The Hong Kong Polytechnic University
Department of Applied Mathematics

AMA1007 (Calculus and Linear Algebra)

Assignment 01 (Solution)

Question 1

(a) $\dfrac{2x^3+x^2+12}{x^2-4}=2x+1+\dfrac{8x+16}{x^2-4}=2x+1+\dfrac{8(x+2)}{(x-2)(x+2)}=2x+1+\dfrac{8}{x-2}$

(b) $\dfrac{x^3}{(x+1)^3}=1+\dfrac{-3x^2-3x-1}{(x+1)^3}=1-\dfrac{3}{x+1}+\dfrac{3}{(x+1)^2}-\dfrac{1}{(x+1)^3}$

(c) $\dfrac{x^2-1}{x^3+3x+4}=\dfrac{(x-1)(x+1)}{(x+1)(x^2-x+4)}=\dfrac{x-1}{x^2-x+4}$

Question 2

For all $x\neq 0$,

$|f(x)|\le M \quad \Rightarrow |x\cdot f(x)|=|x||f(x)|\le |x|M$

$\Rightarrow -|x|M\le x\cdot f(x)\le |x|M$

Since $\lim\limits_{x\to 0}\left(-|x|M\right)=\lim\limits_{x\to 0}\left(|x|M\right)=0$, $\lim\limits_{x\to 0}\left(x\cdot f(x)\right)=0$ (by the Squeeze Theorem).

Question 3

(a) $\lim_{x\to -2}\frac{x^3+2x^2-x+2}{x^2-4}=\lim_{x\to -2}\left(x+2+\frac{3x+10}{x^2-4}\right)$

$\lim_{x\to -2^-}\left(x+2+\frac{3x+10}{x^2-4}\right)=\infty$; $\lim_{x\to -2^+}\left(x+2+\frac{3x+10}{x^2-4}\right)=-\infty$

→ $\lim_{x\to -2}\frac{x^3+2x^2-x+2}{x^2-4}$ does not exist.

(b) $\lim_{x\to 9}\frac{x\left(\sqrt{x}-3\right)}{x-9}=\lim_{x\to 9}\frac{x\left(\sqrt{x}-3\right)}{\left(\sqrt{x}-3\right)\left(\sqrt{x}+3\right)}=\lim_{x\to 9}\frac{x}{\left(\sqrt{x}+3\right)}=\frac{3}{2}$

(c) $\lim_{x\to 0}\frac{1-\cos(ax)}{x^2}=\lim_{x\to 0}\frac{2\sin^2\left(\frac{ax}{2}\right)}{x^2}=\lim_{x\to 0}\frac{2\sin^2\left(\frac{ax}{2}\right)}{\left(\frac{a}{2}\right)^2 x^2}\left(\frac{a}{2}\right)^2=\lim_{x\to 0}2\left(\frac{a}{2}\right)^2=\frac{a^2}{2}$

(d) $\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-cx}\right)$

$=\lim_{x\to\infty}\frac{x(1+c)}{\sqrt{x^2+x}+\sqrt{x^2-cx}}=\lim_{x\to\infty}\frac{x(1+c)}{\sqrt{x^2+x}+\sqrt{x^2-cx}}\cdot\frac{(1/x)}{(1/x)}$

$=\lim_{x\to\infty}\frac{1+c}{\sqrt{1+(1/x)}+\sqrt{1-(c/x)}}=\frac{1+c}{2}$

(e) $\lim_{x\to -\infty}\frac{x^{1/3}-5x+3}{2x+x^{2/3}-4}=\lim_{x\to -\infty}\frac{x^{1/3}-5x+3}{2x+x^{2/3}-4}\cdot\frac{(1/x)}{(1/x)}=\lim_{x\to -\infty}\frac{x^{-2/3}-5+3x^{-1}}{2+x^{-1/3}-4x^{-1}}=-\frac{5}{2}$

(f) $\lim_{x\to\infty}\left(3+\frac{2}{x}\right)\left(\cos\frac{1}{x}\right)=\lim_{y\to 0}\left(3+2y\right)\left(\cos y\right)=3$

Question 4

$|\,x\,|=\begin{cases} x & \text{for } x\geq 0 \\ -x & \phantom{\text{for }} x<0\end{cases}$

$\lim_{x\to 0^+}\left[\left(\frac{2x+x}{x}\right)\left(x^2-2+k\right)\right]=3(k-2)$

$\lim_{x\to 0^-}\left[\left(\frac{2x-x}{x}\right)\left(x^2-2+k\right)\right]=k-2$

Limit exists → $3(k-2)=k-2\ \Rightarrow\ k=2$

Question 5

(a) Given $\varepsilon > 0$, set $\delta = \frac{\varepsilon}{2}$. For $0 < |x-5| < \delta$,

$$|(3-2x)-(-7)| = |10-2x| = |2x-10| = 2|x-5| < 2\delta = 2\frac{\varepsilon}{2} = \varepsilon$$

(b) Given $\varepsilon > 0$, set $\delta = \varepsilon$.

