate.)
- The word '**Rashividya**' is used for mathematics in **Chhandogya Upanishad 歌者奧義書** .
- Some hymns of **Shuklayajureda 夜柔吠陀** reveal the knowledge of odd numbers and tables.
- The **Brahmana 梵書** texts like ; '**ekaya svaha, dvabhyam svaha, tribhyah svaha**' reflect the vedic concept of arithmetical progressions.
- In **Pingal 賓伽羅 Sutra** there is a discourse on the calculation of squares and square roots.

Classical Siddhāntа period 悉檀多時期

古印度數學史: 悉檀多時期 (約公元 5 世紀-12 世紀)
(Classical Hindu astronomy period)
(The end of the 5th century ---- the 12th century)

Introduction

Some of the major mathematicians:

- Aryabhata
- Varahamihira
- Brahmagupta
- Mahavira
- Bhaskara

Introduction

Main characteristics:

- Possible Greek influence through Alexander the Great's army who conquered parts of northeastern India, and Chinese influence through Chinese merchants in Southeast Asia
- No emphasis on logical formalism
- Not afraid to operate on abstract numbers (negative numbers, zero)
- Formalism helps us not to reach conclusions too soon (e.g. The seemingly innocent sentence "The barber shaves all those and only those who do not shave themselves" is logically problematic upon closer analysis), but sometimes we need to be a bit "careless" to move forward.

Hindu-Arabic Numeral System

- First used in India and brought to the West through the Muslims
- Symbols for 1 to 9 originated in the Brahmi writing system, dating back to 3rd century BC
- Learned by the Muslims when they invaded northern India in 8th century BC
- Originally the Indians had symbols for 1 to 9, 10 to 90, 100, 1000, etc.
- To express large numbers they combine symbols for 100 or 1000 with symbols for 1 to 9, in a multiplicative way (e.g. the symbol for 3 and the symbol for 100 together mean 300)
- Similar to the way we say numbers ("three hundred")

- Around 600 AD the Indians dropped all symbols representing numbers bigger than 9 and adopted a *place-value* system
- Place-value: the value a symbol stands for depends on where it is put
- Conjectured to be a Chinese influence
- A manuscript dating back to the 7th century AD contains numbers written in place value system with dots representing 0

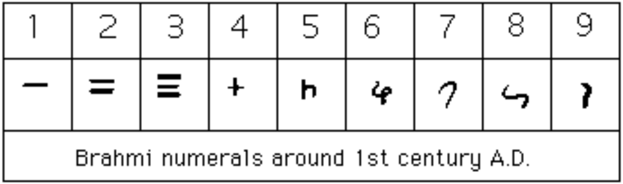


Figure: Brahmi numerals. Source:`http://www-groups.dcs.st-andrews.ac.uk/history/HistTopics/Indian_numerals.html`

Hindu-Arabic Numerals

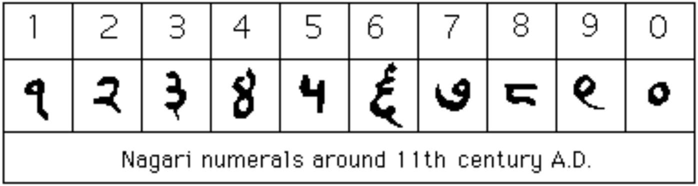


Figure: Nagari numerals. Source:http://www-groups.dcs.st-andrews.ac.uk/history/HistTopics/Indian_numerals.html

Hindu-Arabic Numerals

Figure: Numerals used by Arabic mathematician al-Biruni.
Source:http://www-history.mcs.st-andrews.ac.uk/HistTopics/Arabic_numerals.html

Hindu-Arabic Numerals

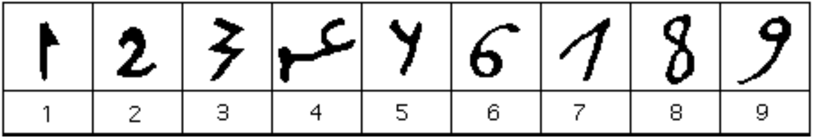


Figure: Numerals used by Arabic mathematician al-Banna.
Source:http://www-history.mcs.st-andrews.ac.uk/HistTopics/Arabic_numerals.html

