The Hong Kong Polytechnic University
Department of Applied Mathematics

AMA1007 (Calculus and Linear Algebra)

Assignment 04 (Solution)

Question 1

(a) $\sum_{n=1}^{\infty}\frac{1-n}{n\,2^n}=\sum_{n=1}^{\infty}\frac{1}{n\,2^n}-\sum_{n=1}^{\infty}\frac{1}{2^n}$

$\sum_{n=1}^{\infty}\frac{1}{n\,2^n}$ converges by the comparison test: $\frac{1}{n\,2^n}<\frac{1}{2^n}$ and $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a convergent geometric series. → $\sum_{n=1}^{\infty}\frac{1-n}{n\,2^n}$ converges.

(b) Diverges by the comparison test: $\frac{1}{n}<\frac{1}{\ln n}<\frac{1}{\ln(\ln n)}$ and $\sum_{n=3}^{\infty}\frac{1}{n}$ diverges.

(c) Converges by the ratio test:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left(\frac{1\times3\times5\times\cdots\times(2n-1)\times(2n+1)}{4^{n+1}\,2^{n+1}\,(n+1)!}\cdot\frac{4^n\,2^n\,n!}{1\times3\times5\times\cdots\times(2n-1)}\right)$$

$$=\lim_{n\to\infty}\frac{2n+1}{8(n+1)}=\frac{1}{4}<1$$

(d) Diverges by the ratio test:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left(\frac{3^{n+1}}{(n+1)^3\,2^{n+1}}\cdot\frac{n^3\,2^n}{3^n}\right)=\lim_{n\to\infty}\frac{3n^3}{2(n+1)^3}=\frac{3}{2}>1$$

(e) Converges by the integral test:

$$\int_3^{\infty}\frac{1/n}{(\ln n)\sqrt{\ln^2 n-1}}\,dx=\int_{\ln 3}^{\infty}\frac{du}{u\sqrt{u^2-1}}=\lim_{t\to\infty}\sec^{-1}|u|\,\Big|_{\ln 3}^{t}=\frac{\pi}{2}-\sec^{-1}(3)$$

(f) Converges by the alternating series test:

Since (1) $f(x)=\dfrac{\sqrt{x}+1}{x+1} \Rightarrow f'(x)=\dfrac{1-x-2\sqrt{x}}{2\sqrt{x}(x+1)^2}<0$ → $f(x)$ is decreasing.

and (2) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\dfrac{\sqrt{n}+1}{n+1}=0$.

(g) Converges absolutely by the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{(100)^{n+1}}{(n+1)!}\cdot\frac{n!}{(100)^n}\right)=\lim_{n\to\infty}\frac{100}{n+1}=0<1$$

(h) Converges by the integral test:

$$\int_1^\infty \frac{1}{n(1+\ln^2 n)}\,dx=\int_1^\infty \frac{1/n}{1+\ln^2 n}\,dx=\int_0^\infty \frac{du}{1+u^2}=\lim_{t\to\infty}\tan^{-1}|u|\Big|_0^t=\frac{\pi}{2}$$

(i) Converges by the alternating series test:

Since (1) $f(x)=\ln\left(1+\dfrac{1}{x}\right) \Rightarrow f'(x)=\dfrac{-1}{x(x+1)}<0$ for $x>0$.

→ $f(x)$ is decreasing for $x>0$

and (2) $\lim\limits_{n\to\infty} a_n=\lim\limits_{n\to\infty}\ln\left(1+\dfrac{1}{n}\right)=\ln\left[\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)\right]=\ln 1=0$.

