AMA1007 Supplementary Notes: To find the closest points between two parabolas

Assuming two non-intersecting parabolas are given by

$$\begin{aligned} P_1: \quad y = f_1(x) &= a_2x^2 + a_1x + a_0, \\ P_2: \quad y = f_2(x) &= b_2x^2 + b_1x + b_0. \end{aligned}$$

Let the closest points be (x_1, y_1) on P_1, and (x_2, y_2) on P_2.

The square of the distance between the two points is given by

$$\begin{aligned} D^2(x_1, x_2) &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (x_2 - x_1)^2 + (b_2x_2^2 + b_1x_2 + b_0 - a_2x_1^2 - a_1x_1 - a_0)^2. \end{aligned}$$

At the closest points, (x_1, y_1) on P_1 and (x_2, y_2) on P_2, the tangent lines on the respective parabolas must have the same slope. Therefore, we have $f_1'(x_1) = f_2'(x_2)$, or

$$2a_2x_1 + a_1 = 2b_2x_2 + b_1.$$

Hence, we have

$$x_2 = \frac{2a_2x_1 + a_1 - b_1}{2b_2}.$$

Substituting this into D^2, the square of the distance would be in terms of x_1 only, i.e., $D^2(x_1)$.

Therefore, minimizing $D^2(x_1)$ would yield the closest distance. So, by solving x_1 from $\frac{d}{dx_1}D^2(x_1) = 0$ would give the candidate position x_1^0. Moreover, check to see if $\frac{d^2}{dx_1^2}D^2(x_1^0) > 0$ to ensure x_1^0 is indeed a minimum.

Example Find the closest points between the two parabolas:

$$\begin{aligned} P_1: \quad y &= x^2, \\ P_2: \quad y &= -x^2 - 16x - 65. \end{aligned}$$

Thus, the square distance is given by

$$D^2(x_1, x_2) = (x_2 - x_1)^2 + (-x_2^2 - 16x_2 - 65 - x_1^2)^2,$$

and at the closest points with equal slope of tangents yield

$$x_2 \quad = \quad \frac{2x_1 - (-16)}{2(-1)} = -x_1 - 8.$$

Hence,

$$\begin{aligned} D^2(x_1) &= (-2x_1 - 8)^2 + (-(-x_1 - 8)^2 - 16(-x_1 - 8) - 65 - x_1^2)^2 \\ &= 4x_1^4 + 8x_1^2 + 32x_1 + 65. \end{aligned}$$

Solving x_1 from $\frac{d}{dx_1} D^2(x_1) = 0$ gives

$$16x_1^3 + 16x_1 + 32 = 16(x_1 + 1)(x_1^2 - x_1 + 2) = \quad = \quad 0.$$

$x_1 = -1$ is the only real solution, thus we have the candidate $x_1 = -1$. Check and find that $\frac{d^2}{dx_1^2} D^2(-1) > 0$, thus, $x_1 = -1$ is the minimum point.

Moreover, $x_2 = -(-1) - 8 = -7$.

The closest points are $(-1, 1)$ on P_1 and $(-7, -2)$ on P_2, and the distance between them is $\sqrt{45} = 3\sqrt{5}$.