THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code:	AMA1007	**Subject Title:**	Calculus and Linear Algebra
Session:	Semester 2, 2019/2020	**Assessment:**	End-of-term Final Exam
Submission:	Blackboard	**Date and Time:**	28 May 2020, 12:30-14:30

This set of question has 12 pages (including this cover page).

Instructions: Attempt **ALL** questions in this paper. Your solutions to each question must be made in the designated area (inside the boxes). Scan your work into one single PDF file with file size not exceeding 6MB, and submit it via Blackboard with file name as your name (surname first). Only **one single submission** will be accepted (it is the students' responsibility to check carefully before submitting). Students are required to sign the statement below.

By submitting this work through the online system,

I affirm on my honour that I am aware of the Regulations on Academic Integrity in Student Handbook and

(i) have not given nor received any unauthorized aid to/from any person or persons, and

(ii) have not used any unauthorized materials in completing my answers to this Exam.

Signature: ______________________________

Name : ______________________________

Student Number : ______________________________

No marks will be given for those not following any of the above instructions.

Subject Examiners: Dr. LEE Heung Wing Joseph

1. Consider the graph of the equation $ax^2 + 2bxy + cy^2 = 1$ where $c > 0$ and $ac - b^2 > 0$. Find the area of the region enclosed by this graph by forming and computing the required definite integral (integrating with respect to x), and indicate clearly the upper and lower limits in the process (no marks will be given if you are using some other techniques). Use the integral formula: $\int \sqrt{r^2 - u^2}du = \frac{u}{2}\sqrt{r^2 - u^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{u}{r}\right) + C$. **[15 points]**

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2. Consider the equation $ax^2 + 2bxy + cy^2 = 1$ where $a > 0$ and $ac - b^2 > 0$. Note that the equation can be written as $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1$. Consider an invertible linear map between (x, y) and (u, v) given by $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$.

(a) Choose a value for e (in terms of a, b and c), so that the given equation on the (u, v) plane becomes $\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} \frac{1}{A^2} & 0 \\ 0 & \frac{1}{B^2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 1$, and find A and B (in terms of a, b and c).

[9 points]

(b) Briefly explain why the linear map between (x, y) and (u, v) is area preserving, that is, a region mapped from the (x, y) plane to the (u, v) plane would have the same area.

[1 points]

3. Consider the integral $\int \frac{1}{1+x^9} dx$, where $1 + x^9 \neq 0$.

(a) Evaluate the integral as a power series, and specify the radius of convergence.

[5 points]

(b) Use the first 4 terms of the result in (a) to approximate $\int_0^{1/2} \frac{1}{1+x^9} dx$, keep your answer as a sum of the 4 rational numbers (marks will not be given for decimals).

[5 points]

4. Consider the function $f(x) = \dfrac{x}{x^2 - 5x + 6} = \dfrac{A}{x-2} + \dfrac{B}{x-3}$ for some constants A and B, $x \neq 2$ and $x \neq 3$. Suppose $f(x)$ is expressed as a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$. Without finding the power series itself, find the interval of convergence when

(a) $c = 0$. **[10 points]**

(b) $c = 5$. **[5 points]**

5. **This is the original version of the question, with the first paragraph about the background of this question. The paragraph was removed in the final version after moderation.** *The HA Employees Alliance (醫管局員工陣線), a relatively new union formed last December, has successfully staged a large scale strike for which several thousands of medical staff have participated in early February. The union demanded a complete closure of Hong Kong's border with the Chinese mainland due to the outbreak of a highly contagious and deadly coronavirus disease spreading wildly in China at the time (the disease was later controversially named COVID 19 by the WHO, and nonetheless, it was being called the Chinese virus by the President of USA publicly afterwards). However, the HKSAR government refused to lay a complete travel ban on the inbounds from that clear outbreak source, despite the very painful and hard-to-forget experience of the deadly SARS in 2003 that was brought to Hong Kong recklessly from the same source (by a Chinese mainlander Liu Jianlun 劉劍倫(廣州中山大學第二附屬醫院退休內科教授)). This was what sparked the strike. The union vice-chairman Ivan Law Cheuk-yiu (羅卓堯), a former External Vice-President of The Hong Kong Polytechnic University Student Union (前理工大學學生會外務副會長) and a former Vice-Chairman of the Representative Council of The Hong Kong Federation of Students (前學聯代表會副主席) said, just within the first week of the outbreak, new members joining the union has jumped to 22.5 thousands, showing the strong and clear view among medical staff.*

