The Hong Kong Polytechnic University
Department of Applied Mathematics

AMA1007 (Calculus and Linear Algebra)

Assignment 03 (Solution)

Question 1

(a) $$\int \sin^2 x\cos^4 x\,dx = \int (\sin^2 x\cos^2 x)\cos^2 x\,dx = \int \left(\frac{1}{2}\sin 2x\right)^2 \left(\frac{1+\cos 2x}{2}\right)dx$$

$$= \frac{1}{8}\int (\sin^2 2x)(1+\cos 2x)\,dx = \frac{1}{8}\int \sin^2 2x\,dx + \frac{1}{8}\int (\sin^2 2x)(\cos 2x)\,dx$$

$$= \frac{1}{8}\int \sin^2 u\left(\frac{1}{2}du\right) + \frac{1}{8}\int v^2\left(\frac{1}{2}dv\right) = \frac{1}{8}\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1}{2}du\right) + \frac{1}{8}\int v^2\left(\frac{1}{2}dv\right)$$

$$= \frac{1}{16}\left(\frac{u}{2} - \frac{\sin 2u}{4}\right) + \frac{1}{16}\left(\frac{v^3}{3}\right) + C = \frac{1}{16}\left(x - \frac{\sin 4x}{4}\right) + \frac{\sin^3 2x}{48} + C$$

(b) $$\int \sin 5x\cos 3x\,dx = \int \left(\frac{1}{2}\sin 2x + \frac{1}{2}\sin 8x\right)dx = -\frac{\cos 2x}{4} - \frac{\cos 8x}{16} + C$$

(c) $$\int \frac{1}{x^2 - 8x + 15}dx = \int \frac{1/2}{x-5}dx - \int \frac{1/2}{x-3}dx = \frac{1}{2}\ln|x-5| - \frac{1}{2}\ln|x-3| + C$$

(d) $$\int \frac{x^2 + 2x + 7}{x^3 + x^2 - 2}dx = \int \frac{2}{x-1}dx - \int \frac{x+3}{x^2 + 2x + 2}dx = \int \frac{2}{x-1}dx - \int \frac{x+3}{(x+1)^2 + 1}dx$$

$$= \int \frac{2}{x-1}dx - \int \frac{u+2}{u^2+1}du = \int \frac{2}{x-1}dx - \left(\int \frac{u}{u^2+1}dx + \int \frac{2}{u^2+1}dx\right)$$

$$= 2\ln|x-1| - \left(\frac{1}{2}\ln(u^2+1) + 2\tan^{-1} u\right) + C$$

$$= 2\ln|x-1| - \frac{1}{2}\ln((x+1)^2 + 1) - 2\tan^{-1}(x+1) + C$$

(e) $\displaystyle\int \frac{1}{x(x^2+1)^2}\,dx = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx - \int \frac{x}{(x^2+1)^2}\,dx$

$$= \int \frac{1}{x}\,dx - \int \frac{1}{u}\left(\frac{1}{2}du\right) - \int \frac{1}{u^2}\left(\frac{1}{2}du\right) = \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + C$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + \frac{1}{2(x^2+1)} + C$$

(f) $\displaystyle\int_{-1}^{2} \frac{x^2}{\sqrt{2+x}}\,dx = \int_1^4 \frac{(u-2)^2}{\sqrt{u}}\,du = \int_1^4 (u^{3/2} - 4u^{1/2} + 4u^{-1/2})\,du$

$$= \left(\frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2} + 8u^{1/2}\right)\Bigg|_1^4 = \frac{26}{15}$$

(g) $\displaystyle\int_1^3 4x\sqrt{6-2x}\,dx = \int_4^0 (12-2u)\sqrt{u}\left(-\frac{1}{2}du\right) = -\int_4^0 (6\sqrt{u} - u^{3/2})\,du$

$$= -\left(4u^{3/2} - \frac{2}{5}u^{5/2}\right)\Bigg|_4^0 = \frac{96}{5}$$

(h) $\displaystyle\int_{-6}^{-3} \frac{\sqrt{x^2-9}}{x}\,dx = \int_{4\pi/3}^{\pi} \frac{\sqrt{9\sec^2 u - 9}}{3\sec u}(3\sec u \tan u\,du) = \int_{4\pi/3}^{\pi} 3\tan^2 u\,du$

$$= \int_{4\pi/3}^{\pi} 3(\sec^2 u - 1)\,du = 3(\tan u - u)\Big|_{4\pi/3}^{\pi} = \pi - 3\sqrt{3}$$

