AMA1007 Supplementary Notes: To find the area of the surface by rotation

Consider the graph of $y = f(x)$ for $a \le x \le b$ where f is differentiable on (a, b). Recall that the arc-length is given by

$$\int ds \quad = \quad \int_a^b \sqrt{1 + \left(\frac{d}{dx} f(x)\right)^2}\; dx.$$

The **Area of the Surface of Rotation about x-axis** (the area of the surface generated by rotating $f(x)$ about the x-axis) is given by

$$\begin{aligned} S &= \int_{x=a}^{x=b} 2\,\pi\, y\; ds \\ &= \int_a^b 2\,\pi\, f(x) \sqrt{1 + \left(\frac{d}{dx} f(x)\right)^2}\; dx. \end{aligned}$$

Suppose f^{-1} exists, i.e., we have $x = f^{-1}(y)$. Then, the **Area of the Surface of Rotation about y-axis** (the area of the surface generated by rotating $f(x)$ about the x-axis) is given by

$$\begin{aligned} S &= \int_{x=a}^{x=b} 2\,\pi\, x\; ds \\ &= \int_a^b 2\,\pi\, x \sqrt{1 + \left(\frac{d}{dx} f(x)\right)^2}\; dx. \end{aligned}$$

Example Consider the graph of $y = \sqrt{x+1}$ where $0 \le x \le 1$. The area of the surface generated by rotation of the graph **about the x-axis** is given by

$$S \quad = \quad \int_{x=0}^{x=1} 2\,\pi\, f(x)\, ds.$$

Finding ds, we have

$$ds \quad = \quad \sqrt{1 + \left(\frac{d}{dx} f(x)\right)^2} \quad = \quad \sqrt{1 + \left(\frac{1}{2\sqrt{x+1}}\right)^2} \quad = \quad \frac{1}{2}\sqrt{\frac{4x+5}{x+1}}.$$

Thus,

$$S \quad = \quad \int_0^1 2\,\pi\, \sqrt{x+1} \cdot \frac{1}{2}\sqrt{\frac{4x+5}{x+1}}\; dx \; = \; \int_0^1 \pi\sqrt{4x+5}\, dx = \frac{1}{6}\,\pi\Big(27 - 5\,\sqrt{5}\Big) \approx 8.2831.$$

The area of the surface generated by rotation of the graph **about the y-axis** is given by

$$S = \int_{x=0}^{x=1} 2\,\pi\,x\,ds.$$

Thus,

$$\begin{aligned} S &= \int_0^1 2\,\pi\,x \cdot \frac{1}{2}\sqrt{\frac{4x+5}{x+1}}\,dx = \int_0^1 \pi x\sqrt{\frac{4x+5}{x+1}}\,dx \\ &= \frac{1}{64}\,\pi\Big(28\,\sqrt{5} + 12\,\sqrt{2} + 17\,\log\Big(4\,\sqrt{5}+9\Big) - 17\,\log\Big(12\,\sqrt{2}+17\Big)\Big) \\ &\approx 3.37380. \end{aligned}$$