Another Four Practice Questions (no need to submit)

Q1. Consider the graph given in polar coordinates as follows

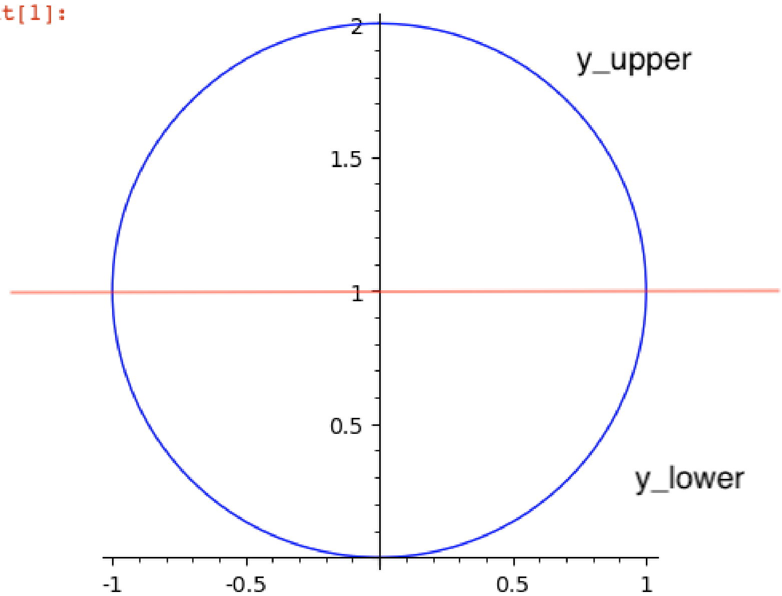
$$r = 2\sin(\theta)$$

where $0 \leq \theta \leq \pi$. (You may pretend you do not recognize it as the unit circle centered at $(0, 1)$ in this question.)

We can also express it in Cartesian coordinates with the parametric equations

$$x(\theta) = r \cdot \cos(\theta) = 2\sin(\theta)\cos(\theta) = \sin(2\theta)$$
$$y(\theta) = r \cdot \sin(\theta) = 2\sin(\theta)\sin(\theta) = 2\sin^2(\theta)$$

In [1]:
```
polar_plot(2*sin(x), (x, -pi/2, pi/2))
```
Out[1]:



(a) By solving $\frac{d}{d\theta}x(\theta) = 0$, we can find the maximum and minimum of x with respect to θ (the righ-tmost and the left-most x-coordinate of the graph). Show that, the right-most at $x = 1$ occurred when $\theta = \frac{\pi}{4}$, and the left-most $x = -1$ occurred when $\theta = \frac{3\pi}{4}$.

(b) Define $f(\theta) = (y(\theta))^2 \cdot \frac{d}{d\theta}x(\theta)$. Show that

$$f(\theta) = 8\pi\sin^4(\theta)\cos(2\theta).$$

(c) Consider the volume of the solid generated by rotating this graph about the x-axis. The solid is of a torus shape. Explain why the volume of the solid can be obtained by

$$\begin{aligned}
V &= \int_{-1}^{1} \pi (y_{upper})^2\,dx - \int_{-1}^{0} \pi (y_{lower})^2\,dx - \int_{0}^{1} \pi (y_{lower})^2\,dx \\
&= \int_{3\pi/4}^{\pi/4} \pi (2\sin^2(\theta))^2\,d(\sin(2\theta)) - \int_{3\pi/4}^{\pi} \pi (2\sin^2(\theta))^2\,d(\sin(2\theta)) \\
&\quad - \int_{0}^{\pi/4} \pi (2\sin^2(\theta))^2\,d(\sin(2\theta)) \\
&= \int_{3\pi/4}^{\pi/4} \pi f(\theta) d\theta - \int_{3\pi/4}^{\pi} \pi f(\theta) d\theta - \int_{0}^{\pi/4} \pi f(\theta) d\theta \\
&= \int_{3\pi/4}^{\pi/4} \pi f(\theta) d\theta + \int_{\pi}^{3\pi/4} \pi f(\theta) d\theta + \int_{\pi/4}^{0} \pi f(\theta) d\theta \\
&= \int_{\pi}^{0} \pi f(\theta) d\theta.
\end{aligned}$$

(d) Note that the expression of $f(\theta)$ given in **(b)** can be reduced for the sake of integration. Consider the CoCalc reduction:

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In [4]: show((8*pi*(sin(x))^4*cos(2*x)).reduce_trig())
```

Out[4]:

$$-2\pi + \frac{1}{2}\pi\cos(6x) - 2\pi\cos(4x) + \frac{7}{2}\pi\cos(2x)$$

Using this, or otherwise, show that the volume of the solid obtained in **(c)** is given by

$$V = 2\pi^2.$$

Keep your workings and answers in terms of π and simplified rational numbers only.

Q2. Consider the integral $\displaystyle\int x\cos^{-1}(x)\,dx$ for $-1 \le x \le 1$.

(a) Use integration by parts to evaluate the indefinite integral.

$$\textbf{Ans}: \frac{x^2}{2}\cos^{-1}(x) - \frac{x}{4}\sqrt{1-x^2} + \frac{1}{4}\sin^{-1}(x) + K.$$

(b) Use **(a)**, or otherwise, to evaluate $\displaystyle\int_0^1 x\cos^{-1}(x)\,dx$.

$$\textbf{Ans}: \frac{\pi}{8}.$$

Q3. Consider the rational function $f(x) = \dfrac{10x - 610}{x^2 - 108x + 1691}$. Note that the denominator can be factorized.
In your answers to this question, keep those values in exact and simplified rational number format only.

(a) Express $f(x)$ in partial fractions, i.e., $\dfrac{a}{x-\alpha} + \dfrac{b}{x-\beta}$.

Ans : $\dfrac{6}{x-19} + \dfrac{4}{x-89}$.

(b) Apart from using Taylor/Maclaurin series expansion, power series of $g(x) = \dfrac{d}{x-\gamma}$ at $x = 0$ can be obtained by letting $\dfrac{d}{x-\gamma} = \sum_{n=0}^{\infty} c_n\, x^n$ first. Then, multiply both sides by $x - \gamma$ to get

$$d \quad = \quad (x-\gamma)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 \dots) \; = \; -\gamma c_0 + \sum_{n=1}^{\infty} (c_{n-1} - \gamma c_n) x^n.$$

By comparing coefficients, we can get $c_0 = -\frac{d}{\gamma}$, and $c_n = \frac{c_{n-1}}{\gamma}$ for $n = 1, 2, ...$ The radius of convergence is given by $\lim_{n\to\infty} \dfrac{c_n}{c_{n+1}} = |\gamma|$.
Find the coefficients of the first three terms (i.e., x^0, x^1, and x^2) of the power series of each of the two partial fractions obtained in **(a)** at $x = 0$, and state the corresponding radius of convergence of each one.

Ans :
$$\frac{6}{x-19} = -\frac{6}{19} - \frac{6}{361}x - \frac{6}{6859}x^2 + \dots$$
$$\frac{4}{x-89} = -\frac{4}{89} - \frac{4}{7921}x - \frac{4}{704969}x^2 + \dots$$

(c) Use the above, or otherwise, find the coefficients of the first three terms of the power series of $f(x)$ at $x = 0$, and state the radius of convergence.

Ans : $f(x) \; = \; -\frac{610}{1691} - \frac{48970}{2859481}x - \frac{4257250}{4835382371}x^2 + \dots$
Radius of Convergence is the overlapping $= \min(|19|, |89|) = 19$.

Q4. Consider the two non-intersecting parabolas:

$$y = f(x) \quad = \quad x^2 - 10x + 25$$
$$y = g(x) \quad = \quad -x^2 - 6x - 10$$

Let u be the x-coordinate of a point on f, and v be the x-coordinate of a point on g.

(a) For any given x-coordinate u on parabola f, there exist a unique v on parabola g, so that the tangent line on f at u, and the tangent line on g at v, are parallel. (Similarly, we have the inverse, that for any given v, there is a unique u, so that the two tangent lines are parallel). Express v as a function of u under this condition.

Ans : $v = -u + 2$.

(b) Let D be the square of the distance between two points, one on each of the two parabolas, $(u, f(u))$ and $(v, g(v))$, and the tangent line at u on f and the tangent line at v on g are parallel. Use the result obtained in **(a)** to express D as a function of u only, i.e. $D(u)$.

Ans : $D(u) = 4u^4 - 80u^3 + 608u^2 - 2048u + 2605.$

(c) Consider the following CoCalc work:

```
In [1]: f(x)=x^2-10*x+25
        g(x)=-x^2-6*x-10
        fdash(x)=diff(f(x),x)
        gdash(x)=diff(g(x),x)
        var('u v')
        parallel=solve(fdash(u)==gdash(v),v)
        v=parallel[0].rhs()
        D(u)=(v-u)^2+(g(v)-f(u))^2
        show(factor(diff(D(u),u)))
Out[1]:
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$$16\left(u^2 - 11\,u + 32\right)(u - 4)$$

Use this result, or otherwise, find the minimum of $D(u)$. Include a second derivative test to confirm it is indeed a minimum.

Ans : $\begin{array}{ll} \textit{minimum} & u = 4. \\ D''(4) = 64 & > 0. \end{array}$

(d) State the minimum location u^* obtained in **(c)**, and the associated v^* according to the parallel condition obtained in **(a)**. Hence, list the points $(u^*, f(u^*))$ and $(v^*, g(v^*))$.

Ans : $(4, -1)$ and $(-2, -2)$.

(e) Briefly explain why the two points listed in **(d)** are the closest points between the two parabolas. Sketch a diagram if necessary.

Ans : Tangents of the closest points on each parabola are parallel.