(i) $\displaystyle\int_0^1 \ln(x+1)\,dx = \big[(x+1)\ln(x+1) - (x+1)\big]_0^1 = 2\ln 2 - 1$

Question 2

Let $u = a - x \Rightarrow du = -dx$; $x = 0 \Rightarrow u = a$ and $x = a \Rightarrow u = 0$.

$$I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = -\int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}\,du$$

$$= \int_0^a \frac{f(a-u)}{f(u)+f(a-u)}\,du = \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx$$

$$\Rightarrow I + I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx + \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx$$

$$= \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,dx = \int_0^a dx = x\Big|_0^a = a \Rightarrow I = \frac{a}{2}$$

Question 3

$$\int_1^2 f(x)\,dx = F(2) - F(1) = F(2) - 6 \;\Rightarrow\; F(2) = \int_1^2 f(x)\,dx + 6$$

$$\int_1^2 f(x)\,dx = \int_1^5 f(x)\,dx + \int_5^6 f(x)\,dx - \int_2^6 f(x)\,dx = 3 + 5 - 4 = 4$$

$$\Rightarrow\; F(2) = \int_1^2 f(x)\,dx + 6 = 4 + 6 = 10$$

Question 4

$$J_0 = \int (\ln x)^0\,dx = x + C$$

$$J_n = \int (\ln x)^n\,dx = x(\ln x)^n - \int n(\ln x)^{n-1}\,dx = x(\ln x)^n - nJ_{n-1}$$

$$J_3 = \int (\ln x)^3\,dx = x(\ln x)^3 - 3J_2 = x(\ln x)^3 - 3\left[x(\ln x)^2 - 2J_1\right]$$

$$= x(\ln x)^3 - 3x(\ln x)^2 + 6\left[x(\ln x) - J_0\right]$$

$$= x(\ln x)^3 - 3x(\ln x)^2 + 6x(\ln x) - 6x + C$$

Question 5

(a) $\displaystyle\int_4^5 \frac{1}{\sqrt{x-4}}\,dx = \lim_{t\to 4^+}\int_t^5 \frac{1}{\sqrt{x-4}}\,dx = \lim_{t\to 4^+}(2\sqrt{x-4})\Big|_t^5 = \lim_{t\to 4^+}(2 - 2\sqrt{t-4}) = 2$

(converge)

(b) $\displaystyle\int_4^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_4^t \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}(2\sqrt{x})\Big|_4^t = \lim_{t\to\infty}(2\sqrt{t} - 4)\Big| = \infty$ (diverge)

(c) $\displaystyle\int_{-1}^2 \frac{1}{x^2}\,dx = \int_{-1}^0 \frac{1}{x^2}\,dx + \int_0^2 \frac{1}{x^2}\,dx$

$$\int_0^2 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\int_t^2 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\left(-\frac{1}{x}\right)\Bigg|_t^2 = \lim_{t\to 0^+}\left(-\frac{1}{2} + \frac{1}{t}\right) = \infty \quad \text{(diverge)}$$

$$\Rightarrow\; \int_{-1}^2 \frac{1}{x^2}\,dx \quad \text{(diverge)}$$

(d) $\int_{-\infty}^{\infty} \frac{1}{4x^2+9}\,dx = 2\int_0^{\infty} \frac{1}{4x^2+9}\,dx = \frac{1}{2}\int_0^{\infty} \frac{1}{x^2+(3/2)^2}\,dx$

$$= \lim_{t\to\infty} \frac{1}{2}\int_0^{t} \frac{1}{x^2+(3/2)^2}\,dx = \frac{1}{2}\lim_{t\to\infty}\left(\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)\right)\Bigg|_0^t$$

$$= \frac{1}{2}\left[\lim_{t\to\infty}\frac{2}{3}\tan^{-1}\left(\frac{2t}{3}\right) - \frac{2}{3}\tan^{-1}0\right] = \frac{1}{2}\left[\frac{2}{3}\cdot\frac{\pi}{2} - \frac{2}{3}\cdot 0\right] = \frac{\pi}{6} \quad \text{(converge)}$$

