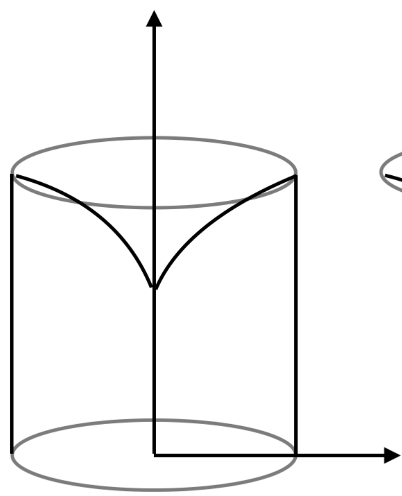
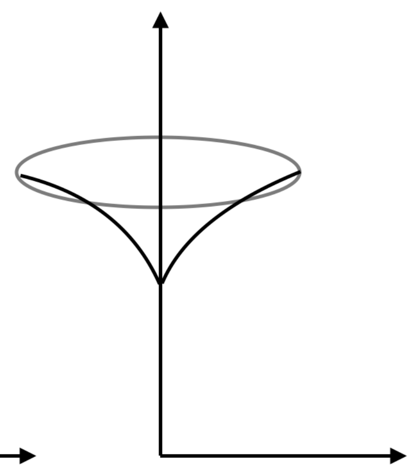
AMA1007 Supplementary Notes: Cylindrical Shell Method to find the volume of rotation

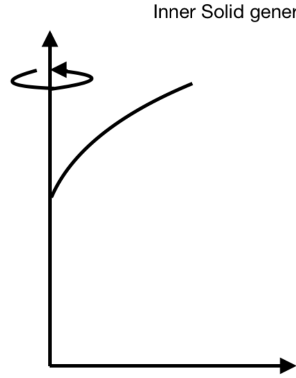
Consider the graph $y = f(x)$ for $0 \leq x \leq b$, where f is continuous and $f \geq 0$. For illustration purposes, let us further impose on f, that f is a one-to-one function. If the graph is rotated about the y-axis, then, there are two solids generated, namely, the Inner Solid and the Outer Solid.



Outer Solid generated



Inner Solid generated



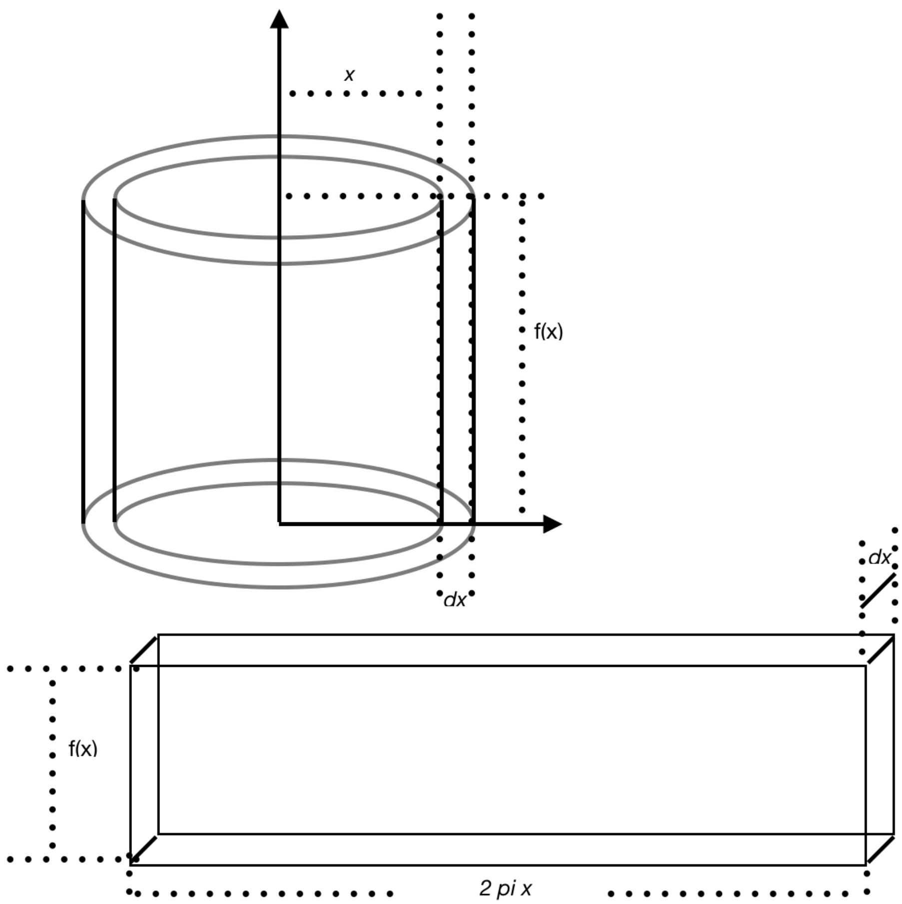
When rotate a piece of graph about the y axis, two solids generated

For the **Inner Solid**, we can find its volume directly using the usual integral (Riemann sum of delta volume of disks), if f is one-to-one, $f(0) = \alpha$, and $f(b) = \beta$.

