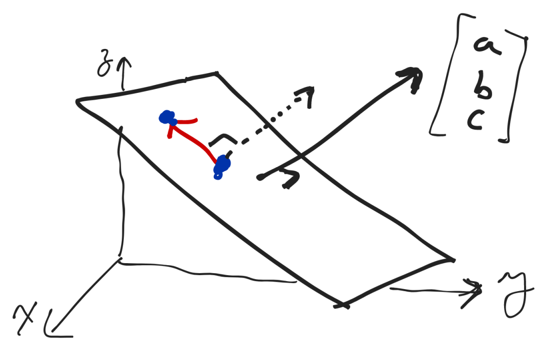
AMA1007 Supplementary Notes: Some common objects in 3D

A plane in 3D in normal form is given by

$$ax + by + cz = d$$

where the coefficients of the x, y and z terms form a vector (a direction vector) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, which gives the normal direction of the plane. Taking any two distinct points on the plane to form a vector (i.e., the direction vector between the two points) would be perpendicular (at right angle) with the normal vector. If $d = 0$, then, the plane passes through the origin.

$$ax + by + cz = d$$



Suppose the plane having a normal direction $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and contains a point (a position vector) $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, then, the value of d can be simply found by

$$ax_0 + by_0 + cz_0 = d.$$

Example Suppose a plane has normal direction $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, and suppose the plane contains the point $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, the equation of the plane in normal form is given by

$$x + 2y - z \;=\; (2) + 2(-1) - (2) \;=\; -2.$$

<u>**A line in 3D in parametric form**</u> is given by

$$\boldsymbol{r}(t) \;=\; \boldsymbol{a} + t\boldsymbol{d}$$

or

$$\begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{bmatrix} \;=\; \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

A line is a $1D$ object, and we need one parameter t. Note that, $\boldsymbol{a}$ is a given fixed point that line passes through, and $\boldsymbol{d}$ is the direction of line is going towards (when t is increasing). We take values of t from $-\infty$ to $+\infty$ and $\boldsymbol{r}(t)$ traces out the line. Note that this respresentation is not unique, any point on the plane can be $\boldsymbol{a}$, and any non-zero scalar multiple of $\boldsymbol{d}$ is also a direction vector.

Example If two planes in $3D$ intercept, and the intersection is a line in $3D$. Suppose the two planes are given by

$$\begin{aligned} x + y + z &= 1 \\ -x + 2y - z &= -1. \end{aligned}$$

Solving the two planes using the Gauss-Jordan method:
$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ -1 & 2 & -1 & | & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 3 & 0 & | & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 0 & | & 0 \end{bmatrix} \Longrightarrow$ $\begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 0 & | & 0 \end{bmatrix}$ this is in reduced row-echelon form. There are two leading 1s. Column of z has no leading 1, so we let $z = t$. Thus, from the first row, we can see $x + z = 1$, therefore, $x = 1 - t$. From the second row, we can see $y = 0$. Hence, the parametric equation of the line is given by

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \;=\; \begin{bmatrix} 1-t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

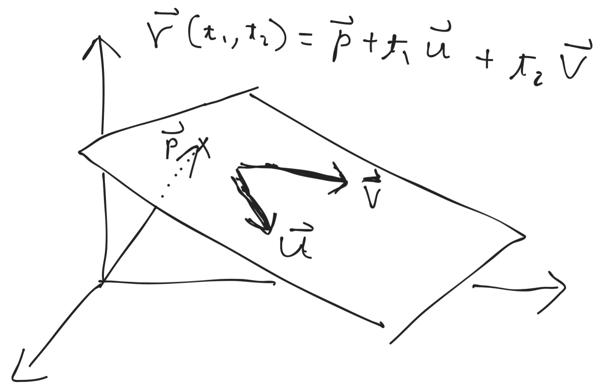
A plane in 3D in parametric form is given by

$$\boldsymbol{r}(t_1, t_2) = \boldsymbol{p} + t_1\boldsymbol{u} + t_2\boldsymbol{v}$$

or

$$\begin{bmatrix} r_1(t_1, t_2) \\ r_2(t_1, t_2) \\ r_3(t_1, t_2) \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + t_1 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + t_2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

A plane is a $2D$ object in $3D$ space, and we need two parameters t_1 and t_2. Note that, $\boldsymbol{p}$ is a given fixed point the plane passes through, and $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent vector directions spanning the plane. Again, the representation is not unique.



To convert a plane given in parametric form back to normal form, we can obtain the direction of normal by performing a cross product $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \boldsymbol{u} \times \boldsymbol{v}$, then $d = ap_1 + bp_2 + cp_3$.

Example Given a plane in parametric form:

$$\boldsymbol{r}(t_1, t_2) = \boldsymbol{p} + t_1\boldsymbol{u} + t_2\boldsymbol{v}$$

where $\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The normal direction of the plane is given by $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \boldsymbol{u} \times \boldsymbol{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, then $d = ap_1 + bp_2 + cp_3 = (-1)(1) + (0)(2) + (1)(3) = 2$. Therefore, the plane in normal form is given by $-x + z = 2$.

To convert a plane given in normal form to parametric form, we can let any two of the three variables x, y, z be the parameters, and then express the remaining variable in terms of the parameters.

Example Suppose a plane given in normal from is $x + y + z = 1$. Let $y = t_1$, and $z = t_2$. Then, $x = 1 - y - z = 1 - t_1 - t_2$. Therefore, we have

$$\begin{bmatrix} x(t_1, t_2) \\ y(t_1, t_2) \\ z(t_1, t_2) \end{bmatrix} = \begin{bmatrix} 1 - t_1 - t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$