Geometric Progression in matrix form

Recall that the sum of the geometric progression is given by
$1 + x + x^2 + x^3 + ... + x^n = \dfrac{1 - x^{n+1}}{1 - x} = (1-x)^{-1}(1-x^{n+1})$ for $(1-x) \neq 0$.
This can be easily verified by multiplying both sides with $1 - x$.

Now, consider the matrix version of the sum:

$$\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \cdots + \boldsymbol{A}^n = (\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{I} - \boldsymbol{A}^{n+1}).$$

where $\boldsymbol{I} - \boldsymbol{A}$ is invertible.
We multiply the left hand side from the left with $(\boldsymbol{I} - \boldsymbol{A})$, we have

$$\begin{aligned}(\boldsymbol{I} - \boldsymbol{A})(\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \cdots + \boldsymbol{A}^n) &= \boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \cdots + \boldsymbol{A}^n \\ &\quad - \boldsymbol{A} - \boldsymbol{A}^2 - \boldsymbol{A}^3 - \cdots - \boldsymbol{A}^n - \boldsymbol{A}^{n+1} \\ &= \boldsymbol{I} - \boldsymbol{A}^{n+1}.\end{aligned}$$

We multiply the right hand side from the left with $(\boldsymbol{I} - \boldsymbol{A})$, we have

$$\begin{aligned}(\boldsymbol{I} - \boldsymbol{A})(\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{I} - \boldsymbol{A}^{n+1}) &= \boldsymbol{I}\ (\boldsymbol{I} - \boldsymbol{A}^{n+1}) \\ &= (\boldsymbol{I} - \boldsymbol{A}^{n+1}).\end{aligned}$$

Thus, LHS=RHS.

Suppose $\boldsymbol{A}^{n+1}$ is given, then, we can compute the sum very easily by

$$\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \cdots + \boldsymbol{A}^n = (\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{I} - \boldsymbol{A}^{n+1}).$$

Example

Consider $\boldsymbol{A} = \begin{bmatrix} \frac{1}{4} & 2 & \frac{1}{5} \\ 0 & \frac{1}{2} & 2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$, and suppose given that $\boldsymbol{A}^8 = \begin{bmatrix} \frac{1}{65536} & \frac{255}{8129} & \frac{12355}{16384} \\ 0 & \frac{1}{256} & \frac{1}{8} \\ 0 & 0 & \frac{1}{256} \end{bmatrix}$.

So $\boldsymbol{I} - \boldsymbol{A}^8 = \begin{bmatrix} \frac{65535}{65536} & -\frac{255}{8192} & -\frac{12355}{16384} \\ 0 & \frac{255}{256} & -\frac{1}{8} \\ 0 & 0 & \frac{255}{256} \end{bmatrix}$.

Moreover, we can compute $(\boldsymbol{I} - \boldsymbol{A})^{-1} = \begin{bmatrix} \frac{4}{3} & \frac{16}{3} & \frac{328}{15} \\ 0 & 2 & 8 \\ 0 & 0 & 2 \end{bmatrix}$. Thus,

$$\begin{aligned}\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \cdots + \boldsymbol{A}^7 &= (\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{I} - \boldsymbol{A}^8) \\ &= \begin{bmatrix} \frac{21845}{16384} & \frac{10795}{2048} & \frac{82367}{4096} \\ 0 & \frac{255}{128} & \frac{247}{32} \\ 0 & 0 & \frac{255}{128} \end{bmatrix}.\end{aligned}$$

Suppose further that, $|\lambda_i| < 1$ for every eigenvalue λ_i of a matrix $\boldsymbol{A}$, then $\lim_{n\to\infty} \boldsymbol{A}^n = \boldsymbol{O}$.
We can then use this result to find the sum

$$\sum_{n=1}^{\infty} \boldsymbol{A}^n \quad = \quad \boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \ldots \; = (\boldsymbol{I} - \boldsymbol{A})^{-1}.$$

In the above example, characteristic polynomial of $\boldsymbol{A}$ is given by
$\lambda^3 - \frac{5}{4}\lambda^2 + \frac{1}{2}\lambda - \frac{1}{16} = (\lambda - \frac{1}{4})(\lambda - \frac{1}{2})^2$.

Therefore, we have $|\lambda_i| < 1$ for $i = 1, 2, 3$.
Thus $\sum_{n=1}^{\infty} \boldsymbol{A}^n = \boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \cdots = (\boldsymbol{I} - \boldsymbol{A})^{-1} = \begin{bmatrix} \frac{4}{3} & \frac{16}{3} & \frac{328}{15} \\ 0 & 2 & 8 \\ 0 & 0 & 2 \end{bmatrix}$.