AMA1D01C Supplementary Notes on Ancient Babylonian Sexagesimal (base 60) Division, Reciprocal, and Regular Numbers

In our usual decimal (base 10) system, there are some "quick tricks" for us to use when we do multiplications or divisions. For example, if we multiply a with a power of 10, say, 10^m, then, we just need to shift the decimal point to the right by m places. Similarly, when we divide a with a power of 10, say, 10^n, then, we just need to shift the decimal point the the left by n places.

For division, there is another useful "quick trick", although it is now less commonly used. This trick is for dividing by a number for which the number itself divides a power of 10. Suppose b is the divisor, and b itself divides a power of 10, say $b \times k = 10^p$, then, when we divide a by b, we just need to multiply a by k, and then shift the decimal point to the left by p places. For example, the number 25 divides a power of 10.

$$25 \times 400 = 10000 = 10^4.$$

Then, we can see the **reciprocal** of 25 is

$$\frac{1}{25} = \frac{400}{10^4}.$$

Suppose we would like to divide 70 by 25, then, the "quick trick" for this would be

$$\frac{70}{25} = \frac{70 \times 400}{10^4} = \frac{28000}{10000} = 2.8.$$

Thus, as illustrated, when a number divides a power of 10, then, doing a division by that number can be replaced by a multiplication procedure, together with a shift in decimal place. (Of course, generally, multiplications are deemed easier to do than divisions, and that's why there are "quick tricks".)

Of course, in Sexagesimal (base 60) system, the ancient Babylonian also took advantage of the "quick trick" specific to their base 60 system.

Those numbers that divides a power of 60 are called **regular numbers**. They are special numbers. Their reciprocal are exact (only having finite number of terms). They are also numbers with only prime divisors 2, 3, and 5. The ancient Babylonian have tables showing the **reciprocal** of the regular numbers.

n	$\bar{n}$	n	$\bar{n}$	n	$\bar{n}$
2	30	16	3.45	45	1.20
3	20	18	3.20	48	1.15
4	15	20	3	50	1.12
5	12	24	2.30	54	1.06.40
6	10	25	2.24	1	1
8	7.30	27	2.13.20	1.04	56.15
9	6.40	30	2	1.12	50
10	6	32	1.52.30	1.15	48
12	5	36	1.40	1.20	45
15	4	40	1.30	1.21	44.26.40

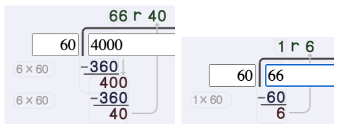
Consider the number 54. It is a regular number, since

$$54 \times 4000 \;=\; 216000 \;=\; 60^3$$

(you may also check that $54 = 2 \times 3^3$).

We can express 4000 in Sexagesimal (base 60) (i.e., divide 4000 by 60, and then divide the quotient by 60, and each time get the remainder)

$$\begin{aligned} 4000 &= 66 \times 60 + 40 \;=\; (1 \times 60 + 6) \times 60 + 40 \\ &= 1 \times (60)^2 + 6 \times (60) + 40 \;=\; [1, 6, 40], \end{aligned}$$



$$(60)^3 \;=\; 1 \times (60)^3 + 0 \times (60)^2 + 0 \times (60) + 0 \;=\; [1, 0, 0, 0].$$

The **reciprocal** of 54 is thus given by

$$\frac{1}{54} = \frac{4000}{60^3} \;=\; \frac{[1, 6, 40]}{[1, 0, 0, 0]} \;=\; [0; 1, 6, 40].$$

Suppose we would like to divide 20 by 54, then, all we need to do is to multiply 20 by [1, 6, 40], and then shifted to the left 3 places.

		1		6		40	
	x					20	
			2		13		
		22		13		20	
				20x6=120		20x40=800=13*60+20	
				120+13=133			
				133=2x60+13			

$$20 \times 4000 \;=\; [22, 13, 20].$$

Thus, we have

$$\frac{20}{54} \;=\; \frac{[22, 13, 20]}{[1, 0, 0, 0]} \;=\; [0; 22, 13, 20].$$

Consider a non-integer (not a whole number) reciprocal example, $[1;20]$:

n	$\overline{n}$	n	$\overline{n}$	n	$\overline{n}$
2	30	16	3.45	45	1.20
3	20	18	3.20	48	1.15
4	15	20	3	50	1.12
5	12	24	2.30	54	1.06.40
6	10	25	2.24	1	1
8	7.30	27	2.13.20	1.04	56.15
9	6.40	30	2	1.12	50
10	6	32	1.52.30	1.15	48
12	5	36	1.40	1.20	45
15	4	40	1.30	1.21	44.26.40

$$[1;20] \quad = \quad 1 + \frac{20}{60} = \frac{80}{60}.$$

Note that 80 is a regular number (i.e., a divisor of power of 60):

$$80 \times 45 \quad = \quad 60^2.$$

So

$$[1;20] \quad = \quad \frac{60}{45}.$$

Thus,

$$\frac{1}{[1;20]} = \frac{45}{60} \quad = \quad [0;45].$$