THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code:	AMA1007 / AMA1120	**Subject Title:**	Calculus and Linear Algebra
Session:	Semester 1, 2022/2023	**Assessment:**	Final Exam
Date:	05 Dec 2022,	**Time:**	12:30-14:30

This set of question has 5 pages (including this cover page).

Instructions: Attempt **ALL** questions in this paper.

DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO.

1. (a) Consider the function $v = f(u) = \ln(1+u)$ where $-1 < u < 1$. Obtain the first five terms (up to degree 4) of the Maclaurin expansion of $f(u)$ (i.e., Taylor Series expansion of $f(u)$ at $u = 0$). **[6 points]**

(b) Obtain the inverse function $u = f^{-1}(v)$, and state its domain. **[2 points]**

(c) Obtain the first five terms (up to degree 4) of the Maclaurin expansion of $f^{-1}(v)$. **[6 points]**

(d) Consider the following expression of y in terms of x only:

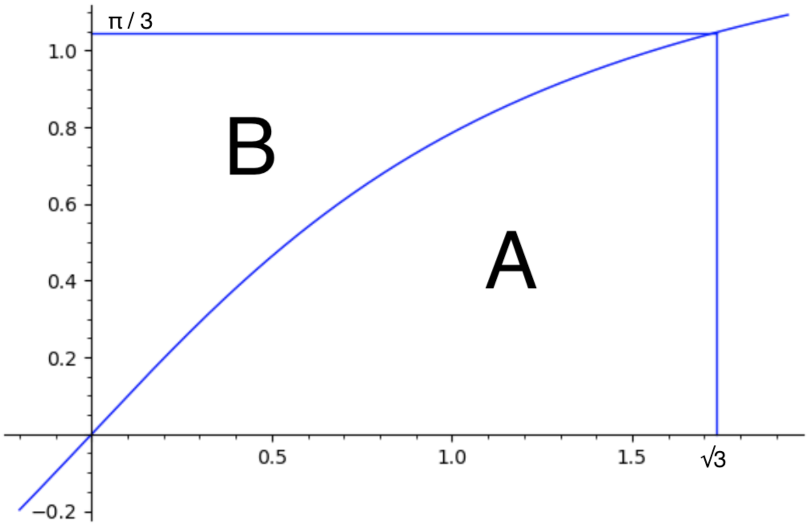
$$y \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \;=\; \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

where $|x| < 1$. Use the results obtained in the previous parts of the question to show that the *inverse* of this expression (that is, the expression of x in terms of y only) is given by

$$x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots \;=\; \sum_{n=1}^{\infty} \frac{y^n}{n!}$$

for $-\infty < y < \ln(2)$. **[6 points]**

2. Consider the graph of $y = \tan^{-1}(x)$ where $0 \le x \le \sqrt{3}$:



(a) Find the area of region A by evaluating the integral $\displaystyle\int_0^{\sqrt{3}} \tan^{-1}(x)\, dx$.
Hint: Let $u = \tan^{-1}(x)$ and let $v = x$, then, use integration by parts. **[10 points]**

(b) Show that $\dfrac{d}{du} \ln(\sec(u)) = \tan(u)$, for $-\frac{\pi}{2} < u < \frac{\pi}{2}$. **[5 points]**

(c) Using part (b), find the area of region B by evaluating the integral $\displaystyle\int_0^{\pi/3} \tan(y)\, dy$. **[5 points]**

3. In this question, we are looking into a simplified version of Page Rank (web page ranking algorithm adopted by Google). Suppose there are only 4 web pages in the world-wide-web, and they are linked to each other by URL in the following way:

- Page 1 has links to Page 2 and Page 4.
- Page 2 has links to Page 1 and Page 3.
- Page 3 has links to Page 1, Page 2, and Page 4.
- Page 4 has links to Page 1 and Page 3.

Thus, a transition matrix can then be written down as $\boldsymbol{A} = \begin{bmatrix} 0 & 1/2 & 1/3 & 1/2 \\ 1/2 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/3 & 0 \end{bmatrix}$.

Note that all 4 column sums of $\boldsymbol{A}$ are 1.

Initially, suppose visitors are evenly distributed among the 4 web pages, that is to say, $\boldsymbol{v^0} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$, and suppose all the visitors simultaneously *click* on one of the link provided on the visiting page, then, after the click, we have a new distribution of visitors over the 4 web pages $\boldsymbol{v^1} = \boldsymbol{A}\boldsymbol{v}^0 = \begin{bmatrix} 1/3 \\ 5/24 \\ 1/4 \\ 5/24 \end{bmatrix}$, and after two *clicks*, then, we have $\boldsymbol{v^2} = \boldsymbol{A}\boldsymbol{v^1} = \boldsymbol{A}^2\boldsymbol{v^0} = \begin{bmatrix} 7/24 \\ 1/4 \\ 5/24 \\ 1/4 \end{bmatrix}$, after three *clicks* $\boldsymbol{v^3} = \boldsymbol{A}^3\boldsymbol{v^0} = \begin{bmatrix} 23/72 \\ 31/144 \\ 1/4 \\ 31/144 \end{bmatrix}$, etc. With a sufficiently large amount of *clicks*, the distribution will be settled at an equilibrium $\boldsymbol{v} = \lim_{n\to\infty} \boldsymbol{A}^n\boldsymbol{v^0}$. Moreover, at equilibrium, any *clicks* afterwards is not changing the distribution, that is to say, $\boldsymbol{v} = \boldsymbol{A}\boldsymbol{v}$. Note that this equilibrium vector $\boldsymbol{v}$ is the Page Rank of the four web pages, and it is also the eigenvector of $\boldsymbol{A}$ associated with eigenvalue 1, and with non-negative entries, and with column sum 1.

