The Hong Kong Polytechnic University
Department of Applied Mathematics

AMA1007 (Calculus and Linear Algebra)

Assignment 02 (Solution)

Question 1

(a) $\ln f(x) = (\cos x)\ln(\cos x)$

$$\Rightarrow \frac{f'(x)}{f(x)} = (-\sin x)\ln(\cos x) - \sin x \Rightarrow f'(x) = -(\cos x)^{\cos x}(\sin x)(\ln(\cos x)+1)$$

(b) $f'(x) = -\dfrac{-3}{\sqrt{1-(-3x)^2}} = \dfrac{3}{\sqrt{1-9x^2}}$

(c) $f'(x) = \dfrac{2^x \ln 2}{1+2^x}$

(d) $f'(x) = \dfrac{1}{(\tan^{-1} x^2)\ln 4} \cdot \dfrac{1}{x^4+1} \cdot 2x = \dfrac{2x}{(x^4+1)(\tan^{-1} x^2)\ln 4}$

Question 2

(a) $y = \left(\dfrac{x+1}{x-1}\right)^r \Rightarrow \ln y = r\ln(x+1) - r\ln(x-1)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{r}{x+1} - \frac{r}{x-1} = \frac{-2r}{x^2-1} \Rightarrow \frac{dy}{dx} = \frac{-2ry}{x^2-1}.$$

(b) Rearrange the expression in (a) gives $(x^2-1)\dfrac{dy}{dx} = -2ry$.

By Leibniz's Formula, $\sum_{r=0}^{n} C_r^n (x^2-1)^{(r)}(y')^{(n-r)} = -2ry^{(n)}$

$$C_0^n(x^2-1)(y')^{(n)} + C_1^n(2x)(y')^{(n-1)} + C_2^n(2)(y')^{(n-2)} = -2ry^{(n)}$$

$$C_0^n(x^2-1)y^{(n+1)} + C_1^n(2x)y^{(n)} + C_2^n 2y^{(n-1)} = -2ry^{(n)}$$

$$(x^2-1)y^{(n+1)}+2nxy^{(n)}+n(n-1)y^{(n-1)}=-2ry^{(n)}$$

$$(x^2-1)y^{(n+1)}+2(nx+r)y^{(n)}+(n^2-n)y^{(n-1)}=0.$$

Question 3

(a) $\lim_{x\to\pi/2}\frac{\cot x}{\cot 3x} \qquad \left(\frac{0}{0}\right)$

$$=\lim_{x\to\pi/2}\frac{-\csc^2 x}{-3\csc^2 3x}=\lim_{x\to\pi/2}\frac{\sin^2 3x}{3\sin^2 x}=\frac{1}{3}$$

(b) $\lim_{x\to\infty}\frac{x+\sqrt{x}}{x+1} \qquad \left(\frac{\infty}{\infty}\right)$

$$=\lim_{x\to\infty}\frac{1+\frac{1}{2\sqrt{x}}}{1}=1$$

(c) $\lim_{x\to 0}\frac{1}{x}\tan\frac{x}{2} \qquad (\infty\cdot 0)$

$$=\lim_{x\to 0}\frac{\tan\frac{x}{2}}{x} \qquad \left(\frac{0}{0}\right) \qquad =\lim_{x\to 0}\frac{\frac{1}{2}\sec^2\frac{x}{2}}{1}=\frac{1}{2}$$

(d) $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) \qquad (\infty-\infty)$

$$=\lim_{x\to 1}\frac{x\ln x-(x-1)}{(x-1)\ln x} \qquad \left(\frac{0}{0}\right)$$

$$=\lim_{x\to 1}\frac{x\frac{1}{x}+\ln x-1}{(x-1)\frac{1}{x}+\ln x}=\lim_{x\to 1}\frac{\ln x}{1-\frac{1}{x}+\ln x} \qquad \left(\frac{0}{0}\right) \qquad =\lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x^2}+\frac{1}{x}}=\frac{1}{2}$$

Question 4

(a)

The domain is $\mathbb{R}$ since $f(x)$ is a polynomial.

