THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code:	AMA1007 / AMA1008	**Subject Title:**	Calculus and Linear Algebra
Session:	Semester 1, 2020/2021	**Assessment:**	Final Exam
Submission:	Blackboard	**Date and Time:**	09 Dec 2020, 12:30-14:30

This set of question has 5 pages (including this cover page).

Instructions: Attempt **ALL** questions in this paper. Your solutions to each question must be made in the designated area (inside the boxes) of the set of **Answer Sheets**. Scan your work into **one single PDF file** with file size not exceeding 6MB, and submit it via Blackboard with file name as your name (surname first). It is recommended for students to use the app Microsoft Office Lens to scan multiple pages into one single pdf file. Only one single submission will be accepted (it is the students' responsibility to check carefully before submitting). Late submissions will not be accepted. Students are required to sign the covering statement of the set of Answer Sheets for submission.

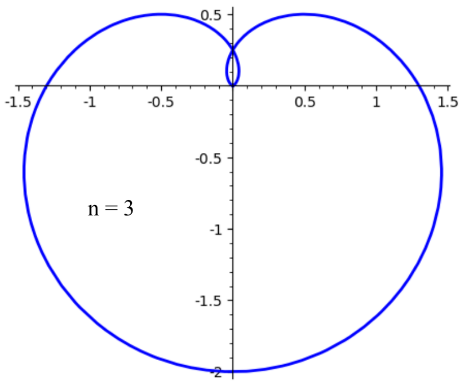
No marks will be given for those not following any of the above instructions.

Subject Examiners: Dr. LEE Heung Wing Joseph

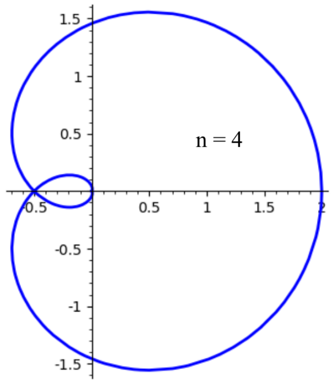
1. (a) Use integration by parts to find the reduction formula for the anti-derivative of $\sin^n(x)$, that is, find the expression to reduce $\int \sin^n(x)\, dx$ to $\int \sin^{n-2}(x)\, dx$ for positive integer $n \geq 2$. Moreover, state specifically the results for $n = 2$, and $n = 3$. **[8 points]**

   (b) Consider the graphs expressed in polar coordinates $r_n(\theta) = 2\sin^n\left(\frac{\theta}{n}\right)$ where $0 \leq \theta \leq n\pi$, for positive integer $n \geq 3$. The graphs for different values of n would be different. For example:

In [1]:
```
polar_plot(2*sin(x/3)^3, (x, 0, 3*pi), thickness=2)
```
Out[1]:



In [2]:
```
polar_plot(2*sin(x/4)^4, (x, 0, 4*pi), thickness=2)
```
Out[2]:



The arc-length of the graph of r_n from $\theta = 0$ to $\theta = n\pi$ is given by $\int_0^{n\pi} \sqrt{(r_n(\theta))^2 + (r_n'(\theta))^2}\, d\theta$. Use the results obtained in part **(a)** and evaluate the arc-length for $n = 3$ and $n = 4$. **[12 points]**

2. Consider the a 2×100 matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 & \dots & 99 & 100 \\ 7 & 8 & 9 & \dots & 105 & 106 \end{bmatrix}$. There exist a minimum number m of elementary matrices $\boldsymbol{E_1}, \boldsymbol{E_2}, \dots, \boldsymbol{E_m}$, so that the product $\boldsymbol{B} = \boldsymbol{E_m}\boldsymbol{E_{m-1}} \dots .\boldsymbol{E_2}\boldsymbol{E_1}\boldsymbol{A}$ is in reduced row-echelon form.
Find these m elementary matrices, and state $\boldsymbol{B}$. **[15 points]**

3. Consider two planes $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ in 3-D. Suppose $\boldsymbol{P_1}$ contains a point $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and has normal direction $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and suppose $\boldsymbol{P_2}$ contains a point $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and has normal direction $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$. The two planes intersect, and form a line of intersection $\boldsymbol{L_1}$.

