Power Series of Rational Functions

Consider the rational function $f(x) = \dfrac{x+1}{x^2-2x-7}$. Suppose the power series of $f(x)$ at $x = 0$ is given by $f(x) = \sum_{n=0}^{\infty} a_n x^n$. We can obtain a_n by the usual Taylor/Maclaurin Series expansion, that is, by computing $\dfrac{f^{(n)}(0)}{n!}$. However, we can also obtain a_n by an alternative and simpler approach.

Consider this example, $\dfrac{x+1}{x^2-2x-7} = \sum_{n=0}^{\infty} a_n x^n$. We would like to find the a_n without the need to differentiate. By rearranging it, we have

$$\begin{aligned}
1+x &= \left(x^2-2x-7\right)\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+2} - 2\sum_{n=0}^{\infty} a_n x^{n+1} - 7\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} a_{n-2}x^n - 2\sum_{n=1}^{\infty} a_{n-1}x^n - 7\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} a_{n-2}x^n - 2\left(a_0 x + \sum_{n=2}^{\infty} a_{n-1}x^n\right) - 7\left(a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n\right) \\
&= -7a_0 - (2a_0 + 7a_1)x + \sum_{n=2}^{\infty}(a_{n-2} - 2a_{n-1} - 7a_n)x^n.
\end{aligned}$$

By comparing the undetermined coefficients of x^0 and x^1 of the right hand side to the left hand side, we have

$$\begin{aligned}
x^0 \quad term: \qquad 1 &= -7a_0 \\
x^1 \quad term: \qquad 1 &= -(2a_0 + 7a_1).
\end{aligned}$$

Solving the two, we have

$$\begin{aligned}
a_0 &= -\frac{1}{7}, \\
a_1 &= -\frac{5}{49}.
\end{aligned}$$

Moreover, for $n \geq 2$ the coefficients on the left hand said are all zeros, thus, for $n \geq 2$, we have

$$x^n \quad term: \qquad 0 = a_{n-2} - 2a_{n-1} - 7a_n.$$

Or, by re-arranging it, we have $a_n = \dfrac{a_{n-2} - 2a_{n-1}}{7}$. This is a second order linear difference equation.

We can use a_0 and a_1 to compute the value of a_2, and use a_1 and a_2 to compute a_3, ... etc without the need to differentiate:

$$\begin{aligned}
a_2 &= \frac{a_0 - 2a_1}{7} = \frac{3}{343} \\
a_3 &= \frac{a_1 - 2a_2}{7} = -\frac{41}{2401} \\
a_4 &= \frac{a_2 - 2a_3}{7} = \frac{103}{16807} \\
a_5 &= \frac{a_3 - 2a_4}{7} = -\frac{493}{117649} \\
a_6 &= \frac{a_4 - 2a_5}{7} = \frac{1707}{823543} \\
a_7 &= \frac{a_5 - 2a_6}{7} = -\frac{6865}{5764801} \\
a_8 &= \frac{a_6 - 2a_7}{7} = \frac{25679}{40353607} \\
&\vdots
\end{aligned}$$

In this case, we can also find the radius of convergence by solving a quadratic equation. Suppose $L = \lim_{n\to\infty} \frac{a_n}{a_{n-1}}$ exist and non-zero. Note that $\frac{1}{L} = \lim_{n\to\infty} \frac{a_{n-2}}{a_{n-1}}$. From the linear equation obtained above, $a_n = \frac{a_{n-2} - 2a_{n-1}}{7}$, divide it by a_{n-1}, and then take limit for $n \to \infty$, and multiply both sides by L.

$$\begin{aligned}
L = \lim_{n\to\infty} \frac{a_n}{a_{n-1}} &= \lim_{n\to\infty} \frac{a_{n-2} - 2a_{n-1}}{7a_{n-1}} \\
&= \frac{1}{7}\left(\lim_{n\to\infty} \frac{a_{n-2}}{a_{n-1}} - 2\right) = \frac{1}{7}\left(\frac{1}{L} - 2\right)
\end{aligned}$$

So, we have a quadratic equation of L

$$7L^2 + 2L - 1 = 0.$$

Solve for L, we have $L = \frac{-2 \pm \sqrt{4 - 4(7)(-1)}}{14} = \frac{-2 \pm \sqrt{32}}{14}$. One positive and one negative. Since we can see that, in this particular example, when $n \geq 2$, the coefficients a_n are alternating in signs when n increases. Thus, $L = \lim_{n\to\infty} \frac{a_n}{a_{n-1}}$ **is negative**. Therefore, we reject the positive one. Thus $L = \frac{-2 - \sqrt{32}}{14}$.

Radius of convergence is thus given by

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{1}{L}\right| = \left|\frac{14}{-2 - \sqrt{32}}\right| = \frac{14}{2 + \sqrt{32}} = 2\sqrt{2} - 1.$$

If the rational function can be expressed into a sum of partial fractions, one can first find power series for each of the partial fractions, and then find the intersection of the intervals of convergence of each of the power series.

Consider the partial fraction expansion

$$\frac{x+1}{x^2-2x-7} \quad = \quad \frac{a}{x-\beta} + \frac{b}{x-\alpha}$$

where $a = \frac{1}{2} + \frac{1}{4}\sqrt{2}$, $b = \frac{1}{2} - \frac{1}{4}\sqrt{2}$, $, \beta = 1 + 2\sqrt{2}$, $\alpha = 1 - 2\sqrt{2}$.

To the first term, power series centered at $x = 0$ would be $\frac{a}{x-\beta} = \sum_{n=0}^{\infty} c_n x^n$, we can follow the above procedures and get $c_0 = \frac{-a}{\beta} = -\frac{\sqrt{2}+2}{4(2\sqrt{2}-1)}$, and $c_{n-1} - \beta c_n = 0$ for $n = 1, 2, \ldots.$

Thus, $\frac{c_{n-1}}{c_{n-1}} - \beta \frac{c_n}{c_{n-1}} = 0$, that is to say, $1 - \beta L = 0$, or $L = \frac{1}{\beta}$.
The radius of convergence is thus given by $\lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \left|\frac{1}{L}\right| = |\beta| = 1 + 2\sqrt{2}$.

To the second term, we have $\frac{b}{x-\alpha} = \sum_{n=0}^{\infty} d_n x^n$, we also follow the same procedures and can find the radius of convergence to be $|\alpha| = |1 - 2\sqrt{2}| = 2\sqrt{2} - 1$.

Taking intersections of the two intervals, we have the radius of convergence for the sum of the two power series $R = 2\sqrt{2} - 1$.