Aryabhata

Aryabhata (born in 476 AD)

- His book, *Aryabhatiya*, was the earliest Indian work in mathematics with an identifiable author
- Consisted of 4 sections and 123 stanzas (stanza: an arrangement of lines, like a poem), and the second section (with 33 stanzas) was on mathematics
- Gave a method to find sines of angles in steps of 3 degrees and 45 minutes ($3\frac{3}{4}$ degrees, $\frac{1}{12}$ of 45 degrees)
- Also gave a rule for calculating cube roots (Stanza II, 5)
- And a method for finding the height of a pole by measuring shadows at different places
- Similar to methods given in the Sea Island Mathematical Manual from China

Aryabhata (continued)

- Gave a formula for the sum of an arithmetic progression

$$a+(a+d)+(a+2d)+\ldots+(a+(n-1)d)=n\left(\frac{a+(a+(n-1)d)}{2}\right)$$

- He also gave a formula for the number of terms n given that the sum is known. Basically he solved a quadratic equation, but he did not give the general quadratic formula.
- He also gave the following formulae:

$$1^2+2^2+\ldots+n^2=\frac{1}{6}n(n+1)(2n+1)$$

$$1^3+2^3+\ldots+n^3=(1+2+\ldots+n)^2$$

- However we do not know how he discovered them.
- India's first artificial satellite was named Aryabhata (launched by the USSR in 1975)

Varahamihira (born in 505 AD)

- Gave a cosine table
- Gave an example on the choosing function: “If a quantity of 16 substances is varied in four different ways, the result will be 1820.” (I.e., $C_4^{16} = 1820$.)
- No general formula was given, but people believed they used something similar to Pascal’s triangle.

Brahmagupta

Brahmagupta (born in 598 AD)

- Major work was *Brahmasphutasiddhanta* ("*Correct Astronomical System of Brahma*")
- Also written in stanzas
- Stated that any number multiplied by 0 is 0
- Gave rules of operation with positive and negative numbers
- Gave an interpolation formula for sine
- Solved linear congruences in the style of the Chinese Remainder Theorem
- However, Brahmagupta only solved two congruences at a time. For a system with more than two congruences, he solved it step-by-step where each step consists of two congruences.

Rules of opeartion with positive and negative numbers:

- "The sum of two positive quantities is positive; of two negative is negative; of a positive and a negative is their difference; or, if they are equal, zero...In subtraction, the less is to be taken from the greater, positive from positive, negative from negative. When the greater, however, is subtracted from the less, the difference is reversed...When positive is to be subtracted from negative, and negative from positive, they must be thrown together. The product of a negative quantity and a positive is negative; of two negative, is positive; of two positive, is positive.... Positive divided by poistive or negative by negative is positive....Positive divided by negative is negative. Negative divided by positive is negative."

- Gave the quadratic formula in the same form as we know it: "Take absolute number on the side opposite to that on which the square and simple unknown are. To the absolute number multiplied by four times the [coefficient] of the square, add the square of the [coefficient of the] unknown; the square root of the same, less the [coefficient of the] unknown, being divided by twice the [coefficient of the]square is the [value of the] unknown."
- In modern notation, if $ax^2 + bx = c$, $x = \dfrac{\sqrt{4ac + b^2} - b}{2a}$

Maharvira (born around 800 AD)

- Gave a rule for dividing by a fraction, i.e., invert and multiply
- Gave (in words) a method which was equivalent to the formula

$$C_r^n = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

- However he did not give a proof/derivation

Bhaskara

Bhaskara (born in 1114 AD)

- A fraction whose denominator is 0 "is termed an infinite quantity. In this quantity...there is no alteration, though many be inserted or extracted; as no change takes place in the infinite and immutable God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth."
- i.e., infinity plus/minus any real number is infinity
- Pointed out that the square root of a positive number may be positive or negative, and also a negative number does not have a square root

Bhaskara (continued)

- Considered equations of the form $ax^2 + bx = c$ by completing the square to get $(rx + s)^2 = d$, and $x = \frac{s \pm \sqrt{d}}{r}$
- Unlike the Babylonians, he accepted that one equation may have more than one solutions.
- However, he only calculated positive roots, and he never gave examples of equations with two negative roots or no real roots. He did not give examples of equations with irrational roots either.
- Possibly, to him, negative numbers are just tools which are allowed in the middle of a calculation, but not something which is worthy enough to be answers.
- Also, possibly because problems were asked in real-life settings.