$f(-x) = -(-x)^4 + 6(-x)^2 - 4 = -x^4 + 6x^2 - 4 = f(x)$ → Even function

$f'(x) = -4x^3 + 12x = -4x(x+\sqrt{3})(x-\sqrt{3})$ $\qquad$ $f'(x) = 0 \Rightarrow x = 0, \pm\sqrt{3}$

$f''(x) = -12x^2 + 12 = -12(x-1)(x+1)$ $\qquad$ $f''(x) = 0 \Rightarrow x = \pm 1$

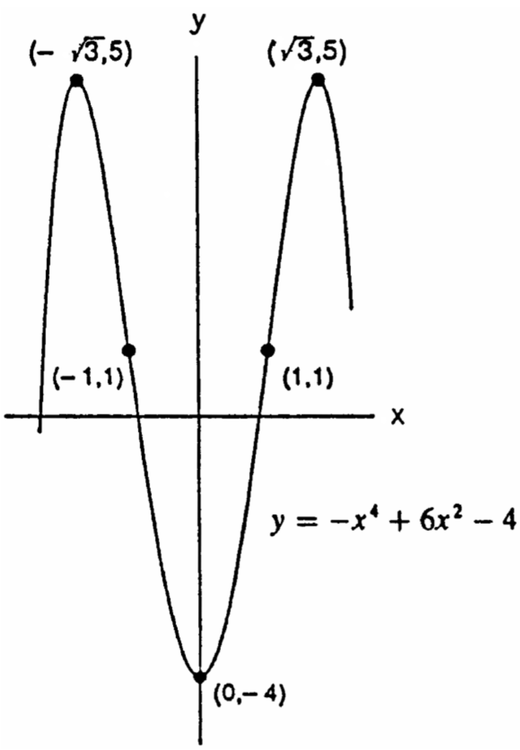
	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},\infty)$
$f'(x)$	+	0	–			0	+			0	–
$f''(x)$	–			0	+			0	–		

There are two local maxima at $(\pm\sqrt{3},5)$, a local minimum at $(0,-4)$, and two inflection points at $(\pm 1,1)$.

The x-intercepts are $(\pm\sqrt{3-\sqrt{5}},0)$ and $(\pm\sqrt{3+\sqrt{5}},0)$.

The y-intercept is $(0,-4)$.

There is no asymptote.



(b)

The domain is $|x| \le \sqrt{8}$.

$f(-x) = (-x)\sqrt{8-(-x)^2} = -x\sqrt{8-x^2} = -f(x)$ → Odd function

$$f'(x) = \frac{2(2-x)(2+x)}{\sqrt{(2\sqrt{2}+x)(2\sqrt{2}-x)}} \qquad f'(x) = 0 \Rightarrow x = \pm 2$$

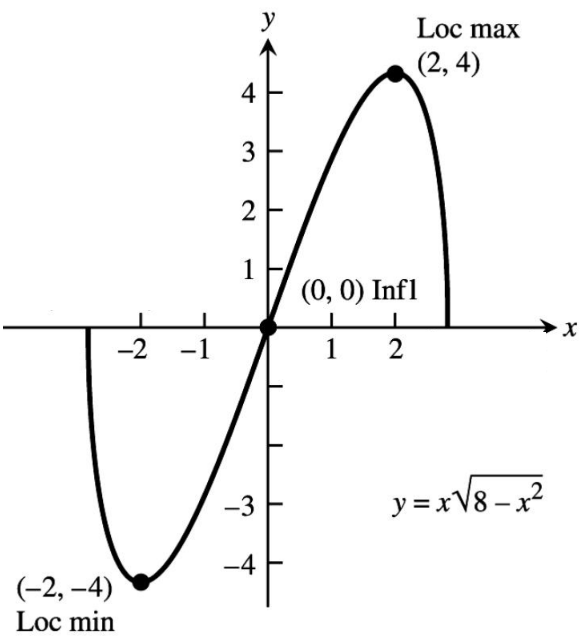
$$f''(x) = \frac{2x(x^2-12)}{\sqrt{(8-x^2)^3}} \qquad f''(x) = 0 \Rightarrow x = 0 \text{ for } |x| \le \sqrt{8}$$

	$(-2\sqrt{2},-2)$	-2	$(-2,0)$	0	$(0,-2)$	2	$(2,2\sqrt{2})$
$f'(x)$	$-$	0		$+$		0	$-$
$f''(x)$		$+$		0		$-$	

There is a local maximum at $(2,4)$, a local minimum at $(-2,-4)$, and a inflection poins at $(0,0)$.