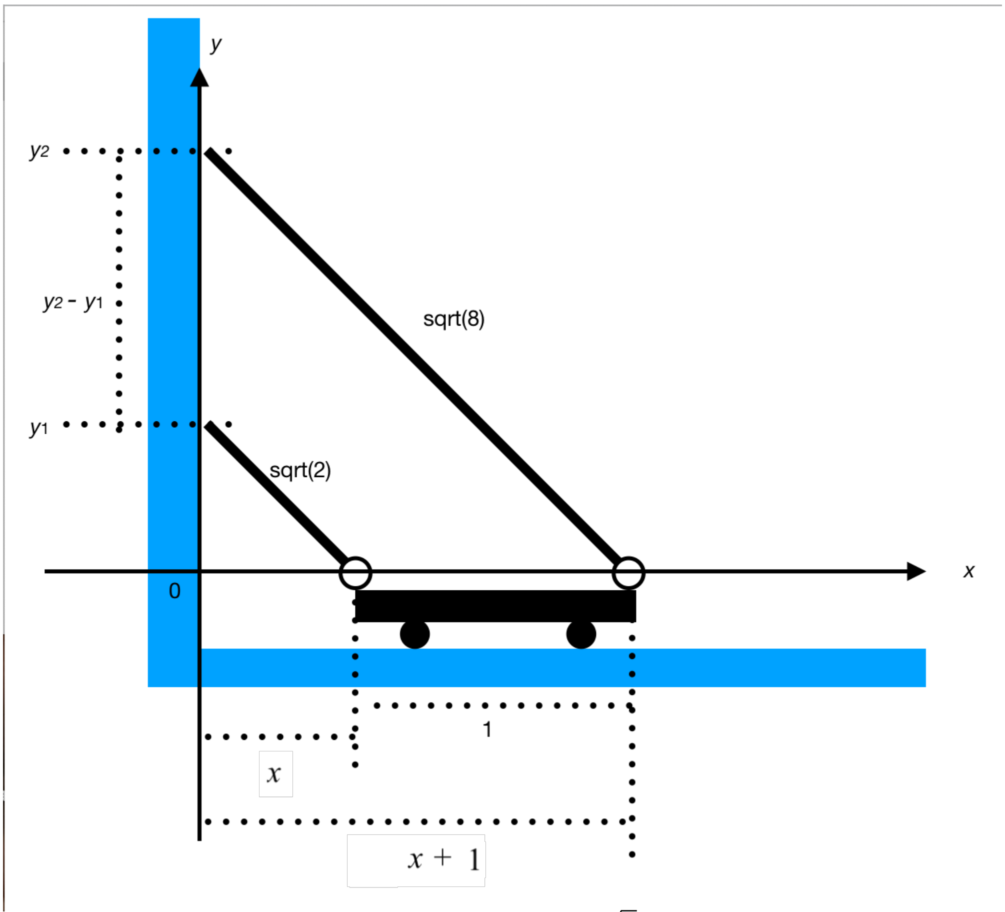
   (a) Obtain the equation of $\boldsymbol{P_1}$ in normal form (i.e. $a_1x + b_1y + c_1z = d_1$). **[3 points]**

   (b) Obtain the equation of $\boldsymbol{P_2}$ in normal form (i.e. $a_2x + b_2y + c_2z = d_2$). **[3 points]**

   (c) $\boldsymbol{L_1}$ intersects on the xz-plane. Find this location of intersection by the Gauss-Jordan method only, as a system of 3 linear equations of 3 unknowns, and keep all the values in your solution as rational numbers only. **[14 points]**

4. Determine the convergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n+\sin(x)}$ for $x \in [0,\, 2\pi]$. **[15 points]**

5. (a) Consider two ladders with lengths $\sqrt{2}$m and $\sqrt{8}$m respectively, with their bottom-ends hinged 1 m apart on a cart which could move horizontally. Suppose the top-ends of both ladders are rested against the vertical wall as shown in the diagram. Starting from $x = 0$ when the shorter ladder is vertical, if the cart moves to the right (increase in x), the top-ends of both ladders will slide downwards, in different amounts. The distance between the two top-ends is given by $y_2 - y_1$.