Question 6

(a) $A = \int_{-1}^{0}\left[(y^3-y^2)-(2y)\right]dy + \int_0^2\left[(2y)-(y^3-y^2)\right]dy$

$$= \left(\frac{y^4}{4}-\frac{y^3}{3}-y^2\right)\Bigg|_{-1}^{0} + \left(y^2-\frac{y^4}{4}+\frac{y^3}{3}\right)\Bigg|_0^2 = \frac{5}{12}+\frac{8}{3} = \frac{37}{12}$$

(b) $A = \int_0^1 (x-0)dx + \int_1^2 (\frac{1}{x}-0)dx = \frac{x^2}{2}\Big|_0^1 + log(x)\Big|_1^2 = \frac{1}{2} + \log(2).$

Question 7

(a) $V = \int_{-3}^{3}\pi\left[\left(12-\frac{1}{2}x^2\right)^2 - \left(\frac{1}{2}x^2+3\right)^2\right]dx = \pi\int_{-3}^{3}\left(135-15x^2\right)dx$

$$= \pi\left(135x-5x^3\right)\Big|_{-3}^{3} = 540\pi$$

(b) $V = \int_0^1 \pi\left(\sec^2 x - \tan^2 x\right)dx = \pi\int_0^1 dx = \pi\, x\Big|_0^1 = \pi$

Question 8

(a) $f(x) = x^{3/2}+1 \Rightarrow f'(x) = \frac{3}{2}x^{1/2}$

$$L = \int_0^{3/4}\sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx = \int_0^{3/4}\sqrt{1+\frac{9}{4}x}\,dx \quad (\text{Let } u = 1+\frac{9}{4}x)$$

$$= \int_1^{43/16}\sqrt{u}\,\frac{4}{9}du = \frac{8}{27}u^{3/2}\Big|_1^{43/16} = \frac{8}{27}\left[\left(\frac{43}{16}\right)^{3/2} - 1\right]$$

(b) $f(x) = \ln x - \frac{1}{8}x^2 \Rightarrow f'(x) = \frac{1}{x} - \frac{x}{4}$

$$L = \int_1^2 \sqrt{1 + \left(\frac{1}{x} - \frac{x}{4}\right)^2}\, dx = \int_1^2 \sqrt{\left(\frac{1}{x} + \frac{x}{4}\right)^2}\, dx = \int_1^2 \left(\frac{1}{x} + \frac{x}{4}\right) dx$$

$$= \left(\ln x + \frac{x^2}{8}\right)\Bigg|_1^2 = \ln 2 + \frac{3}{8}$$

Question 9

Let $u = 1 - x \Rightarrow du = -dx$; $x = 0 \Rightarrow u = 1$ and $x = 1 \Rightarrow u = 0$.

$$\int_0^1 f(1-x)\, dx = -\int_1^0 f(u)\, du = \int_0^1 f(u)\, du = \int_0^1 f(x)\, dx$$

2013-2014 Sem 1 Final Exam Q5 (b), (c)

(b)

$$\begin{aligned} L &= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_\alpha^\beta \sqrt{1+\left(\frac{dy/d\theta}{dx/d\theta}\right)^2}\frac{dx}{d\theta}\,d\theta = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta \\ &= \int_\alpha^\beta \sqrt{(r'(\theta))^2+(r(\theta))^2}d\theta. \end{aligned}$$

(c)

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(r'(\theta))^2+(r(\theta))^2}d\theta = \int_0^{2\pi}\sqrt{(1-\cos\theta)^2+\sin^2\theta}d\theta \\ &= \int_0^{2\pi}\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}d\theta = \int_0^{2\pi}\sqrt{2-2\cos\theta}d\theta \\ &= \int_0^{2\pi}\sqrt{4\sin^2(\frac{\theta}{2})}d\theta = \int_0^{2\pi} 2\left|\sin(\frac{\theta}{2})\right| d\theta \\ &= \int_0^{2\pi} 2\sin\frac{\theta}{2}d\theta = -4\cos\frac{\theta}{2}\Big]_0^{2\pi} = 4+4=8. \end{aligned}$$

2013-2014 Sem 1 Final Exam Q10

Let $x=\tan\theta$.