The x-intercepts are $(\pm 2\sqrt{2},0)$. The y-intercept is $(0,0)$.

There is no asymptote.



(c)

The domain is $\mathbb{R} \setminus \pm\sqrt{3}$.

$$f(-x) = \frac{(-x)^2 - 4}{(-x)^2 - 3} = \frac{x^2 - 4}{x^2 - 3} = f(x)$$ → Even function

$$f'(x) = \frac{2x}{(x^2 - 3)^2} \qquad f'(x) = 0 \Rightarrow x = 0$$

$$f''(x) = \frac{-6(x^2 + 1)}{(x^2 - 3)^3} \qquad f''(x) \neq 0 \ \ \forall x$$

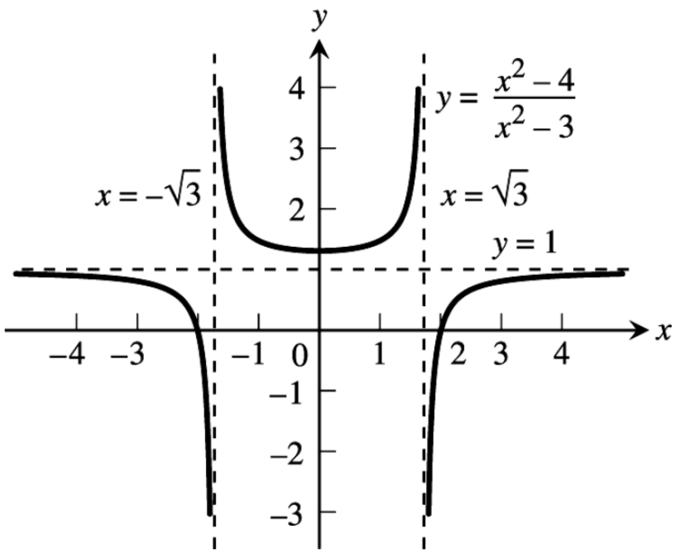
	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, 0)$	0	$(0, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
$f'(x)$	–	N. A.	–	0	+	N. A.	+
$f''(x)$	–	N. A.	+			N. A.	–

There is a local minimum at $\left(0, \frac{4}{3}\right)$

The x-intercept is $(\pm 2, 0)$. The y-intercept is $\left(0, \frac{4}{3}\right)$.

There are two vertical asymptotes: $x = -\sqrt{3}$ and $x = \sqrt{3}$.

There is one horizontal asymptote: $y = 1$



(d)

The domain is $\mathbb{R}$.

$$f'(x)=\frac{5}{3}x^{2/3}-\frac{10}{3}x^{-1/3}=\frac{5}{3}x^{-1/3}(x-2) \qquad f'(x)=0 \Rightarrow x=2$$

$$f''(x)=\frac{10}{9}x^{-1/3}+\frac{10}{9}x^{-4/3}=\frac{10}{9}x^{-4/3}(x+1) \qquad f''(x)=0 \Rightarrow x=-1$$