Bhaskara (Continued)

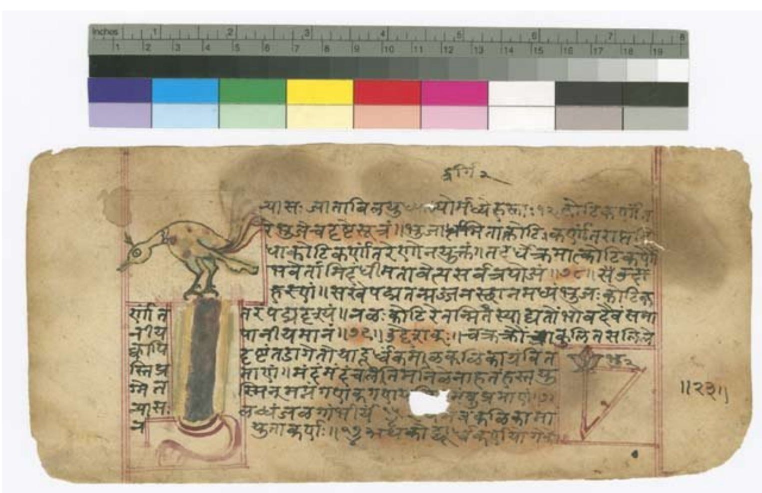
- Invented symbols for negative numbers, square and cubes
- Negative: a dot over a number
- Square: an initial for the word "square"
- Cube: an initial for the word "solid"
- Variables: iniital syllables of words for colours
- Something like r, b, g, etc. (We use these letters for combinatorics questions where we draws balls of different colours from a bag.)

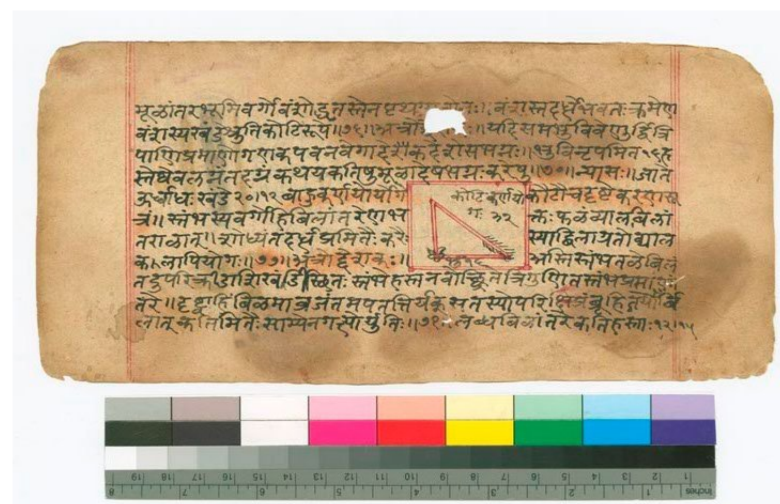
Bhaskaracharya
(Bhāskara，1114年－1185年)
(婆什迦羅第二 Bhaskara II)
(**婆什迦羅老師** Bhāskara Achārya)

《Lilavati 莉拉沃蒂》

The work forms the first part of a larger work of the author called the Siddhanta-siromani《天文系統之冠》This part is called by the author, Pdtiganita or Arithmetic; but this name has not been properly given.

The name Lilavati was Bhaskara II's daughter.



