AMA1007 Supplementary Notes: Find the linear map matrix in 2-D

Consider a 2×2 linear map $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that maps a point (location) $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ on the $x - y$ plane to another point (image) on the $x - y$ plane $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, i.e. $\boldsymbol{Au} = \boldsymbol{v}$, or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Suppose we do not know the 2×2 linear map matrix $\boldsymbol{A}$, but we know the images of the two locations $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$, i.e.

$$\begin{aligned} \boldsymbol{Au_1} &= \boldsymbol{v_1}, \\ \boldsymbol{Au_2} &= \boldsymbol{v_2}, \end{aligned}$$

where the locations $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ would satisfy a certain condition we will describe below, then, it is possible to find $\boldsymbol{A}$.

In particular, if we pick $\boldsymbol{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{u_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then

$$\begin{aligned} \boldsymbol{Au_1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \boldsymbol{v_1}, \\ \boldsymbol{Au_2} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \boldsymbol{v_2}. \end{aligned}$$

Therefore, the columns of $\boldsymbol{A}$ are just $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ in this ideal choice of $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$.

Suppose $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are in other "no-so-ideal" general positions, that satisfy a certain condition, we can still find $\boldsymbol{A}$, but with a bit of more work.

Let $\boldsymbol{u_1} = \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}$, $\boldsymbol{u_2} = \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}$, and $\boldsymbol{v_1} = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$, $\boldsymbol{v_2} = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$.

$$\begin{aligned} \boldsymbol{Au_1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \\ \boldsymbol{Au_2} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}. \end{aligned}$$

That means,

$$\begin{aligned} u_{11}a + u_{12}b & & &= v_{11} \\ u_{11}c + u_{12}d & & &= v_{12} \\ & & u_{21}a + u_{22}b &= v_{21} \\ & & u_{21}c + u_{22}d &= v_{22} \end{aligned}$$

or

$$\begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} \\ u_{21} & u_{22} & 0 & 0 \\ 0 & 0 & u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{12} \\ v_{21} \\ v_{22} \end{bmatrix}.$$

Therefore, if

$$\begin{aligned} det\left(\begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} \\ u_{21} & u_{22} & 0 & 0 \\ 0 & 0 & u_{21} & u_{22} \end{bmatrix}\right) &= -det\left(\begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ u_{21} & u_{22} & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} \\ 0 & 0 & u_{21} & u_{22} \end{bmatrix}\right) \\ &= -\left(det\left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}\right)\right)^2 \neq 0, \end{aligned}$$

then, we have a unique solution for $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

Example

Find the 2×2 linear map $\boldsymbol{A}$ such that $\boldsymbol{Au_1} = \boldsymbol{v_1}$, and $\boldsymbol{Au_2} = \boldsymbol{v_2}$, where $\boldsymbol{u_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{u_2} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{v_2} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

We check that $det\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}\right) = -\left(det\left(\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\right)\right)^2 = -1 \neq 0$, thus it is possible to find $\boldsymbol{A}$. Using the technique presented above, we solve the system

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

and obtain $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$. Therefore, the linear map is given by $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$.