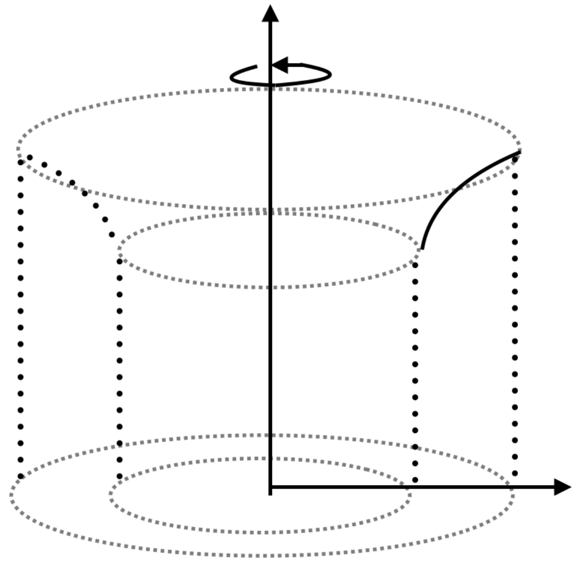
$$V = \int_{\alpha}^{\beta} \pi(x)^2\, dy \quad = \quad \int_{\alpha}^{\beta} \pi(f^{-1}(y))^2\, dy.$$

For the **Outer Solid**, we can also find its volume directly using a method called the **Cylindrical Shell method** (which is the Riemann sum of delta volume of shells of cylinders):

$$V \quad = \quad \int_0^b 2\pi\, x\, f(x)\, dx.$$



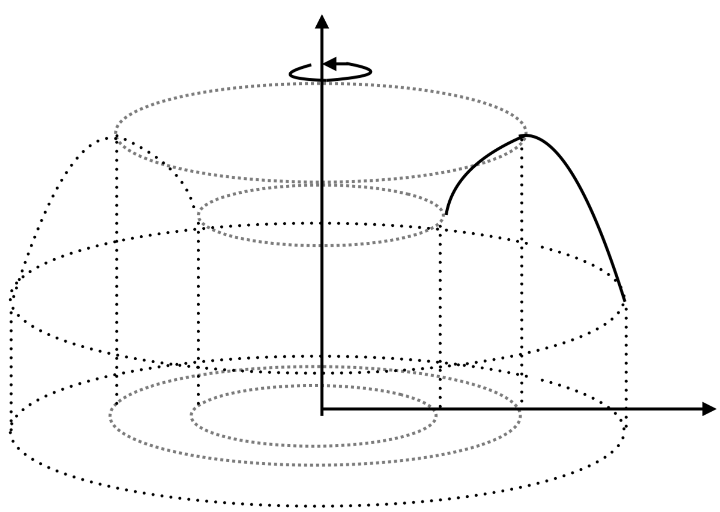
Volume of the Shell cylinder is given by 2 pi x f(x) dx



In general, the volume of the "Outer Solid" generated by rotating $y = f(x)$ for $0 \le a \le x \le b$ about the y-axis is given by

$$V = \int_a^b 2\pi\, x\, f(x)\, dx.$$

Note that the Cylindrical Shell method can be applied to f of which it is not even a one-to-one function:



Example

Consider $y = \sqrt{x+1}$ for $0 \leq x \leq 1$. The volume of the "Outer Solid" generated by rotating the graph about the y-axis is given by

$$V_1 \quad = \quad \int_0^1 2\pi\, x\, \sqrt{x+1}\, dx = \frac{8}{15}\,\pi\Big(\sqrt{2}+1\Big).$$

Of course we can also check the volume of the "Inner Solid" directly by the disks method. Since $y = \sqrt{x+1}$, then, $y^2 = x+1$, or $x = y^2-1$. Thus, $x^2 = (y^2-1)^2$. When $x = 0$, $y = 1$. And when $x = 1$, $y = \sqrt{2}$. Thus, the volume of the "Inner Solid" is given by

$$V_2 = \int_1^{\sqrt{2}} \pi\, x^2 \;dy \quad = \quad \int_1^{\sqrt{2}} \pi(y^2-1)^2\, dy \;=\; \frac{1}{15}\pi(7\sqrt{2}-8).$$

We can check with the volume of the upright Cylinder made-up by these two solids: a cylinder with height $\sqrt{2}$, and cross-section a circle with radius 1, the volume is given by $\sqrt{2}\pi$. And of course, $V_1 + V_2 = \sqrt{2}\pi$.