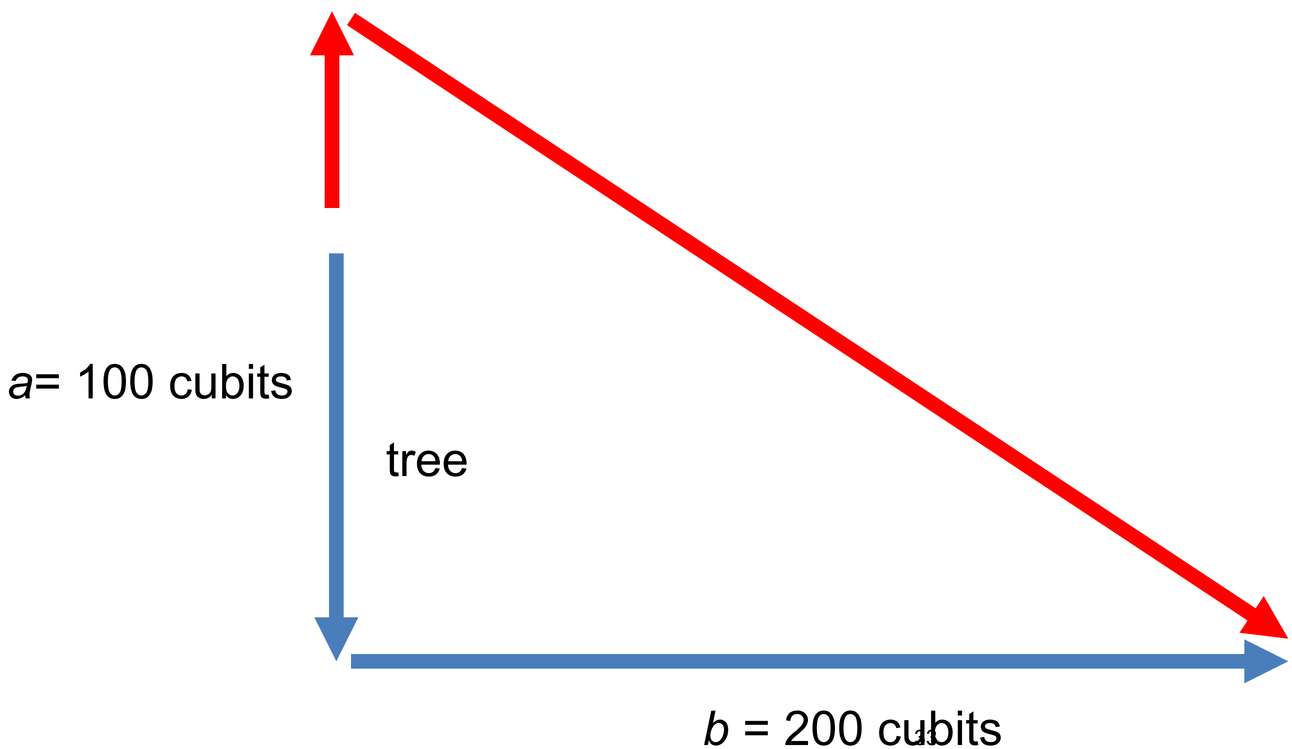


COLEBROOKE'S TRANSLATION OF THE LILAVATI

WITH NOTES BY H.C. BANERJI SECOND EDITION
The Book Company, Ltd 1927
(1st published 1892 reprint 1993)
https://ia600903.us.archive.org/30/items/lilavati00bhas/lilavati00bhas.pdf

[155] (this problem is often referred to as "*the problem of two monkeys*").

Example (of rule of **[154]**)- From a tree a hundred cubits high, an ape descended and went to a pond two hundred cubits distant : while another ape, vaulting to some height off the tree, proceeded with velocity diagonally to the same spot. If the space travelled by them be equal, tell me quickly, learned man, the height of the leap, if thou have diligently studied calculation.

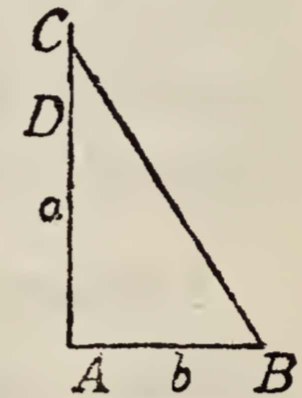


Statement. Tree 100 cubits. Distance of it from the pond 200. Proceeding as directed (rule **[154]**), the height of the leap comes out 50.

$$x = \frac{ab}{2a+b}.$$

154. Rule.[1] The height of the tree multiplied by its distance from the pond, is divided by twice the height of the tree added to the space between the tree and the pond : the quotient will be the measure of the leap.

[The rule refers to the example which follows.



