THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code:	AMA1007	**Subject Title:**	Calculus and Linear Algebra
Session:	Semester 2, 2021/2022	**Assessment:**	Final Exam
Submission:	Blackboard	**Date and Time:**	10 May 2022, 12:30-14:30

This set of question has 5 pages (including this cover page).

Instructions: Attempt **ALL** questions in this paper. Your solutions to each question must be made in the designated area (inside the boxes) of the set of **Answer Sheets**. Scan your work properly into **one clear and readable single PDF file** with file size not exceeding 10MB, and submit it via Blackboard with file name as your name (surname first). Use the app Microsoft Office Lens to scan multiple pages into one single pdf file. Pages must be in the right order in sequence, with one page image per page in the submission file. Only one single submission will be accepted (it is the students' responsibility to check carefully before submitting). Late submissions will not be accepted. Students are required to sign the covering statement of the set of Answer Sheets for submission.

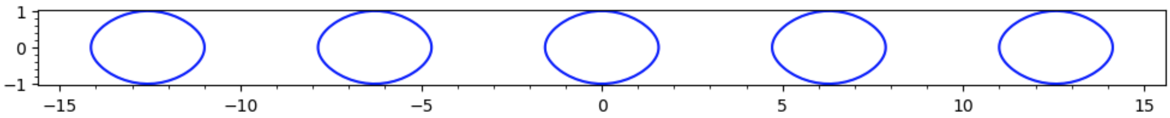
No marks will be given for those not following any of the above instructions.

Subject Examiners: Dr. LEE Heung Wing Joseph

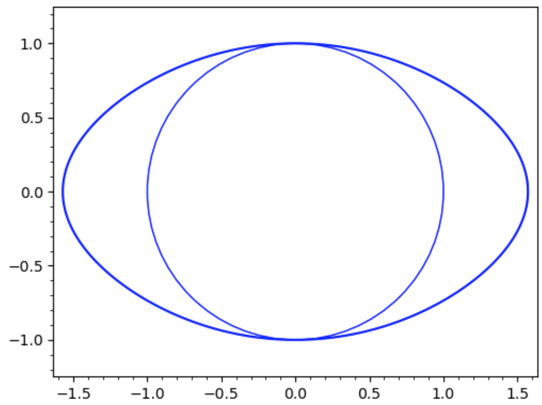
1. Consider the graph of $y^2 = \cos(x)$ (you may consider this graph as a combination of the two halves, namely, the upper half $y = +\sqrt{\cos(x)}$ and the lower half $y = -\sqrt{\cos(x)}$). Note that not all real value of x would have an associated real value of y, for example, there is no real y when $\frac{\pi}{2} < x < \frac{3\pi}{2}$. Consider the implicit plot using CoCalc:

In [1]:
```
var('x,y')
implicit_plot(y^2 == cos(x), (x,-15,15), (y,-1,1),ticks=[None,1.0],figsize=10)
```

Out[1]:



As shown in the graph, there are infinitely many closed curves, or loops. If we restrict values of x at $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$, then, there is only one loop centered at the origin. We consider this *centered* loop, and on the same graph, we also plot the unit circle centered at the origin, that is, $x^2 + y^2 = 1$. See the plot below, it resembles an *eye*. Note that the two closed curves touch each other tangentially at $(0, -1)$ and $(0, 1)$.



If we rotate these two closed curves about the x-axis, then, two closed surfaces are generated. Show that the total volume of the space in between the two surfaces is given by $\dfrac{2}{3}\pi$. **[10 points]**

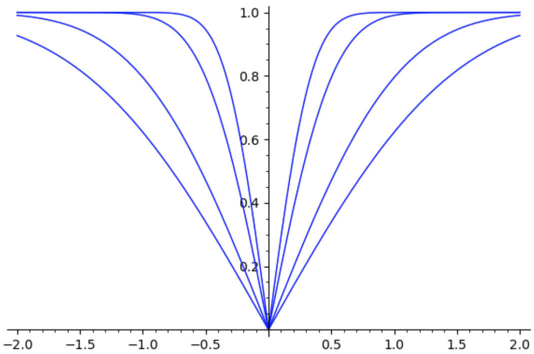
2. Consider a 3×100 matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 5 & 7 & \dots & 199 & 201 \\ 11 & 13 & 15 & 17 & \dots & 209 & 211 \\ 21 & 23 & 25 & 27 & \dots & 219 & 221 \end{bmatrix}$. There exist a minimum number of m 3×3 elementary matrices $\boldsymbol{E_1}$, $\boldsymbol{E_2}$, $\dots$, $\boldsymbol{E_m}$, so that the product $\boldsymbol{BA}$ is in reduced row-echelon form, where $\boldsymbol{B} = \boldsymbol{E_m E_{m-1}} \dots \boldsymbol{E_2 E_1}$.