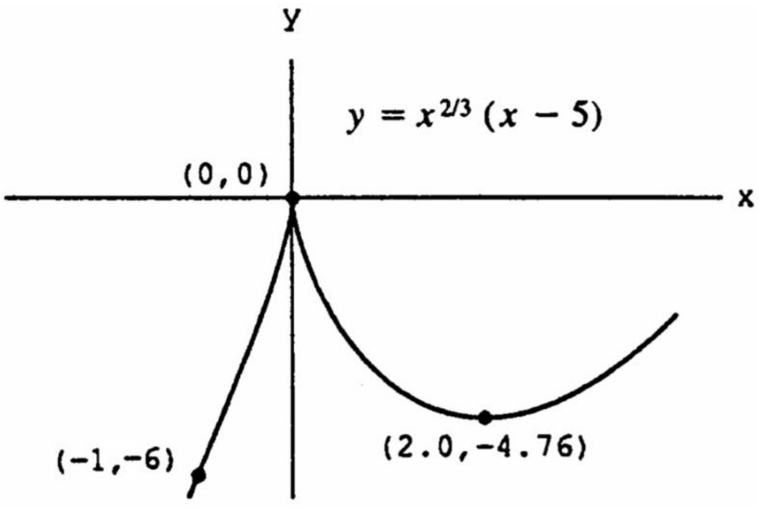
Note that $f'(x)$ and $f''(x)$ are not defined at $x=0$ but $f(0)$ is defined.

	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,2)$	2	$(2,\infty)$
$f'(x)$	+			undefined	–	0	+
$f''(x)$	–	0	+	undefined	+		

There is a local minimum at $(2,-3\cdot 2^{2/3})$, a local maximum at $(0,0)$, and an inflection point at $(-1,-6)$.

The x-intercepts are $(0,0)$ and $(5,0)$. The y-intercept is $(0,0)$.

There is no asymptote.



Question 5

Circumference of the circle: $60-4x$; Radius of the circle: $\dfrac{60-4x}{2\pi}$;

Area of the circle: $\pi\cdot\left(\dfrac{60-4x}{2\pi}\right)^2=\dfrac{(30-2x)^2}{\pi}$.

Objective: Minimize the total area $A(x)=x^2+\dfrac{(30-2x)^2}{\pi}$

$$\frac{dA}{dx}=2x-\frac{4(30-2x)}{\pi} \quad\text{and}\quad \frac{d^2A}{dx^2}=2+\frac{8}{\pi}$$

$$\frac{dA}{dx}=0 \Rightarrow 2\pi x-120+8x=0 \Rightarrow x=\frac{60}{\pi+4}$$

$$\left.\frac{d^2A}{dx^2}\right|_{x=\frac{60}{\pi+4}}=2+\frac{8}{\pi}>0$$ → Global minimum ($\because$ Only one stationary point)

$$\underset{0\le x\le 60}{Min}\ A(x)=\left(\frac{60}{\pi+4}\right)^2+\frac{\left[30-2\left(\frac{60}{\pi+4}\right)\right]^2}{\pi}=\frac{3600+900\pi}{(\pi+4)^2}=\frac{900}{\pi+4}\approx 126.02cm^2$$

Question 6

Let r be the radius of the cylinder where $0<r<10$.

Then the height of the cylinder is $2\sqrt{100-r^2}$.

Objective: Maximize the volume $V(r)=2\pi r^2\sqrt{100-r^2}$

$$\frac{dV}{dr}=\frac{2\pi r(200-3r^2)}{\sqrt{100-r^2}}$$

$$\frac{dV}{dr}=0 \Rightarrow \frac{2\pi r(200-3r^2)}{\sqrt{100-r^2}}=0 \Rightarrow r=0 \text{ (rejected)}, \sqrt{\frac{200}{3}}$$

$$\frac{dV}{dr}>0 \text{ when } r<\sqrt{\frac{200}{3}} \text{ and } \frac{dV}{dr}<0 \text{ when } r>\sqrt{\frac{200}{3}}$$

$$r=\sqrt{\frac{200}{3}}$$ → Global maximum ($\because$ Only one stationary point)

$$\underset{0<x<10}{Max}\ V(r)=2\pi\left(\sqrt{\frac{200}{3}}\right)^2\sqrt{100-\left(\sqrt{\frac{200}{3}}\right)^2}=\frac{4000\pi}{3\sqrt{3}}\approx 2418.40cm^3$$

Question 7

Since $f(x)$ must be continuous at $x=0$ and $x=1$, we must have $\lim_{x\to 0^+} f(x) = f(0)$

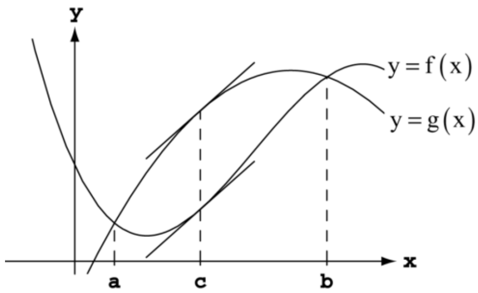
which implies $a=3$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$ which implies $m+b=5$.

