THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code: AMA1007 **Subject Title:** Calculus and Linear Algebra

Session: Semester 2, 2018/2019

Date: May 04, 2019 **Time:** 12:30 - 14:30

Time Allowed: 2 hours

Section B has 4 pages (including this page)

Section A has 5 pages (including a cover page)

Instructions: This paper has two sections, Section A and Section B.

Attempt **ALL** questions in this paper.

Answers to Section B must be made in the Answer Book.

Both Section A and Section B question papers will be collected.

Put both sets of question papers inside the answer book at the end.

You are not allowed to detach any pages out.

Marks will be deducted if you have detached pages.

Subject Examiners: Dr. LEE Heung Wing Joseph

DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO.

Section B

11. It was reported that, on 9 April 2019, nine Umbrella Movement 雨傘運動 leaders were found guilty over their role in the 79 days protest held in 2014 (Occupy Central with Love and Peace 讓愛與和平佔領中環, the largest civil disobedience event in the history of Hong Kong). The charges against them were not all very commonly heard of: (I) conspiracy to commit public nuisance 串謀犯公眾妨擾罪, (II) incitement to commit public nuisance 煽惑他人犯公眾妨擾罪, and (III) inciting people to incite others to commit public nuisance 煽惑他人煽惑公眾妨擾罪. It was also reported that the 28th and final Governor of British Hong Kong Lord Patten (Christopher Francis Patten 彭定康) said the trial was the result of a *vengeful campaign* (報復) by the authorities.

    Suppose there are two different types of people, Type 1 is the participating type of a civil disobedience event, and Type 2 is the non-participating type. Type 1 members can incite Type 2 members to become Type 1, and vice versa. Let $x_1(k)$ be the proportion of the population of Type 1 at time k, and $x_2(k)$ be the proportion of the population of Type 2 at time k. Suppose the transition of the proportions after one incitement from $\boldsymbol{x}_k = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$ to $\boldsymbol{x}_{k+1} = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix}$ is given by $\boldsymbol{T}\boldsymbol{x}_k = \boldsymbol{x}_{k+1}$, where $\boldsymbol{T} = \begin{bmatrix} 9/10 & 1/5 \\ 1/10 & 4/5 \end{bmatrix}$, ($\boldsymbol{T}$ is called the transition matrix). Thus, $\begin{bmatrix} 9/10 & 1/5 \\ 1/10 & 4/5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix}$. If such incitement occurred infinitely many times (and given in this case, the column sums of $\boldsymbol{T}$ are 1), it can be shown that the limiting equilibrium proportions $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ exists, i.e. for any initial proportion $\boldsymbol{x}_0$, we have $\lim_{n\to\infty} \boldsymbol{T}^n \boldsymbol{x}_0 = \boldsymbol{T}\boldsymbol{v} = \boldsymbol{v}$. Note that one of the eigenvalue of $\boldsymbol{T}$ is 1, and its associated eigenvector is in the same direction of $\boldsymbol{v}$.

    (a) Find $\boldsymbol{v}$ (i.e. find the eigenvector of $\boldsymbol{T}$ associated with the eigenvalue 1). Keep your answer normalised, i.e. keep $v_1 + v_2 = 1$, $v_1 \geq 0$, and $v_2 \geq 0$. **[7 points]**

    (b) Consider if there is double incitement (inciting people to incite others). If a portion $p = 1/2$ is incited to incite others, the transition matrix is then given by $\boldsymbol{S} = p\boldsymbol{T} + (1-p)\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity. Again, it can be shown that limiting equilibrium proportions $\boldsymbol{u}$ exists, (i.e. $\boldsymbol{S}\boldsymbol{u} = \boldsymbol{u}$), and one of the eigenvalue of $\boldsymbol{S}$ is 1, and its associated eigenvector is in the same direction of $\boldsymbol{u}$. Find $\boldsymbol{u}$ (i.e. find the eigenvector of $\boldsymbol{S}$ associated with the eigenvalue 1). Keep your answer normalised, i.e. keep $u_1 + u_2 = 1$, $u_1 \geq 0$, and $u_2 \geq 0$. Briefly explain why considering double incitement is redundant in this scenario of finding limiting equilibrium. **[8 points]**

    This question is written by the Subject Lecturer Dr. Joseph Lee. It does not represent the political stance of The Department of Applied Mathematics.