Consider a simple infectious disease model. Let $S(t) \geq 0$ be the number of susceptible at time t (individuals whom possible to be infected but not (or not yet) infected with the disease at t), $I(k) \geq 0$ be the number of infective at time t (those who are infected and be able to transmit the disease), $R(t) \geq 0$ be the number of the recovered or deceased at time t (and those recovered are assumed immuned and not transmitting anymore). The dynamics of the model (i.e., the rate of change of S, I, and R with respect to time) is given by:

$$\begin{aligned} \frac{dS}{dt} &= -\beta I(t)S(t) + \mu_1, \\ \frac{dI}{dt} &= \beta I(t)S(t) - \gamma I(t) + \mu_2 \\ \frac{dR}{dt} &= \gamma I(t). \end{aligned}$$

where $\beta > 0$, $\gamma > 0$ are constants, and μ_1 is the rate of imported susceptible, and μ_2 is the rate of imported infective. Suppose $\beta = \frac{1}{1000}$, and $\gamma = \frac{51}{50}$, and at a particular time $t > 0$, if $S = 1000$, $I = 3$, and $R = 0$.

This question is written by the Subject Lecturer Dr. Joseph Lee. It does not represent the political position of The Department of Applied Mathematics.

(a) Suppose travel bans are imposed, i.e., $\mu_1 = 0$ and $\mu_2 = 0$. Explain each of S, I and R if they are increasing, decreasing, or stationary at time t. **[5 points]**

(b) Suppose travel bans are not imposed on travellers from regions with the outbreak, say, $\mu_1 = \frac{12}{5}$ and $\mu_2 = \frac{6}{5}$. Explain each of S, I and R if they are increasing, decreasing, or stationary at time t. **[5 points]**

6. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $n \times n$ matrices. They are said to be **similar** if there exists a non-singular matrix $\boldsymbol{P}$ such that $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$. Moreover, similar matrices have the following three properties:

   (i) $Tr(\boldsymbol{A}) = Tr(\boldsymbol{B})$, [i.e. sum of all the diagonal elements of $\boldsymbol{A}$ and sum of all the diagonal elements of $\boldsymbol{B}$ are the same],

   (ii) $det(\boldsymbol{A}) = det(\boldsymbol{B})$,

   (iii) $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same characteristic polynomials.

   Given two similar matrices, $\boldsymbol{A} = \begin{bmatrix} -4 & -10 & 0 \\ a & a+2 & 0 \\ 3 & 6 & 1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b \end{bmatrix}$ where a and b are some real constants. Use the three properties of similar matrices stated above to find a and b, and all the eigenvalues of $\boldsymbol{A}$. **[15 points]**

7. Consider the linear system

$$\begin{aligned} x_1 + x_2 - 2x_3 + 3x_4 &= 0 \\ 2x_1 + x_2 - 6x_3 + 4x_4 &= -1 \\ 3x_1 + 2x_2 + px_3 + 7x_4 &= -1 \\ x_1 - x_2 - 6x_3 - x_4 &= t. \end{aligned}$$

Find the conditions (on t and p) that the system is consistent, and inconsistent. If the system is consistent, find all the possible solutions (including stating the dimension of the solution space(s) and describe the solution space(s) in parametric form). **[15 points]**

8. Evaluate $\int_{-1}^{1} \frac{d}{dx}\left(\frac{1}{1+2^{1/x}}\right) dx$, note that $\frac{d}{dx}\left(\frac{1}{1+2^{1/x}}\right)$ is not continuous at $x = 0$.

[10 points]

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