Since $f(x)$ must be differentiable at $x=1$, we must have $\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^-} f'(x)$

which implies $m=1$ and hence, $b=4$.

Question 8

Consider the function $h(x) = f(x) - g(x)$.



$h(x)$ is continuous and differentiable on $[a,b]$, and since $h(a) = f(a) - g(a) = 0$ and $h(b) = f(b) - g(b) = 0$, by the Mean Value Theorem, there must be a point $c \in (a,b)$ where $h'(c) = 0$.

Since, $h'(x) = f'(x) - g'(x) \Rightarrow h'(c) = f'(c) - g'(c) = 0$, this implies $f'(c) = g'(c)$ and c is a point where the graphs of $f(x)$ and $g(x)$ have tangent lines with the same slope, so these lines are either parallel or in fact the same line.

Question 9

(a) $\int x^5 \sqrt{x^2-1}\, dx$ Let $u = x^2 - 1 \Rightarrow du = 2xdx$ and $x^2 = u+1$

$$= \frac{1}{2}\int (u+1)^2 \sqrt{u}\, du = \frac{1}{2}\int (u^{5/2} + 2u^{3/2} + u^{1/2})\, du$$

$$= \frac{1}{2}\left(\frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C = \frac{1}{7}u^{7/2} + \frac{2}{5}u^{5/2} + \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{7}(x^2-1)^{7/2} + \frac{2}{5}(x^2-1)^{5/2} + \frac{1}{3}(x^2-1)^{3/2} + C$$

(b) $\int x \cos x^2\, dx$ Let $u = x^2 \Rightarrow du = 2xdx$

$$= \frac{1}{2}\int \cos u\, du = \frac{1}{2}(\sin u) + C = \frac{1}{2}(\sin x^2) + C$$

(c) $\int \frac{1}{\sqrt{4x-x^2}}\,dx$

By completing the square: $4x-x^2 \Rightarrow 4-4+4x-x^2 \Rightarrow 4-(x-2)^2$

$$\int \frac{1}{\sqrt{4x-x^2}}\,dx = \int \frac{1}{\sqrt{4-(x-2)^2}}\,dx \qquad \text{Let } u = x-2 \Rightarrow du = dx$$

$$= \int \frac{1}{\sqrt{4-u^2}}\,du = \sin^{-1}\left(\frac{u}{2}\right) + C = \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

(d) $\int x^2 e^{4x}\,dx$ Let $u = x^2,\ dv = e^{4x}dx \Rightarrow du = 2x\,dx,\ v = \frac{1}{4}e^{4x}$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{2}\int xe^{4x}\,dx \qquad \text{Let } u = x,\ dv = e^{4x}dx \Rightarrow du = dx,\ v = \frac{1}{4}e^{4x}$$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{2}\left(\frac{1}{4}xe^{4x} - \frac{1}{4}\int e^{4x}\,dx\right) = \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C$$

$$= \frac{1}{32}e^{4x}(8x^2-4x+1) + C$$

(e) $\int e^x \ln(1+e^x)\,dx$ Let $u = 1+e^x \Rightarrow du = e^x dx$

$$= \int \ln u\,du \qquad \text{Let } f = \ln u,\ dg = du \Rightarrow df = \frac{1}{u}du,\ g = u$$

