THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code:	AMA1007	**Subject Title:**	Calculus and Linear Algebra
Session:	Semester 2, 2020/2021	**Assessment:**	Final Exam
Submission:	Blackboard	**Date and Time:**	15 May 2021, 12:30-14:30

This set of question has 5 pages (including this cover page).

Instructions: Attempt **ALL** questions in this paper. Your solutions to each question must be made in the designated area (inside the boxes) of the set of **Answer Sheets**. Scan your work into **one single PDF file** with file size not exceeding 10MB, and submit it via Blackboard with file name as your name (surname first). Use the app Microsoft Office Lens to scan multiple pages into one single pdf file. Pages must be in the right order. Only one single submission will be accepted (it is the students' responsibility to check carefully before submitting). Late submissions will not be accepted. Students are required to sign the covering statement of the set of Answer Sheets for submission.

No marks will be given for those not following any of the above instructions.

Subject Examiners: Dr. LEE Heung Wing Joseph

1. Consider $f(x) = \dfrac{1}{3}(x-3)\sqrt{x}$ for $x \geq 0$.

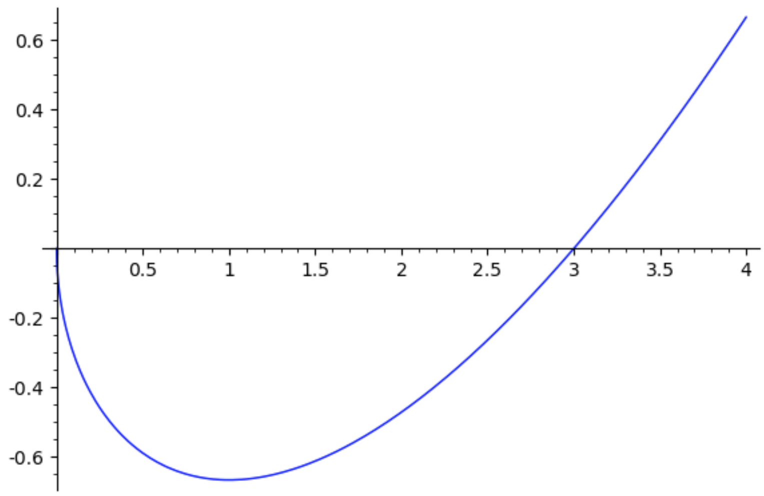
   Suppose a student was using CoCalc to find $\sqrt{1+(f'(x))^2}$ and obtaind the results:

In [1]:
```
f(x)=1/3*sqrt(x)*(x-3)
show(f)
```

$$x \mapsto \frac{1}{3}\,(x-3)\sqrt{x}$$

In [2]:
```
plot(f(x),x,0,4)
```

Out[2]:



In [3]:
```
fdash(x)=diff(f(x),x)
show(fdash)
```

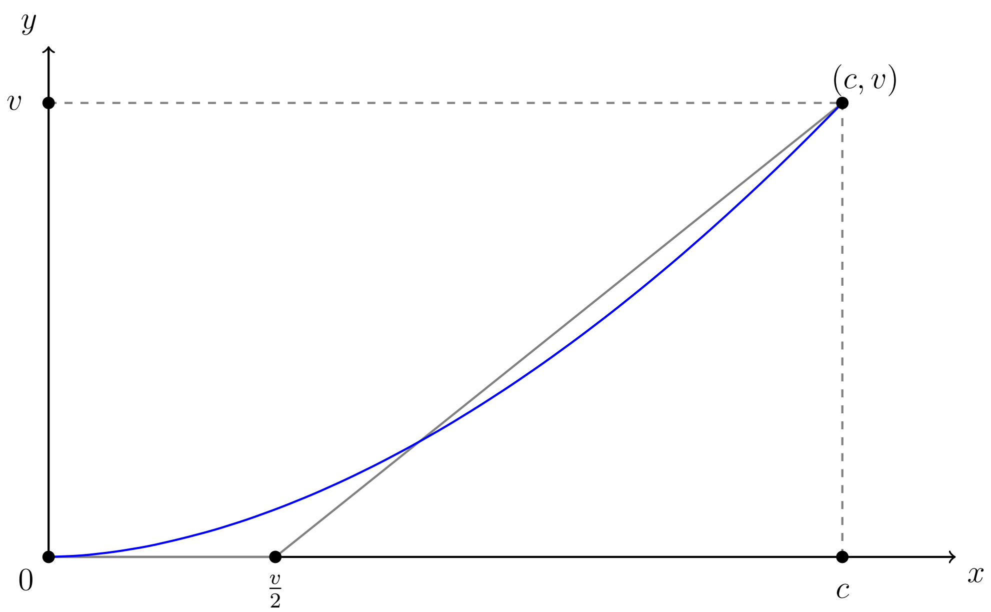
$$x \mapsto \frac{x-3}{6\sqrt{x}} + \frac{1}{3}\sqrt{x}$$