For $0 < |x-0| < \delta$, together with the fact that $\left|\cos\left(\frac{1}{x}\right)\right| \le 1$,

$$\left|x\cos\left(\frac{1}{x}\right)\right| = |x|\left|\cos\left(\frac{1}{x}\right)\right| \le |x| < \delta = \varepsilon$$

Question 6

$$\lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = f'(0) = 1$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x)+f(h)+2xh-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(h)+2xh}{h} = \lim_{h\to 0}\frac{f(h)}{h} + 2x = 1+2x$$

Question 7

(a) $\lim_{\Delta x\to 0}\frac{\sqrt{1-2(x+\Delta x)}-\sqrt{1-2x}}{\Delta x}$

$$= \lim_{\Delta x\to 0}\frac{\sqrt{1-2(x+\Delta x)}-\sqrt{1-2x}}{\Delta x}\cdot\frac{\sqrt{1-2(x+\Delta x)}+\sqrt{1-2x}}{\sqrt{1-2(x+\Delta x)}+\sqrt{1-2x}}$$

$$= \lim_{\Delta x\to 0}\frac{-2\Delta x}{\Delta x\left(\sqrt{1-2(x+\Delta x)}+\sqrt{1-2x}\right)}$$

$$= \lim_{\Delta x\to 0}\frac{-2}{\sqrt{1-2(x+\Delta x)}+\sqrt{1-2x}} = \lim_{\Delta x\to 0}\frac{-2}{2\sqrt{1-2x}} = -\frac{1}{\sqrt{1-2x}}$$

(b) $\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{x+\Delta x}{(x+\Delta x)-1}-\frac{x}{x-1}\right)$

$$= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{(x+\Delta x)(x-1)-x(x+\Delta x-1)}{(x-1)(x+\Delta x-1)}\right) = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{-\Delta x}{(x-1)(x+\Delta x-1)}\right)$$

$$= \lim_{\Delta x\to 0}\frac{-1}{(x-1)(x+\Delta x-1)} = \frac{-1}{(x-1)^2}$$

(c) $\lim_{\Delta x \to 0} \frac{\sin(2(x+\Delta x)) - \sin 2x}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\sin(2x+2\Delta x) - \sin 2x}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[2\sin\left(\frac{2\Delta x}{2}\right)\cos\left(\frac{4x+2\Delta x}{2}\right)\right]$$

$$= \lim_{\Delta x \to 0} 2\left[\frac{1}{\Delta x}\sin(\Delta x)\right]\cos\left(\frac{4x+2\Delta x}{2}\right) = \lim_{\Delta x \to 0}\left[2\cos\left(\frac{4x+2\Delta x}{2}\right)\right] = 2\cos 2x$$

Question 8

(a) $\frac{dy}{dx} = 2x\left[\sec^2\left(\sin x^2\right)\right]\left(\cos x^2\right)$

(b) $\frac{dy}{dx} = 2e^{2x}\ln\left(3+e^x\right) + \frac{e^{2x}}{3+e^x}\cdot e^x = 2e^{2x}\ln\left(3+e^x\right) + \frac{e^{3x}}{3+e^x}$

(c) $\frac{dy}{dx} = (e^x + xe^x) - (-5e^{-x}) = e^x + xe^x + 5e^{-x}$

(d) $\frac{dy}{dx} = -(2x+1)\left(e^{\cos\left(x^2+x\right)}\right)\sin\left(x^2+x\right)$

(e) $9y^2\frac{dy}{dx} - \left(8xy + 4x^2\frac{dy}{dx}\right) + \left(y + x\frac{dy}{dx}\right) = 0 \Rightarrow \frac{dy}{dx} = \frac{8xy - y}{9y^2 - 4x^2 + x}$

(f) $3y^2\frac{dy}{dx} + \cos(xy^2)\left(y^2 + 2xy\frac{dy}{dx}\right) = 0 \Rightarrow \frac{dy}{dx} = \frac{-y^2\cos(xy^2)}{3y^2 + 2xy\cos(xy^2)}$

Question 9

$$h'(x) = \left(f'(x)g(x) + f(x)g'(x)\right) + \frac{1}{\sqrt{x}} - \frac{1}{4x^{3/2}} + \frac{xe^x - e^x}{x^2}$$

$$h'(1) = \left(f'(1)g(1) + f(1)g'(1)\right) + \frac{1}{\sqrt{1}} - \frac{1}{4(1)^{3/2}} + \frac{(1)e^1 - e^1}{(1)^2}$$

$$= (-1)(1) + (-3)(3) + 1 - \frac{1}{4} + 0 = -\frac{37}{4}$$

Question 10

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%typeset_mode True
f(x)=(cos(x)+2^x)^x
f
derivative(f(x),x)
plot(f(x),x,-1.5,1.5)
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$$x \mapsto \left(2^{x} + \cos\left(x\right)\right)^{x}$$

$$\left(2^{x} + \cos\left(x\right)\right)^{x} \left(\frac{\left(2^{x} \log\left(2\right) - \sin\left(x\right)\right) x}{2^{x} + \cos\left(x\right)} + \log\left(2^{x} + \cos\left(x\right)\right)\right)$$