$$= u\ln u - \int du = u\ln u - u + C = (1+e^x)\ln(1+e^x) - e^x + C$$

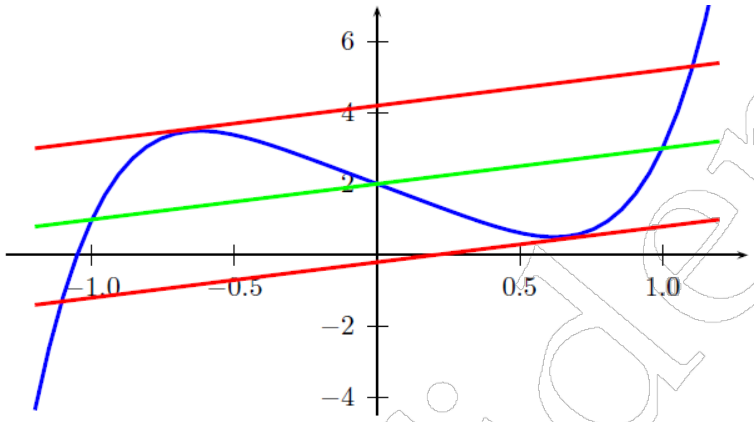
(f) $\int \cos\sqrt{x}\,dx$ Let $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}dx = \frac{1}{2u}dx$

$$= \int 2u\cos u\,du \qquad \text{Let } f = 2u,\ dg = \cos u\,du \Rightarrow df = 2du,\ g = \sin u$$

$$= 2u\sin u - \int 2\sin u\,du = 2u\sin u + 2\cos u + C$$

$$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

2013-2014 Sem 1 Q4



By differenting the polynomial once, we have $f'(x) = 20x^4 - 3$. So solving for $f'(c) = 1$, we have $20c^4 - 3 = 1$, which implies $c = \pm\frac{1}{\sqrt[4]{5}}$.

2013-2014 Sem 1 Q5(a)

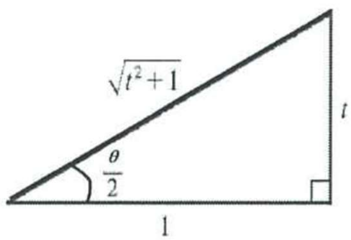
By differentiating these using Product Rule, we have $\frac{dx}{d\theta} = r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)$ and $\frac{dy}{d\theta} = r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)$. Therefore,

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= (r'(\theta))^2\cos^2(\theta) - 2r(\theta)r'(\theta)\cos(\theta)\sin(\theta) + (r(\theta))^2\sin^2(\theta) \\ &\quad + (r'(\theta))^2\sin^2(\theta) + 2r(\theta)r'(\theta)\sin(\theta)\cos(\theta) + (r(\theta))^2\cos^2(\theta)\end{aligned}$$

Using the trigonometric identity $\sin^2(\theta) + \cos^2(\theta) = 1$, we can simplified the expression into $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (r'(\theta))^2 + (r(\theta))^2$.

2014/2015 Sem 2 Final Exam Q4

4. (a) $t = \tan(\theta/2)$, the figure gives



Therefore $\cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{1+t^2}}$, and $\sin\left(\frac{\theta}{2}\right) = \frac{t}{\sqrt{1+t^2}}$.

(b)

$$\begin{aligned}
\cos(\theta) &= \cos\left(2\cdot\frac{\theta}{2}\right) = 2\cos^2\left(\frac{\theta}{2}\right) - 1 = 2\left(\frac{1}{\sqrt{1+t^2}}\right)^2 - 1 \\
&= \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}. \\
\sin(\theta) &= \sin\left(2\cdot\frac{\theta}{2}\right) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\
&= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right) = \frac{2t}{1+t^2}.
\end{aligned}$$

(c) $\frac{\theta}{2} = \tan^{-1}(t)$, therefore $d\theta = \frac{2}{1+t^2}\,dt$.