In [4]:
```
af(x)=sqrt((1+(fdash(x))^2).factor())
show(af)
```

$$x \mapsto \frac{1}{2}\sqrt{\frac{(x+1)^2}{x}}$$

   Use these CoCalc outputs, or otherwise, find the arc-length of $f(x)$ from $x = 0$ to $x = 3$. Keep your answer in surd form only. **[20 points]**

2. Problem 35 of the first chapter of the ancient Chinese text *The Nine Chapters of Mathematical Art* (九章算術) was equating the area of a flat-based-crescent shape (or bow shape) field (弧田) by the area of a trapezium (with the same base, same height, and the top side the same length as the height). It was not very specific about the crescent shape (bow shape), but many believed that it was referring to a circular segment. If the text was indeed referring to a circular segment, then the answer provided therein would not be correct (it implies that the text was incorrectly taking the value of π as 3). What if the text was not referring to a circular segment? Can we consider some other shapes other than the circular segment but matching the trapezium area exactly? Consider half of the field as shown in the diagram.



Suppose $0 < v < c$. The four vertices of the trapezium are given by $(0,0)$, $(\frac{v}{2}, 0)$, (c, v) and $(0, v)$. Now, consider the graph of

$$y = f(x) \quad = \quad v\left(\frac{x}{c}\right)^{\alpha} \qquad for \quad x \in [0, c], \quad , \; and \quad \alpha > 1.$$

Note that the graph $y = f(x)$ starts from $(0,0)$ and ends at (c, v).

(a) Show briefly that the area of the trapezium is given by $\dfrac{2cv + v^2}{4}$. **[2 points]**

(b) By integrating $x = f^{-1}(y)$ with respect to y, find the area of the region bounded by $y = v$ and the graph of f where $x \in [0, c]$. Express your answer in terms of v, c and α only. **[13 points]**

(c) Suppose the two areas obtained in **(a)** and **(b)** are the same, find α in terms of v and c. **[5 points]**

3. Consider the rational function $y = \dfrac{x+1}{x^2+1}$. The function has three inflection points. A student was trying to use CoCalc to get $y''(x)$, and obtained the output:

In [1]:
```
numerator(x)=x+1
denominator(x)=x^2+1
f(x)=numerator(x)/denominator(x)
show(f)
```

$$x \mapsto \frac{x+1}{x^2+1}$$

In [2]:
```
fddash(x)=diff(f(x),x,2).factor()
show(fddash)
```

$$x \mapsto \frac{2\left(x^2+4x+1\right)(x-1)}{\left(x^2+1\right)^3}$$

(a) From the output, or otherwise, find all inflection points of the given rational function, list them one by one by their xy-coordinates.
Keep your answers in surd form. **[6 points]**

(b) Given three points (x_0, y_0), (x_1, y_1), (x_2, y_2) in the Cartesian plane, they are collinear (contained in one straight line) if the determinant $\begin{vmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix}$ is zero. By computing the value of this determinant, determine if the three inflection points are collinear or not. **[14 points]**