Then, $x=-\infty$ when $\theta=-\frac{\pi}{2}$, and $x=\infty$ when $\theta=\frac{\pi}{2}$.

Moreover, $dx=\sec^2\theta d\theta$.

Using the fact that

$$1+\tan^2\theta = 1+\frac{\sin^2\theta}{\cos^2\theta}=\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}=\sec^2\theta,$$

we have

$$\begin{aligned} \int_{-\infty}^{\infty}\frac{1}{1+x^2}dx &= \int_{-\pi/2}^{\pi/2}\frac{\sec^2\theta}{1+\tan^2\theta}d\theta = \int_{-\pi/2}^{\pi/2}\frac{\sec^2\theta}{\sec^2\theta}d\theta \\ &= \int_{-\pi/2}^{\pi/2} d\theta = [\theta]_{-\pi/2}^{\pi/2} = \frac{\pi}{2}-(-\frac{\pi}{2}) = \pi. \end{aligned}$$

2013/14 Sem2 Final Exam Q2

2. (a) Volume of the Gabriel's Horn is given by

$$V = \lim_{b\to\infty}\int_1^b \pi\left(\frac{1}{x}\right)^2 dx = \lim_{b\to\infty}\pi\left(1-\frac{1}{b}\right) = \pi.$$

(b) Surface area of the Gabriel's Horn:

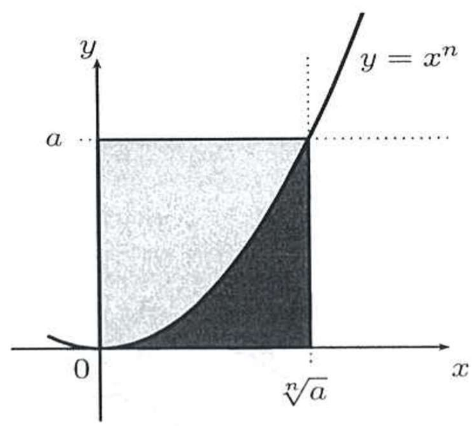
$$A = \lim_{b\to\infty} 2\pi\int_1^b \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx \geq \lim_{b\to\infty} 2\pi\int_1^b \frac{1}{x}dx = \lim_{b\to\infty} 2\pi\ln(b).$$

Thus, Gabrial's Horn has an infinite surface area.

2013/14 Sem2 Final Exam Q7

7.

$$\int_0^{\sqrt[n]{a}} x^n\,dx = \left.\frac{x^{n+1}}{n+1}\right]_0^{\sqrt[n]{a}} = \frac{a^{1+\frac{1}{n}}}{n+1}$$



Let $y = x^n$. Then, $x = \sqrt[n]{y}$. Therefore,

$$\int_{y=0}^{y=a} \sqrt[n]{y}\,dy = a\cdot\sqrt[n]{a} - \int_{x=0}^{x=\sqrt[n]{a}} x^n\,dx$$

$$= a\cdot\sqrt[n]{a} - \frac{a^{1+\frac{1}{n}}}{n+1}$$

Question 12

```
%typeset_mode True
h(x)=5*x^2-3*x+4
h
integral(sqrt(1+derivative(h,x)^2),x,0,3)
plot(h(x),x,0,3)
```

$$x \mapsto 5\,x^2 - 3\,x + 4$$

$$\frac{27}{20}\,\sqrt{730} + \frac{3}{20}\,\sqrt{10} + \frac{1}{20}\,\operatorname{arsinh}\left(27\right) + \frac{1}{20}\,\operatorname{arsinh}\left(3\right)$$