Let D be the top of the tree, and B the position of the pond. The first ape is supposed to descend from D to A, and then to go from A to B; while the second ape is supposed to jump vertically upwards from D to C, and then to leap directly from C to B. Now let $DA = a$, $AB = b$, and $CD = x$, which is required. Then by the question, we have $x + \sqrt{(a+x)^2 + b^2} = a + b$;

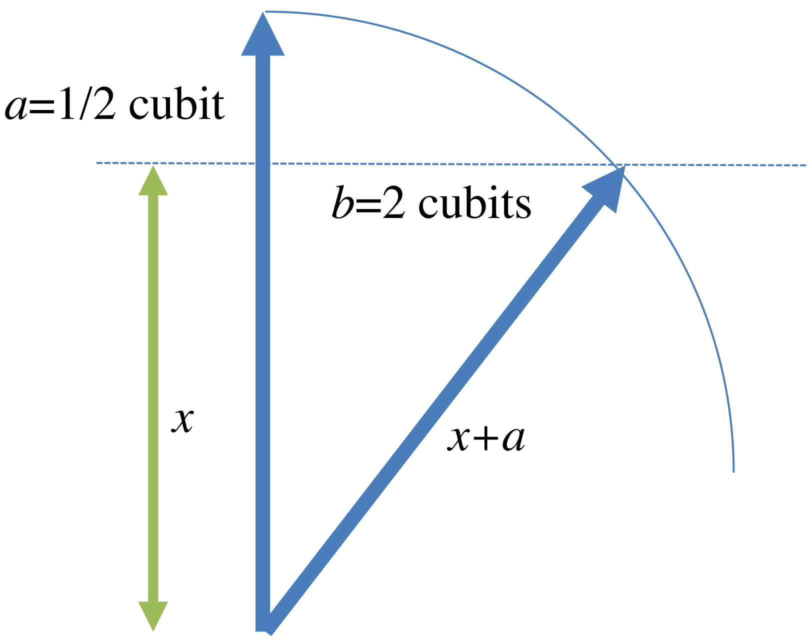
$$\therefore\ (a+x)^2 + b^2 = (a+b)^2 - 2(a+b)x + x^2,$$

whence $x = \frac{ab}{2a+b}$. Hence the rule.]

[153] (this problem is often referred to as "*a Lotus bud dips in lake*").

Example. In a certain lake swarming with ruddy geese and cranes, the tip of a bud of lotus was seen a span (that means ½ cubits) above the surface of the water. Forced by the wind it gradually advanced, and was submerged at the distance of two cubits. Compute quickly, mathematician, the depth of the water.

Statement. Diff. of hypotenuse and upright, 1/2 cubit. Side 2 cubits. Proceeding as directed (rule **[151]**), the upright is found 15/4. It is the depth of the water. Adding to it the height of the bud, the hypotenuse comes out 17/4.



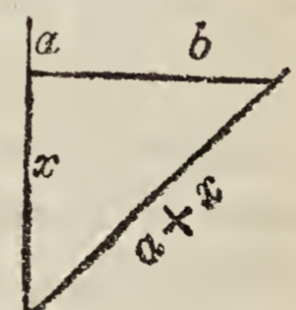
$$x = \frac{1}{2}\left(\frac{b^2}{a} - a\right),$$

151. Rule.[1] The quotient of the square of the side divided by the difference between the hypotenuse and upright is twice set down; and the difference is subtracted from the quotient (in one place) and added to it (in the other). The moieties (of the remainder and sum) are in their order the upright and hypotenuse.[2]

This[3] is to be generally applied by the intelligent mathematician.

[The demonstration of the rule on § 147 applies to this rule as Ganesa observes.

Let a denote the difference between hypotenuse and upright, b the side, and x the upright.



Then, $b^2=(a+x)^2-x^2$, whence as in § 147, $x=\frac{1}{2}\left(\frac{b^2}{a}-a\right)$, and $a+x=\frac{1}{2}\left(\frac{b^2}{a}+a\right)$. Hence the rule.]

152. Friend, the space between the lotus (as it stood) and the spot where it submerged, is the side. The lotus as seen (above water) is the difference between the hypotenuse and upright. The stalk is the upright, for the depth of water is measured by it. Say, what the depth of the water is.[1]