Question 2

(a)

$f(x)=2^x$	$f(1)=2$
$f'(x)=\ln 2\cdot 2^x$	$f'(1)=\ln 2\cdot 2$
$f''(x)=(\ln 2)^2\cdot 2^x$	$f''(1)=(\ln 2)^2\cdot 2$
$f'''(x)=(\ln 2)^3\cdot 2^x$	$f'''(1)=(\ln 2)^3\cdot 2$
$\vdots$	$\vdots$
$f^{(n)}(x)=(\ln 2)^n\cdot 2^x$	$f^{(n)}(1)=(\ln 2)^n\cdot 2$

Taylor series at $x=1$:

$$2+(2\ln 2)(x-1)+\frac{2(\ln 2)^2}{2!}(x-1)^2+\frac{2(\ln 2)^3}{3!}(x-1)^3+\cdots=\sum_{n=0}^{\infty}\frac{2(\ln 2)^n(x-1)^n}{n!}$$

(b) $f(x) = 2x^3 + x^2 + 3x - 8$ $\quad$ $f(1) = -2$

$f'(x) = 6x^2 + 2x + 3$ $\quad$ $f'(1) = 11$

$f''(x) = 12x + 2$ $\quad$ $f''(1) = 14$

$f'''(x) = 12$ $\quad$ $f'''(1) = 12$

$f^{(n)}(x) = 0$ for $n \geq 4$ $\quad$ $f^{(n)}(1) = 0$ for $n \geq 4$

Taylor series at $x = 1$:

$$-2 + 11(x-1) + \frac{14}{2!}(x-1)^2 + \frac{12}{3!}(x-1)^3 = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3$$

Question 3

(a) $f(x) = (1+x)e^{2x}$ $\quad$ $f(0) = 1$

$f'(x) = (3+2x)e^{2x}$ $\quad$ $f'(0) = 3$

$f''(x) = (8+4x)e^{2x}$ $\quad$ $f''(0) = 8$

$f'''(x) = (20+8x)e^{2x}$ $\quad$ $f'''(0) = 20$

Maclaurin's polynomial (up to degree 3):

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 = 1 + 3x + 4x^2 + \frac{10}{3}x^3$$

(b) $f(x) = \ln(3+e^x)$ $\quad$ $f(0) = \ln 4$

$f'(x) = \dfrac{e^x}{3+e^x}$ $\quad$ $f'(0) = \dfrac{1}{4}$

$f''(x) = \dfrac{3e^x}{(3+e^x)^2}$ $\quad$ $f''(0) = \dfrac{3}{16}$

$f'''(x) = \dfrac{9e^x - 3e^{2x}}{(3+e^x)^3}$ $\quad$ $f'''(0) = \dfrac{3}{32}$

Maclaurin's polynomial (up to degree 3):

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 = \ln 4 + \frac{1}{4}x + \frac{3}{32}x^2 + \frac{1}{64}x^3$$

Question 4

(a) By Sarrus' Rule: $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = (-20-7+72)-(20+84+6) = -65$

(b) Expand along the second row:

$$\begin{vmatrix} 1 & 3 & 2 & 5 \\ 0 & 0 & 3 & 2 \\ 1 & 5 & 4 & 0 \\ 1 & 2 & 1 & 1 \end{vmatrix} = (-3)\begin{vmatrix} 1 & 3 & 5 \\ 1 & 5 & 0 \\ 1 & 2 & 1 \end{vmatrix} + (2)\begin{vmatrix} 1 & 3 & 2 \\ 1 & 5 & 4 \\ 1 & 2 & 1 \end{vmatrix}$$

By Sarrus' Rule:

$$= (-3)\big[(5+0+10)-(25+0+3)\big] + (2)\big[(10+8+3)-(5+12+4)\big] = 39$$

(c) Expand along the third row:

$$\begin{vmatrix} 0 & -1 & 2 & 1 \\ -4 & 3 & -3 & 5 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 1 \end{vmatrix} = (1)\begin{vmatrix} -1 & 2 & 1 \\ 3 & -3 & 5 \\ 1 & 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 & 2 \\ -4 & 3 & -3 \\ -1 & 1 & 0 \end{vmatrix}$$

By Sarrus' Rule:

$$= (1)\big[(3+10+0)-(-3+0+6)\big] + (1)\big[(0-3-8)-(-6+0+0)\big] = 5$$

Question 5

$$\det(\mathbf{A}-4\mathbf{I}) = \begin{vmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 4 & -17 & 4 \end{vmatrix} = -(0)\begin{vmatrix} 1 & 0 \\ -17 & 4 \end{vmatrix} + (-4)\begin{vmatrix} -4 & 0 \\ 4 & 4 \end{vmatrix} - (1)\begin{vmatrix} -4 & 1 \\ 4 & -17 \end{vmatrix} = 0.$$