Consider $f(x) = y_2 - y_1$ where $0 \le x \le \sqrt{2}$. Find all maximum / minimum of f on the interval. **[15 points]**

(b) Suppose a student is using CoCalc (Jupyter with Kernel Sage 9.1) to investigate a function $g(x)$ defined on $x \in [0, \sqrt{2}]$ as the vertical distance between two circles at x (distance is measured in the positive y direction), where the two circles are given by

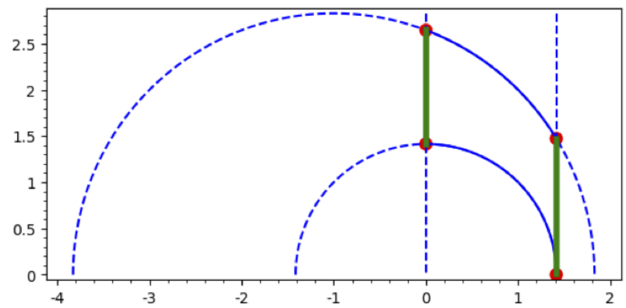
$$\begin{aligned} x^2 + y^2 &= 2 \\ (x+1)^2 + y^2 &= 8 \end{aligned}$$

and plot out the derivative of g with respect to x **twice**, that is, $\dfrac{d^2}{dx^2} g(x)$.

In [1]:

```
var('x,y')
p1=implicit_plot(y^2+x^2 == 2, (x,0,sqrt(2)), (y,0,sqrt(2)) ,aspect_ratio=1 )
p2=implicit_plot(y^2+(x+1)^2 == 8, (x,0,sqrt(2)), (y,0,sqrt(8)), aspect_ratio=1)
p3=implicit_plot(y^2+x^2 == 2, (x,-2,sqrt(2)), (y,0,sqrt(2)) ,aspect_ratio=1 ,linestyle = "dashed")
p4=implicit_plot(y^2+(x+1)^2 == 8, (x,-4,sqrt(8)), (y,0,sqrt(8)), aspect_ratio=1, linestyle = "dashed")
p5=implicit_plot(x == 0, (x,0,2),(y,0,sqrt(8)) ,linestyle = "dashed" )
p6=implicit_plot(x == sqrt(2), (x,0,2),(y,0,sqrt(8)) ,linestyle = "dashed" )
p7=line([(0,sqrt(2)), (0,sqrt(8-1))], rgbcolor='green',thickness=4)
p8=line([(sqrt(2),0), (sqrt(2),sqrt(8-(sqrt(2)+1)^2))], rgbcolor='green', thickness=4)
pt1=point((0,sqrt(2)), rgbcolor='red', pointsize=80)
pt2=point((0,sqrt(8-1)), rgbcolor='red', pointsize=80)
pt3=point((sqrt(2),0), rgbcolor='red', pointsize=80)
pt4=point((sqrt(2), sqrt((8-(sqrt(2)+1)^2))), rgbcolor='red', pointsize=80)
(p1+p2+p3+p4+p5+p6+p7+p8+pt1+pt2+pt3+pt4).show(xmin=-4,xmax=2,ymin=0,ymax=sqrt(8))
```

Out[1]:



In [2]:

```
g(x)=sqrt(8-(x+1)^2)-sqrt(2-x^2)
show(g)
```

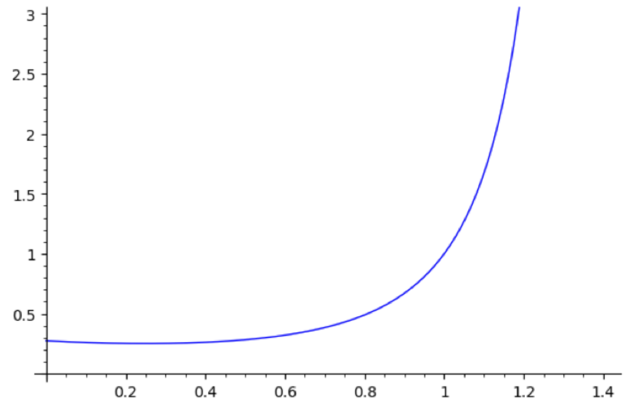
Out[2]:

$$x \mapsto \sqrt{-(x+1)^2+8} - \sqrt{-x^2+2}$$

In [3]:

```
plot(diff(g(x),x,2),x,0,sqrt(2)).show(ymax=3,ymin=0)
```

Out[3]:



Note that this plot of $\dfrac{d^2}{dx^2}g(x)$ is above the x-axis on the interval. In view of this, what can you conclude about the convexity of $f(x)$ in part **(a)** on the interval from $x = 0$ to $x = \sqrt{2}$? (There is no need to get to use CoCalc to answer this question). **[5 points]**

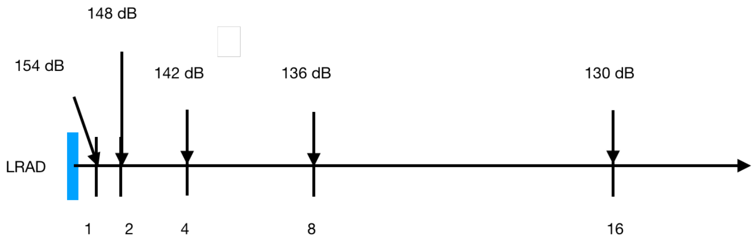
6. It was reported that, on 17 Nov 2019 at around 2:28pm, the Hong Kong Police was using the controversial Long Range Acoustic Device (LRAD) against protesters, just outside of PolyU. LRAD is an array of piezoelectric transducers (acting as point source speakers), and by making use of the interference of wave from these specially arranged point sources, it can produce very narrow beam of directional (targeting) loud sound wave (focused within 30^0).

Photo Source : Appledaily
圖片由《蘋果日報》提供
(photo taken by 翁志偉)

LRAD is the hexagonal-shaped plate device mounted on top of the police armoured vehicle.

If the plate size of the LRAD is not too large, the attenuation behaves in a similar way as a point source (radiator) along the target front direction when the target is sufficiently far away (compared to the plate size), i.e. every doubling of the distance away along the target direction reduces the Sound Pressure Level (SPL) by 6 dB. Suppose SPL at 1 m away from the LRAD direct front blast is 154 dB. Thus, we can construct data points:

Distance (m)	SPL (dB)
$1 = 2^0$	154
$2 = 2^1$	148 = 154-6
$4 = 2^2$	142 = 148-6
$8 = 2^3$	136 = 142-6
$\vdots$	$\vdots$
$512 = 2^9$	100
$1024 = 2^{10}$	94 = 100-6



Note that even at 512 m away from the LRAD, SPL is still as high as 100 dB.

Let x be the distance, and let y be the SPL. Assuming y is a differentiable function of x for $x \geq 1$. For $n \geq 1$, we have a pair of equations $\begin{cases} x = 2^n \\ y = 154 - 6n \end{cases}$. Note that there is no need for n to be restricted as an integer. By eliminating n from the pair of equations, or otherwise, construct y as a continuous function of x over the closed interval $[1, 1024]$, and differentiable over the open interval $(1, 1024)$.

Moreover, find the expression $\dfrac{dy}{dx}$ in terms of x. Further more, explain why there exist a location c so that $y'(c) = \dfrac{y(1024) - y(1)}{1024 - 1} = \dfrac{94 - 154}{1023} = -\dfrac{60}{1023}$, and find c. **[10 points]**

This question is written by the Subject Lecturer Dr. Joseph Lee. It does not represent the political position of The Department of Applied Mathematics.

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