4. Consider the rational function $f(x) = \dfrac{x+2}{x^2-2x-3}$. Suppose the power series of $f(x)$ at $x = 0$ is given by $f(x) = \sum_{n=0}^{\infty} a_n x^n$. We can obtain a_n by the usual Taylor/Maclaurin Series expansion, that is, by computing $\dfrac{f^{(n)}(0)}{n!}$. However, we can also obtain a_n by an alternative and simpler approach as shown in lecture. Consider the expression we are constructing $\dfrac{x+2}{x^2-2x-3} = \sum_{n=0}^{\infty} a_n x^n$. By rearranging it, we have

$$\begin{aligned}
2 + x &= \left(x^2 - 2x - 3\right) \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+2} - 2\sum_{n=0}^{\infty} a_n x^{n+1} - 3\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} a_{n-2} x^n - 2\sum_{n=1}^{\infty} a_{n-1} x^n - 3\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} a_{n-2} x^n - 2\left(a_0 x + \sum_{n=2}^{\infty} a_{n-1} x^n\right) - 3\left(a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n\right) \\
&= -3a_0 - (2a_0 + 3a_1)x + \sum_{n=2}^{\infty} (a_{n-2} - 2a_{n-1} - 3a_n) x^n.
\end{aligned}$$

(a) By comparing the undetermined coefficients of x^0 and x^1 of the right hand side to the left hand side, show that $a_0 = \frac{-2}{3}$ and $a_1 = \frac{1}{9}$. **[2 points]**

(b) From the left hand of the equation, clearly there are no x^n terms for $n \geq 2$. For $n \geq 2$, obtain the linear equation of a_n in terms of a_{n-1} and a_{n-2}. Moreover, use this equation and the results obtained in **(a)** to find a_2, a_3, a_4, a_5. Express your answers as rational numbers only. **[8 points]**

(c) Suppose $L = \lim_{n\to\infty} \dfrac{a_n}{a_{n-1}}$ exist and non-zero. Note that $\dfrac{1}{L} = \lim_{n\to\infty} \dfrac{a_{n-2}}{a_{n-1}}$. From the linear equation obtained in **(b)**, divide it by a_{n-1}, and then take limit for $n \to \infty$, and form a quadratic equation of L. Then, find the two possible values of L (note that one is positive, and one is negative). Then, by observing values of a_0, a_1, a_2, a_3, a_4 obtained above, determine if L is positive or negative, and conclude which is the value of L. **[5 points]**

(d) Find the radius of convergence of the power series. **[5 points]**

5. Recall that the sum of the geometric progression is given by
$1 + x + x^2 + x^3 + ... + x^n = \dfrac{1 - x^{n+1}}{1 - x} = (1 - x)^{-1}(1 - x^{n+1}) \qquad \text{for } (1 - x) \neq 0.$
This can be easily verified by multiplying both sides with $1 - x$. Now, consider the matrix version of the sum:

$$\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \cdots + \boldsymbol{A}^n \;=\; (\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{I} - \boldsymbol{A}^{n+1}).$$

where $\boldsymbol{I} - \boldsymbol{A}$ is invertible.

(a) Explain why this matrix version of the sum holds. **[2 points]**

(b) Consider $\boldsymbol{A} = \begin{bmatrix} \frac{1}{4} & 2 \\ 0 & \frac{1}{2} \end{bmatrix}$. By evaluating $det(\boldsymbol{I} - \boldsymbol{A})$, show that $\boldsymbol{I} - \boldsymbol{A}$ is invertible.
[2 points]

(c) Consider the CoCalc output:

In [1]:
```
A=matrix([[1/4,2],[0,1/2]])
show(A)
```

$$\begin{pmatrix} \frac{1}{4} & 2 \\ 0 & \frac{1}{2} \end{pmatrix}$$

In [2]:
```
show(identity_matrix(2)-A^(5))
```

$$\begin{pmatrix} \frac{1023}{1024} & -\frac{31}{128} \\ 0 & \frac{31}{32} \end{pmatrix}$$

In [3]:
```
show((identity_matrix(2)-A)^(-1))
```

$$\begin{pmatrix} \frac{4}{3} & \frac{16}{3} \\ 0 & 2 \end{pmatrix}$$

Use the results of the output to evaluate the sum $\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \boldsymbol{A}^4$.
Keep your answers in rational numbers only. **[6 points]**

(d) Obtain the characteristic polynomial of $\boldsymbol{A}$.
Then, use it to find all the eigenvalues of $\boldsymbol{A}$. **[5 points]**

(e) If $|\lambda_i| < 1$ for every eigenvalue λ_i of a matrix $\boldsymbol{B}$, then $\lim_{n \to \infty} \boldsymbol{B}^n = \boldsymbol{O}$.
Use this result, find the sum $\sum_{n=1}^{\infty} \boldsymbol{A}^n = \boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \ldots$. **[5 points]**

*** END ***