(d)

$$\begin{aligned}
\int \frac{1}{3\sin(\theta) - 4\cos(\theta)}\,d\theta &= \int \frac{\frac{2}{1+t^2}}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right)}\,dt \\
&= \int \frac{1}{2t^2+3t-2}dt = \int \frac{1}{(2t-1)(t+2)}dt \\
&= \int \left[\frac{\frac{2}{5}}{2t-1} - \frac{\frac{1}{5}}{t+2}\right] dt \\
&= \frac{1}{5}ln|2t-1| - \frac{1}{5}ln|t+2| + C = \frac{1}{5}ln\left|\frac{2t-1}{t+2}\right| + C \\
&= \frac{1}{5}ln\left|\frac{2\tan(\theta/2)-1}{\tan(\theta/2)+2}\right| + C.
\end{aligned}$$

2014/2015 Sem 2 Final Exam Q5

5. (a) Since $y = \dfrac{1-x}{1+x^2}$, thus $\dfrac{dy}{dx} = \dfrac{x^2-2x-1}{(1+x^2)^2}$.

(b) Given that $\dfrac{d^2y}{dx^2} = \dfrac{-2x^3+6x^2+6x-2}{(1+x^2)^3}$, so, $\dfrac{d^2y}{dx^2} = 0$ gives $2x^3-6x^2-6x+2=0$, or $2(x+1)(x^2-4x+1)=0$.

Therefore, the candidate locations for the points of inflection are at $x=-1$, and $x = 2\pm\sqrt{3}$. Since $\dfrac{d^3y}{dx^3} = \dfrac{(1+x^2)(-6x^2+12x^6)-(-2x^3+6x^2+6x-2)(6x)}{(1+x^2)^4}$ are nonzeros for all three locations, and thus, all three candidates are points of inflection.
When $x=-1$, $y=1$.
When $x = 2+\sqrt{3}$, $y = \dfrac{-\sqrt{3}-1}{4(2+\sqrt{3})}$.
When $x = 2-\sqrt{3}$, $y = \dfrac{\sqrt{3}-1}{4(2-\sqrt{3})}$.

(c) Using two-point-form on the last two points of inflexion, we have

$$\frac{y-\left(\frac{\sqrt{3}-1}{4(2-\sqrt{3})}\right)}{x-(2-\sqrt{3})} = \frac{\left(\frac{-\sqrt{3}-1}{4(2+\sqrt{3})}\right)-\left(\frac{\sqrt{3}-1}{4(2-\sqrt{3})}\right)}{(2+\sqrt{3})-(2-\sqrt{3})} = \frac{\frac{1}{4}\left(\left(\frac{-\sqrt{3}-1}{(2+\sqrt{3})}\right)-\left(\frac{\sqrt{3}-1}{(2-\sqrt{3})}\right)\right)}{2\sqrt{3}}$$
$$= \frac{1}{8\sqrt{3}}\left(\frac{(-\sqrt{3}-1)(2-\sqrt{3})-(\sqrt{3}-1)(2+\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}\right)$$
$$= \frac{1}{8\sqrt{3}}\left(\frac{(-\sqrt{3}+1)-(\sqrt{3}+1)}{4-3}\right) = \frac{-2\sqrt{3}}{8\sqrt{3}} = \frac{-1}{4}.$$

By re-arranging, we have

$$4y - \left(\frac{\sqrt{3}-1}{2-\sqrt{3}}\right) = -x+2-\sqrt{3},$$

or

$$x+4y = 2-\sqrt{3}+\left(\frac{\sqrt{3}-1}{2-\sqrt{3}}\right) = \frac{(4-4\sqrt{3}+3)+(\sqrt{3}-1)}{2-\sqrt{3}}$$
$$= \frac{(6-3\sqrt{3})}{2-\sqrt{3}} = \frac{3(2-\sqrt{3})}{2-\sqrt{3}} = 3.$$

Check that the first given inflection point $(-1,1)$ also satisfies the equation $(-1)+4(1)=3$.
Therefore, all the three points lie on the same straight line $x+4y=3$.

Question 12

```
%typeset_mode True
g(x)=(x+1)/(x^2+9)
g
integral(g(x),x)
plot(g(x),x,-15,15)
```

$$x \mapsto \frac{x+1}{x^2+9}$$

$$\frac{1}{3}\,\arctan\left(\frac{1}{3}\,x\right) + \frac{1}{2}\,\log\left(x^2+9\right)$$