Three Practice Questions (no need to submit)

Q1. Consider the rational function $f(x) = \dfrac{x-3}{x^2-3x-3}$, and the power series of this rational function centered at $x = 0$, that is, $\sum_{n=0}^{\infty} a_n x^n$.

Consider the technique presented in supplementary notes:
`https://www.polyu.edu.hk/ama/profile/hwlee/AMA1007/supplementary10.pdf`

(a) Find a_0 and a_1. **Ans:** $a_0 = 1$, $a_1 = \dfrac{-4}{3}$.

(b) For $n \geq 2$, obtain the linear equation of a_n in terms of a_{n-1} and a_{n-2}. Moreover, use this equation and the results obtained in (**a**) to find a_2, a_3, a_4, a_5. Express your answers as rational numbers only.

Ans: $a_n = \dfrac{a_{n-2} - 3a_{n-1}}{3}$.

$a_2 = \dfrac{5}{3}$, $a_3 = \dfrac{-19}{9}$, $a_4 = \dfrac{8}{3}$, $a_5 = \dfrac{-91}{27}$.

(c) Let $L = \lim_{n\to\infty} \dfrac{a_n}{a_{n-1}}$. Determine the value of L.

Ans: L is negative, and $L = -\dfrac{\sqrt{21}}{6} - \dfrac{1}{2}$.

(d) Find the radius of convergence of the power series.

Ans: $\dfrac{6}{\sqrt{3}\sqrt{7}+3}$.

Q2. Consider the rational function $y = \dfrac{16x}{4x^2+1}$. The function has three inflection points.

(a) Find all inflection points of the given rational function, list them one by one by their xy-coordinates.

Ans: $(x_0, y_0) = (\dfrac{-\sqrt{3}}{2}, -2\sqrt{3})$

$(x_1, y_1) = (\dfrac{\sqrt{3}}{2}, 2\sqrt{3})$

$(x_2, y_2) = (0, 0)$

(b) Use the technique presented in supplementary notes (Determinant):
`https://www.polyu.edu.hk/ama/profile/hwlee/AMA1007/supplementary12.pdf`
determine if the three inflection points are collinear or not.

Ans: $$\begin{vmatrix} 1 & 1 & 1 \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2}\sqrt{3} & 0 \\ -2\sqrt{3} & 2\sqrt{3} & 0 \end{vmatrix} = 0.$$

Thus, the three inflection points are collinear.

Q3. Consider $\boldsymbol{A} = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{3} \end{bmatrix}$, and suppose given that $\boldsymbol{A}^8 = \begin{bmatrix} \frac{1}{256} & \frac{6305}{279936} \\ 0 & \frac{1}{6561} \end{bmatrix}$.

Consider the technique presented in supplementary notes:
`https://www.polyu.edu.hk/ama/profile/hwlee/AMA1007/supplementary11.pdf`

(a) Find $\boldsymbol{I} - \boldsymbol{A}^8$ and $(\boldsymbol{I} - \boldsymbol{A})^{-1}$.

Ans: $\boldsymbol{I} - \boldsymbol{A}^8 = \begin{pmatrix} \frac{255}{256} & -\frac{6305}{279936} \\ 0 & \frac{6560}{6561} \end{pmatrix}$,

$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \begin{pmatrix} 2 & 3 \\ 0 & \frac{3}{2} \end{pmatrix}$.

(b) Compute $\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \cdots + \boldsymbol{A}^7$.

Ans: $\begin{pmatrix} \frac{255}{128} & \frac{137845}{46656} \\ 0 & \frac{3280}{2187} \end{pmatrix}$.

(c) Find all eigenvalues of $\boldsymbol{A}$.

Ans: $\lambda_1 = \frac{1}{2}$, and $\lambda_2 = \frac{1}{3}$.

(d) Evaluate $\displaystyle\sum_{n=0}^{\infty} \boldsymbol{A}^n$.

Ans: $\begin{pmatrix} 2 & 3 \\ 0 & \frac{3}{2} \end{pmatrix}$.