→ 4 is a root of $\det(\mathbf{A}-x\mathbf{I}) = 0$.

$$\begin{vmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 4 & -17 & 8-x \end{vmatrix} = -(0)\begin{vmatrix} 1 & 0 \\ -17 & 8-x \end{vmatrix} + (-x)\begin{vmatrix} -x & 0 \\ 4 & 8-x \end{vmatrix} - (1)\begin{vmatrix} -x & 1 \\ 4 & -17 \end{vmatrix}$$

$$= (-x)(-x)(8-x) - (1)(17x-4) = -x^3 + 8x^2 - 17x + 4 = (x-4)(x^2 - 4x + 1)$$

$$\det(\mathbf{A} - x\mathbf{I}) = (x-4)(x^2 - 4x + 1) = 0 \Rightarrow x = 4,\ 2 \pm \sqrt{3}\,.$$

Question 6

(a) $\begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

$$\Delta = \begin{vmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \\ 6 & 2 & 4 \end{vmatrix} = -144 \quad (\neq 0)\,.$$

$$\Delta_1 = \begin{vmatrix} 1 & 4 & 6 \\ 3 & 6 & 2 \\ 5 & 2 & 4 \end{vmatrix} = -132\,, \quad \Delta_2 = \begin{vmatrix} 2 & 1 & 6 \\ 4 & 3 & 2 \\ 6 & 5 & 4 \end{vmatrix} = 12\,, \text{ and } \quad \Delta_3 = \begin{vmatrix} 2 & 4 & 1 \\ 4 & 6 & 3 \\ 6 & 2 & 5 \end{vmatrix} = 12\,.$$

By Cramer's rule, we have $x_1 = \dfrac{\Delta_1}{\Delta} = \dfrac{-132}{-144} = \dfrac{11}{12}$, $x_2 = \dfrac{\Delta_2}{\Delta} = \dfrac{12}{-144} = -\dfrac{1}{12}$, and $x_3 = \dfrac{\Delta_3}{\Delta} = \dfrac{12}{-144} = -\dfrac{1}{12}$.

(b) $\begin{cases} 2x_1 + 2x_2 - x_3 = 3 \\ 3x_1 - 4x_2 + 2x_3 = 1 \\ 8x_1 - x_2 - 3x_3 = 4 \end{cases}$

→ $x_1 = \dfrac{\Delta_1}{\Delta} = \dfrac{49}{49} = 1$, $x_2 = \dfrac{\Delta_2}{\Delta} = \dfrac{49}{49} = 1$, and $x_3 = \dfrac{\Delta_3}{\Delta} = \dfrac{49}{49} = 1$.

Question 7

(a) $\mathbf{I} = \mathbf{I}^3 = \mathbf{I}^3 - \mathbf{A}^3 = (\mathbf{I} - \mathbf{A})(\mathbf{I}^2 + \mathbf{A} + \mathbf{A}^2) = (\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2)$

$\Rightarrow (\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2$

(b) $\mathbf{A}^3 - \mathbf{A} + \mathbf{I} = \mathbf{0} \Rightarrow \mathbf{I} = \mathbf{A} - \mathbf{A}^3 = \mathbf{A}(\mathbf{I} - \mathbf{A}^2) \Rightarrow \mathbf{A}^{-1} = \mathbf{I} - \mathbf{A}^2$

Question 8

$$\frac{dW}{dx} = \frac{d}{dx}\begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix}$$

$$= \frac{d}{dx}\left(f_1(x)g_2(x) - f_2(x)g_1(x) \right)$$

$$= \left(f_1'(x)g_2(x) + f_1(x)g_2'(x) \right) - \left(f_2'(x)g_1(x) + f_2(x)g_1'(x) \right)$$

$$= \left(f_1'(x)g_2(x) - f_2'(x)g_1(x) \right) + \left(f_1(x)g_2'(x) - f_2(x)g_1'(x) \right)$$

$$= \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}$$

2013-2014 Sem 1 Final Exam Q6

Let x, y, and z be the measures of the first, second, and third classes of corn. We thus have

