The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1007 Calculus and Linear Algebra

Tutorial 11

Infinite Series

1. Let $I_n = \int_0^{\pi} x^n \cos x dx$ and $J_n = \int_0^{\pi} x^n \sin x dx$, where n is a non-negative integer. Using integration by parts, express I_n in terms of J_{n-1} and J_n in terms of I_{n-1} for $n \geq 1$. Hence, compute $\int_0^{\pi} x^3 \cos x dx$.

2. If $\sum_{i=1}^{\infty} a_n$ converges and $a_n > 0$ for all n, can anything be said about $\sum_{i=1}^{\infty} \frac{1}{a_n}$. Give your reason to the answer.

3. (Logarithmic p-series)
   (a) Show that $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$ (p is a positive constant) converges if and only if $p > 1$
   (b) What implication does the fact in part (a) have for the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$? Give your reason to the answer.

4. (a) If $\sum_{i=1}^{\infty} a_n$ is a convergent series of non-negative numbers, can anything be said about the convergence of $\sum_{i=1}^{\infty} \frac{a_n}{n}$. Explain briefly.
   (b) Prove that if $\sum_{i=1}^{\infty} a_n$ is a convergent series of non-negative terms, then $\sum_{i=1}^{\infty} a_n^2$ converges.

5. Show that the Ratio test fails to provide information about convergence of p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$.

6. The series $\tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\dots$ converges to $\tan x$ for $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$.

   (a) Find the first five terms of the series for $\ln\left|\sec x\right|$. For what values of x should be the series converge? Check the series you obtained with CoCalc Jupyter.

   (b) Find the first five terms of the series for $\sec^2 x$. For what values of x should the series converge?

   (c) Check your result in (b) by squaring the series

$$\sec x = 1+\frac{x^2}{2}+\frac{5x^4}{24}+\frac{61x^6}{720}+\frac{277x^8}{8064}+\dots$$

7. (Taylor and Maclaurin Expansions)

   (a) State the similarity and difference between Taylor and Maclaurin Expansions, if any. [Hint: compare the explicit formula of the two equations]

   (b) Find the Maclaurin series for the following functions:

      (i) e^{ax}

      (ii) $\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$

   (c) Expand the following functions into Taylor series about $x=a$:

      (i) $x^{-1}; a=-1;$

      (ii) $\sin \pi x; x=\frac{1}{2}.$

8. Determine the interval of convergence of the following series

   (a) $\sum \frac{x^n}{n^s}, s>0$;

   (b) $\sum \frac{n!x^n}{n^n}$;

   (c) $\sum nx^n$;

   (d) $\sum \frac{(-1)^{n-1}}{n}(x+1)^n$;

   (e) $\sum \frac{n^2}{2^n}(x+2)^n$.

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