AMA1007 Supplementary Notes: another nice application of linear algebra: To find the closest points between 2 lines in 3D

Suppose two straight lines in 3D are given in parametric form (and assuming they are not intersecting):

$$\begin{aligned} \textit{line } L_1: \quad \boldsymbol{r_1}(t) &= \boldsymbol{a_1} + t\boldsymbol{d_1} \\ \textit{line } L_2: \quad \boldsymbol{r_2}(t) &= \boldsymbol{a_2} + t\boldsymbol{d_2} \end{aligned}$$

or

$$\begin{aligned} \textit{line } L_1: \begin{bmatrix} r_{11}(t) \\ r_{12}(t) \\ r_{13}(t) \end{bmatrix} &= \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} + t \begin{bmatrix} d_{11} \\ d_{12} \\ d_{13} \end{bmatrix} \\ \textit{line } L_2: \begin{bmatrix} r_{21}(t) \\ r_{22}(t) \\ r_{23}(t) \end{bmatrix} &= \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} + t \begin{bmatrix} d_{21} \\ d_{22} \\ d_{23} \end{bmatrix} \end{aligned}$$

Note that, $\boldsymbol{a_1}$ is a given fixed point that line L_1 passes through, and $\boldsymbol{a_2}$ is a given fixed point that line L_2 passes through. Moreover, $\boldsymbol{d_1}$ is direction of line L_1 is going towards (when t is increasing), and $\boldsymbol{d_2}$ is the direction of line L_2 is going towards (when t is increasing). We take values of t from $-\infty$ to $+\infty$ of each of $\boldsymbol{r_1}(t)$ and $\boldsymbol{r_2}(t)$ to trace out the two lines.

Suppose $\boldsymbol{P_1}$ is the point on line L_1 closest to L_2, and $\boldsymbol{P_2}$ is the point in line L_2 closest to L_1. Let t_1 be the parameter that $\boldsymbol{r_1}(t_1) = \boldsymbol{P_1}$, and t_2 be the parameter that $\boldsymbol{r_2}(t_2) = \boldsymbol{P_2}$.

The line segment $\boldsymbol{P_1} - \boldsymbol{P_2}$ must be perpendicular to both lines L_1 and L_2. Let the direction of this line segment be $\boldsymbol{d_3}$. Hence, the direction of this line segment can be obtained by performing a cross product of $\boldsymbol{d_1}$ and $\boldsymbol{d_2}$, i.e., $\boldsymbol{d_3} = \boldsymbol{d_1} \times \boldsymbol{d_2}$.

Let the parametric form of this line segment be $\boldsymbol{r_3}(t) = \boldsymbol{P_1} + t\boldsymbol{d_3}$. At $t = 0$, $\boldsymbol{r_3}(0) = \boldsymbol{P_1}$, and suppose at $t = t_3$, $\boldsymbol{r_3}(t_3) = \boldsymbol{P_2}$.

Therefore, we have

$$\boldsymbol{P_1} + t_3\boldsymbol{d_3} \quad = \quad \boldsymbol{P_2},$$

or

$$\boldsymbol{a_1} + t_1\boldsymbol{d_1} + t_3\boldsymbol{d_3} \quad = \quad \boldsymbol{a_2} + t_2\boldsymbol{d_2}.$$

After re-arranging it, we have

$$t_1\boldsymbol{d_1} - t_2\boldsymbol{d_2} + t_3\boldsymbol{d_3} \quad = \quad \boldsymbol{a_2} - \boldsymbol{a_1}.$$

That is to say

$$\begin{bmatrix} d_{11} & -d_{21} & d_{31} \\ d_{12} & -d_{23} & d_{32} \\ d_{13} & -d_{24} & d_{33} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \quad = \quad \begin{bmatrix} a_{21} - a_{11} \\ a_{22} - a_{12} \\ a_{23} - a_{13} \end{bmatrix}$$

We can then solve for t_1, t_2, and t_3, and thus get the closest points $\boldsymbol{P_1} = \boldsymbol{r_1}(t_1)$, $\boldsymbol{P_2} = \boldsymbol{r_2}(t_2)$. Moreover, the distance between them is given by the norm (length) of $t_3\boldsymbol{d_3}$.

Example We use the technique to find the closest points on these two lines:

$$\textit{line } L_1 : \begin{bmatrix} r_{11}(t) \\ r_{12}(t) \\ r_{13}(t) \end{bmatrix} \quad = \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$\textit{line } L_2 : \begin{bmatrix} r_{21}(t) \\ r_{22}(t) \\ r_{23}(t) \end{bmatrix} \quad = \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

The direction $\boldsymbol{d_3}$ is given by

$$\boldsymbol{d_3} = \boldsymbol{d_1} \times \boldsymbol{d_2} \quad = \quad \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Hence, we solve the system

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \quad = \quad \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}.$$

Solving it yield $t_1 = 0$, $t_2 = -\frac{2}{3}$, and $t_3 = -\frac{2}{3}$. Therefore, the closest point are $\boldsymbol{r_1}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ on line L_1, and $\boldsymbol{r_2}(-\frac{2}{3}) = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ on line L_2.

The distace between them is $||t_3\boldsymbol{d_3}|| = \dfrac{2}{3}\sqrt{3}$.