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{array}\right] \xrightarrow{\frac{1}{3}\mathbf{r}_1 \to \mathbf{r}_1} \left[\begin{array}{ccc|c} 1 & 2/3 & 1/3 & 13 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{array}\right] \xrightarrow[\mathbf{r}_3 - \mathbf{r}_1 \to \mathbf{r}_3]{\mathbf{r}_2 - 2\mathbf{r}_1 \to \mathbf{r}_2} \left[\begin{array}{ccc|c} 1 & 2/3 & 1/3 & 13 \\ 0 & 5/3 & 1/3 & 8 \\ 0 & 4/3 & 8/3 & 13 \end{array}\right]$$

$$\xrightarrow{\frac{3}{5}\mathbf{r}_2 \to \mathbf{r}_2} \left[\begin{array}{ccc|c} 1 & 2/3 & 1/3 & 13 \\ 0 & 1 & 1/5 & 24/5 \\ 0 & 4/3 & 8/3 & 13 \end{array}\right] \xrightarrow{\mathbf{r}_3 - \frac{4}{3}\mathbf{r}_2 \to \mathbf{r}_3} \left[\begin{array}{ccc|c} 1 & 2/3 & 1/3 & 13 \\ 0 & 1 & 1/5 & 24/5 \\ 0 & 0 & 12/5 & 33/5 \end{array}\right] \xrightarrow{\frac{5}{12}\mathbf{r}_3 \to \mathbf{r}_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2/3 & 1/3 & 13 \\ 0 & 1 & 1/5 & 24/5 \\ 0 & 0 & 1 & 11/4 \end{array}\right] \xrightarrow{\mathbf{r}_2 - \frac{1}{5}\mathbf{r}_3 \to \mathbf{r}_2} \left[\begin{array}{ccc|c} 1 & 2/3 & 1/3 & 13 \\ 0 & 1 & 0 & 17/4 \\ 0 & 0 & 1 & 11/4 \end{array}\right]$$

$$\xrightarrow{\mathbf{r}_1 - \frac{2}{3}\mathbf{r}_2 - \frac{1}{3}\mathbf{r}_3 \to \mathbf{r}_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 37/4 \\ 0 & 1 & 0 & 17/4 \\ 0 & 0 & 1 & 11/4 \end{array}\right]$$

Therefore, $x = \frac{37}{4}$, $y = \frac{17}{4}$, and $\frac{11}{4}$.

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$$\left[\begin{array}{ccccc|ccccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{\mathbf{r}_4 - \mathbf{r}_5 \to \mathbf{r}_4}]{\substack{\mathbf{r}_1 - \mathbf{r}_5 \to \mathbf{r}_1 \\ \mathbf{r}_2 - \mathbf{r}_5 \to \mathbf{r}_2 \\ \mathbf{r}_3 - \mathbf{r}_5 \to \mathbf{r}_3}} \left[\begin{array}{ccccc|ccccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\mathbf{r}_1 - \mathbf{r}_4 \to \mathbf{r}_1 \\ \mathbf{r}_2 - \mathbf{r}_4 \to \mathbf{r}_2 \\ \mathbf{r}_3 - \mathbf{r}_4 \to \mathbf{r}_3}} \left[\begin{array}{ccccc|ccccc} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\mathbf{r}_1 - \mathbf{r}_3 \to \mathbf{r}_1 \\ \mathbf{r}_2 - \mathbf{r}_3 \to \mathbf{r}_2}} \left[\begin{array}{ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\mathbf{r}_1 - \mathbf{r}_2 \to \mathbf{r}_1} \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Therefore, the inverse is given by $\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

2013-2014 Sem 1 Final Exam Q9

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda & 0 & -2 & 0 \\ -1 & \lambda & -1 & 0 \\ 0 & -1 & \lambda+2 & 0 \\ 0 & 0 & 0 & \lambda-1 \end{bmatrix}$$

$$= \lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2$$

$$= (\lambda - 1)^2(\lambda + 2)(\lambda + 1)$$

the characteristic equation is

$$(\lambda - 1)^2 (\lambda + 2)(\lambda + 1) = 0$$

The eigenvalues are $\lambda = 1$, $\lambda = -2$, and $\lambda = -1$. If we set $\lambda = 1$, then $(\lambda I - A)\mathbf{x} = \mathbf{0}$ becomes

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix can be reduced to

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $x_1 = 2s$, $x_2 = 3s$, $x_3 = s$, and $x_4 = t$ is a solution for all s and t. In particular, if we let
(i) s=1, t=0
(ii) s=0, t=1
we see that

$$\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for the eigenspace associated with $\lambda = 1$.