(a) Consider the CoCalc output giving the factorized characteristic polynomial of $\boldsymbol{A}$ (that is, $det(\boldsymbol{A} - x\boldsymbol{I})$):

```
In [1]: A=matrix([[0,1/2,1/3,1/2],[1/2,0,1/3,0],[0,1/2,0,1/2],[1/2,0,1/3,0]])

In [2]: show(A.charpoly().factor())
```

Out[2]: $(x-1) \cdot x \cdot (x^2 + x + \frac{1}{6})$

Using the CoCalc output, or otherwise, confirm that there is an eigenvalue 1 of $\boldsymbol{A}$.

[5 points]

(b) Consider the Cocalc work:

In [3]:
```
lamb=1
B=A-lamb*matrix.identity(4)
c=matrix([0,0,0,0]).transpose()
Bc=B.augment(c)
show(Bc)
```

Out[3]:

$$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & -1 & 0 \end{pmatrix}$$

In [4]:
```
show(Bc.rref())
```

Out[4]:

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

.

Note that the output in Cell [4] is showing the **reduced row echelon form** (`.rref()`) of eliminating the augmented matrix $[\boldsymbol{A} - \boldsymbol{I} \mid \boldsymbol{0}]$ by the Gauss-Jordan method. Use the result of the CoCalc output, or otherwise, find the Page Rank, that is, find the eigenvector $\boldsymbol{v}$ associated with the eigenvalue 1 with only non-negative entries, and with column sum 1. **[14 points]**

(c) Among the four web pages, which page has the highest ranking? **[1 point]**

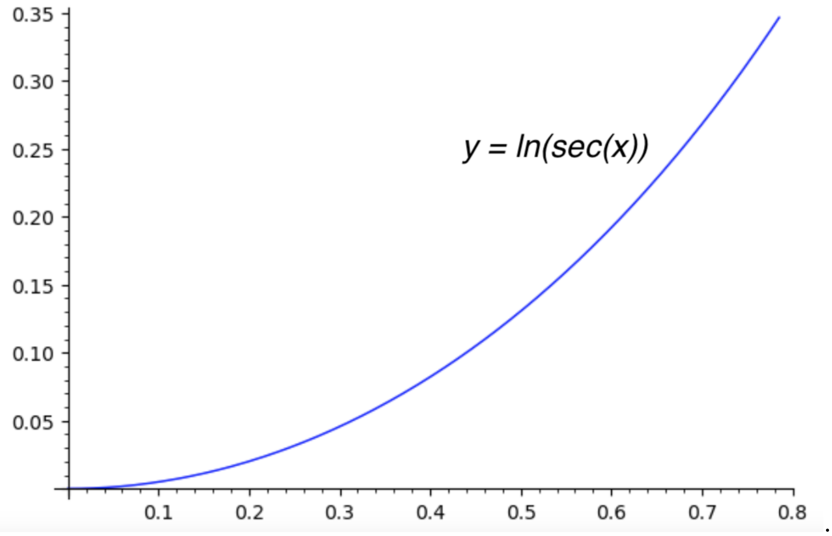
4. (a) Suppose $u = \tan(x) + \sec(x)$. Find $\dfrac{du}{dx}$. **[5 point]**

(b) Show that $\displaystyle\int \sec(x)dx = \ln(\sec(x) + \tan(x)) + C$ for $0 \le x < \frac{\pi}{4}$.

Hint: Rewrite $\sec(x)$ as $\dfrac{\sec(x)\,(\sec(x) + \tan(x))}{(\sec(x) + \tan(x))} = \dfrac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}$.

[7 point]

(c) Consider the graph of $y = \ln(\sec(x))$ for $0 \le x \le \frac{\pi}{4}$.



Make use of the above results, or otherwise, show that, the arc length of the graph is given by $\ln(\sqrt{2}+1)$.
Hint: $\tan^2(x) + 1 = \sec^2(x)$. **[8 point]**

5. Consider $y = \tan^{-1}(x)$ in the neighbourhood of $x = 0$. Observe that, y is an odd function of x. Moreover, note that $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$. Thus, we have $(1+x^2)\dfrac{dy}{dx} = 1$.

   (a) Show that, $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$, where $y_k = \dfrac{d^k y}{dx^k}$.
   Hint 1: Differentiate $n+1$ times the expression $(1+x^2)\dfrac{dy}{dx} = 1$ using Leibniz Rule.
   Hint 2: The third and all higher derivatives of $(1+x^2)$ are zeros. **[5 point]**

   (b) When $x = 0$, the result in Part (a) becomes $y_{n+2} + n(n+1)y_n = 0$.
   Moreover, at $x = 0$, $y_0 = \tan^{-1}(0) = 0$, and $y_1 = \dfrac{1}{1+0} = 1$.
   Show that, at $x = 0$, $y_{2m} = 0$, and $y_{2m+1} = (-1)^m(2m)!$ for all positive integer m.
   [5 point]

   (c) Use the above result only to state the Maclaurin expansion of $\tan^{-1}(x)$. **[3 point]**

   (d) Differentiate the Maclaurin expansion of $\tan^{-1}(x)$ obtained in Part (c) term-by-term to obtain the power series of $\dfrac{1}{1+x^2}$ in the neighbourhood of $x = 0$. **[4 point]**

   (e) Consider the geometric series $\dfrac{1}{1+u} = 1 - u + u^2 - u^3 + u^4 \ldots$ where $|u| < 1$.
   Use this to confirm the power series of $\dfrac{1}{1+x^2}$ obtained above in the neighbourhood of $x = 0$. **[3 point]**

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