12. The 8th chapter of the ancient Chinese text of **The Nine Chapters of the Mathematical Art** (《九章算術》卷八) is dedicated to Rectangular Arrays (方程). The 15th problem (第十五問) in the chapter is as follows:

    今有甲禾二秉、乙禾三秉、丙禾四秉，重皆過於石。甲二重如乙一，乙三重如丙一，丙四重如甲一。問甲、乙、丙禾一秉各重幾何?

    - Note: 秉= 束/ 捆紮 sheaf (for singluar), sheaves (for plural).
    - 石(or 擔) stone, unit of weight, about 31 kg.

    English translation/descriptions to the problem :
    Suppose there are 2 sheaves of grade A rice, 3 sheaves of grade B rice, and 4 sheaves of grade C rice, and each collection of sheaves weighs over a stone. Two sheaves of grade A rice has the same weight as a sheaf of grade B rice and one stone. Three sheaves of grade B rice has the same weight as a sheaf of grade C rice and one stone. Four sheaves of grade C rice has the same weight as a sheaf of grade A rice and one stone. How heavy is a sheaf of each grade of rice?

    Let x be the weight (in unit of stone) of a sheaf of grade A rice, y be the weight of a sheaf of grade B rice, and z be the weight of a sheaf of grade C rice. Thus, we have

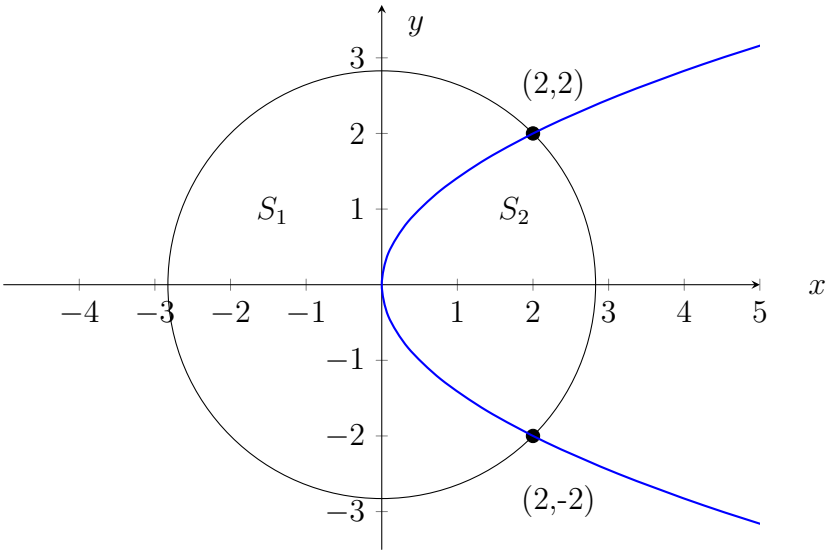
$$\begin{aligned} 2x &= y+1 \\ 3y &= z+1 \\ 4z &= x+1 \end{aligned}$$

    (a) Solve the system using Gauss-Jordan method. **[14 points]**

    (b) Check to confirm the answer you have obtained satisfy $2x > 1$, $3y > 1$, and $4z > 1$ (i.e., Two sheaves of grade A rice, three sheaves of grade B rice, and four sheaves of grade C rice, and each collection of sheaves weighs over a stone.). **[6 points]**

13. Evaluate the indefinite integral $\int \sqrt{x}\ln^2(x)dx$. **[15 points]**

14. Consider the parabola $y^2 = 2x$ and the circle $x^2 + y^2 = 8$. They intersect at $(2, 2)$ and $(2, -2)$. The parabola splits the area enclosed by the circle into two parts S_1 and S_2.



(a) Evaluate $\int \sqrt{8 - y^2}\,dy$. Hint: you may find the integral formula

$$\int \sqrt{a^2 - u^2}\,du = \frac{u\sqrt{a^2 - u^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + C \quad \text{useful.}$$ **[7 points]**

(b) Use the result obtained in **(a)** to evaluate $\int_0^2 (\sqrt{8 - y^2} - \frac{y^2}{2})dy$. Keep your answer in terms of π. [Note: $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$]. **[7 points]**

(c) Use the result obtained in **(b)** to find the area of S_2, and thus, find S_1. Keep your answer in terms of π. **[6 points]**

*** END of Section B ***