If we set $\lambda = -2$, then $(\lambda I - A)\mathbf{x} = \mathbf{0}$ becomes

$$\begin{bmatrix} -2 & 0 & -2 & 0 \\ -1 & -2 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix can be reduced to

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This implies that $x_1 = -s$, $x_2 = x_4 = 0$, and $x_3 = s$. Therefore the vector

$$\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

forms a basis for the eigenspace associated with $\lambda = -2$.

Finally, if we set $\lambda = -1$, then $(\lambda I - A)\mathbf{x} = \mathbf{0}$ becomes

$$\begin{bmatrix} -1 & 0 & -2 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix can be reduced to

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $x_1 = -2s$, $x_2 = s$, $x_3 = s$, and $x_4 = 0$ is a solution. Therefore the vector

$$\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

forms a basis for the eigenspace associated with $\lambda = -1$.

2013/14 Sem2 Final Exam Q3

3. (a)

$$\begin{aligned} det(\mathbf{A} - \lambda\mathbf{I}) &= det \begin{bmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix} \\ &= (5-\lambda)\begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} - 4\begin{vmatrix} 4 & 2 \\ 2 & 2-\lambda \end{vmatrix} + 2\begin{vmatrix} 4 & 5-\lambda \\ 2 & 2 \end{vmatrix} \\ &= (5-\lambda)((5-\lambda)(2-\lambda)-(2)(2)) - 4(4(2-\lambda)-(2)(2)) + 2((4)(2)-(5-\lambda)(2)) \\ &= -\lambda^3 + 12\lambda^2 - 21\lambda + 10 \\ &= -(\lambda-1)^2(\lambda-10). \end{aligned}$$

(b) For $\lambda = 10$, the associated eigen-space can be found by solving

$$\left[\begin{array}{ccc|c} 5-10 & 4 & 2 & 0 \\ 4 & 5-10 & 2 & 0 \\ 2 & 2 & 2-10 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, the required line is given in parametric form by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

(c) For $\lambda = 1$, the associated eigen-space can be found by solving

$$\left[\begin{array}{ccc|c} 5-1 & 4 & 2 & 0 \\ 4 & 5-1 & 2 & 0 \\ 2 & 2 & 2-1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, the required plane is given in parametric form by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -s-\frac{1}{2}t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}.$$

2013/14 Sem2 Final Exam Q5

5. To find the radius of convergence, we use the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(x+1)^{n+1}/[(n+1)2^{n+1}]}{(x+1)^n/[n2^n]}\right| &= \lim_{n\to\infty}\left|\left(\frac{(x+1)^{n+1}}{(n+1)2^{n+1}}\right)\left(\frac{n2^n}{(x+1)^n}\right)\right| \\ &= \lim_{n\to\infty}\left|\frac{n}{2(n+1)}\right||x+1|.\end{aligned}$$

So,

$$\lim_{n\to\infty}\left|\frac{n}{2(n+1)}\right||x+1| \;<\; 1$$

implies that

$$\frac{1}{2}|x+1| \;<\; 1.$$

Thus, the series converges absolutely for $|x+1|<2$, or $-3<x<1$. That is to say, the radius of convergence about $x_0=-1$ is $\rho=2$.

Question 11

```
%typeset_mode True
A=matrix([[1,2,3,4],[0,1,1,8],[1,3,4,7],[1,0,0,7]])
b=vector([4,3,2,1])
AM = A.augment(b)
show(AM)
AM.rref()
```

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 4 \\ 0 & 1 & 1 & 8 & 3 \\ 1 & 3 & 4 & 7 & 2 \\ 1 & 0 & 0 & 7 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 & -21 \\ 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$