The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1007 Calculus and Linear Algebra

Tutorial 6

Differentials and L'Hopital's Rule

1. Let $y = f(x)$ be a differentiable function. Show that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{dy} = 1$$

where Δy is the increment in y and dy is the differential of y.

2. Find the differentials of the following functions:

   (a) $y = \dfrac{5}{\sqrt{x}}$;

   (b) $y = \left(5 - 2x + x^5\right)^4$;

   (c) $y = \sin^{-1} x + \left(\tan^{-1} x\right)^2$;

   (d) $y = 5^{-\frac{1}{x^2}} + \dfrac{2}{x^2} - 5x^2$.

3. Given the function $y = x^3 + 2x$, find Δy and dy at $x = 2$ for

   (a) $\Delta x = 1$;

   (b) $\Delta x = 0.1$;

   (c) $\Delta x = 0.01$.

4. Approximate the following formulae by differentials for small $|x|$.

   (a) $\sqrt{1+x}$;

   (b) e^x;

   (c) $\ln(1+x)$

   (d) $\sin x$

5. In a given circle, arc AB subtend a central angle α. Consider the chord AB and the tangent lines to the circle at A and B. Letting S_1 be the area between the chord and the arc, and S_2 the area between the tangents and the arc, find $\lim_{\alpha \to 0} \frac{S_1}{S_2}$.

6. Find the following values by the concepts of differential (linear approximation):
   (a) $\sin 28^\circ$
   (b) $\sqrt[5]{33}$

7. Evaluate the following limits by L'Hopital's Rule
   (a) $\lim_{x \to 0} \frac{1-\cos nx}{e^x - x - 1}$;
   (b) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$;
   (c) $\lim_{x \to 0+} x^x$;
   (d) $\lim_{x \to 0+} \frac{\ln x}{\ln(\ln(1+x))}$.

8. By L'Hopital's Rule, consider the following statements:

   I. $\lim_{x \to 1} \frac{x^3 + x - 2}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{3x^2 + 1}{2x - 3} = \lim_{x \to 1} \frac{6x}{2} = 3$.

   II. $\lim_{x \to 1} \frac{x^2 + 2}{x - 3} = \lim_{x \to 1} \frac{2x}{1} = 2$.

   III. $\lim_{x \to \infty} \frac{2x + \cos x}{x} = \lim_{x \to \infty} (2 - \sin x)$ doesn't exist. Thus, $\lim_{x \to \infty} \frac{2x + \cos x}{x}$ doesn't exist.

   IV. If f is twice continuously differentiable on $\mathbb{R}$ such that $f(0) = 1$, $f'(0) = 0$ and $f''(0) = -1$, then for $a \in \mathbb{R}$, $\lim_{x \to \infty} \left(f\left(\frac{a}{\sqrt{x}} \right) \right)^x = e^{-\frac{a^2}{2}}$.

   V. $\lim_{x \to 0^+} (\sin x)^{\tan x} = 1$.

   Which of the following statements is true? Briefly explain.
   (a) Only one of the above statements is correct.
   (b) Only two of the above statements are correct.
   (c) Only three of the above statements are correct.
   (d) Only four of the above statements are correct.
   (e) All of the above statements are correct.

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