   (a) Find all these m elementary matrices, and state $\boldsymbol{B}$. **[12 points]**
   Hint: $\boldsymbol{B}$ is a 3×3 matrix, and check the bottom row of $\boldsymbol{B}$ should be $\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$.

   (b) Note that each column of $\boldsymbol{A}$ is given by $\begin{bmatrix} 2k+1 \\ 2k+11 \\ 2k+21 \end{bmatrix}$ for $k = 0, 1, 2, ..., 100$.
   Prove that, the last/bottom row of $\boldsymbol{BA}$ consists entirely of zeros. **[5 points]**

(c) Consider $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{b} = \begin{bmatrix} 1 \\ 10 \\ 20 \end{bmatrix}$. Describe the solution space. **[3 points]**

3. Consider the graph of $f(x) = \sqrt{1 - e^{-a^2x^2}}$ where $0 < a < \infty$. Note that at $x = 0$, $f(x)$ is not differentiable with respect to x for any possible $a > 0$.



(a) Show that $\lim\limits_{u \to 0} \dfrac{u^2}{1 - e^{-u^2}} = 1$. Then, conclude that

$$\lim_{u \to 0^-} \frac{u}{\sqrt{1 - e^{-u^2}}} = \lim_{u \to 0^-} -\sqrt{\frac{u^2}{1 - e^{-u^2}}} = -1, \text{ and}$$

$$\lim_{u \to 0^+} \frac{u}{\sqrt{1 - e^{-u^2}}} = \lim_{u \to 0^+} \sqrt{\frac{u^2}{1 - e^{-u^2}}} = 1.$$ **[4 points]**

(b) For $x \neq 0$, show that $f'(x) = ae^{-a^2x^2} \dfrac{ax}{\sqrt{1 - e^{-a^2x^2}}}$. **[4 points]**

(c) Taking the left limit to $x = 0$ (that is to say, $\lim\limits_{x \to 0^-}$), and using the results from part (a) and part (b), show that the slope of the *left* tangent line of the graph of $f(x)$ at $x = 0$ is $-a$. **[4 points]**

(d) Taking the right limit of $x = 0$ (that is to say, $\lim\limits_{x \to 0^+}$), and using the results from part (a) and part (b), show that the slope of the *right* tangent line of the graph of $f(x)$ at $x = 0$ is a. **[4 points]**

(e) Use the results obtained in part (c) and part (d), show that the acute angle between these two tangent lines is given by **[4 points]**

$\theta = 2\tan^{-1}\left(\dfrac{1}{a}\right)$ when $a > 1$, and $\theta = 2\tan^{-1}(a)$ when $0 < a < 1$.

Students may or may not need the following hints:

Hint 1: Suppose θ is the acute angle between two straight lines with finite slope m_1 and m_2, then $\tan(\theta) = \dfrac{m_1 - m_2}{1 + m_1m_2}$.

Hint 2: The Tangent Half-angle formula is given by $\tan(\theta) = \dfrac{2\tan\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)}$.

Hint 3: If $c > 0$, then $\tan^{-1}(c) + \tan^{-1}\left(\dfrac{1}{c}\right) = \dfrac{\pi}{2}$.

4. Consider the indefinite integral $\displaystyle\int \frac{c\sin(x)+d\cos(x)}{a\sin(x)+b\cos(x)}\,dx$ where a, b, c, d are constants and $a^2+b^2>0$:

   (a) Show that the numerator in the integrand, $c\sin(x)+d\cos(x)$, can be expressed as $A(a\sin(x)+b\cos(x))+B(a\cos(x)-b\sin(x))$ where A and B are constants. Specifically, find A and B in terms of a, b, c, d only. **[7 points]**

   (b) Use the result obtained in part (a), and evaluate the indefinite integral in terms of A, B, a, b only, together with an integration constant. **[7 points]**

   (c) Use the result obtained in part (b) only to evaluate $\displaystyle\int \frac{1}{3+5\tan(x)}\,dx$. **[6 points]**

5. Consider the power series centered at the origin:
$f(x)=\displaystyle\sum_{n=0}^{\infty}\frac{x^{4n+1}}{4n+1}=x+\frac{x^5}{5}+\frac{x^9}{9}+\frac{x^{13}}{13}+\ldots.$
Note that some terms, x^2, x^3, x^4, x^6, x^7, x^8 $\ldots$ are missing, so we cannot use the usual way to compute its radius of convergence without rewriting it in the usual form first.

   (a) By factorizing x^1, rewrite the power series into $\displaystyle x\left(\sum_{n=0}^{\infty}\frac{x^{4n}}{4n+1}\right)$ and consider the series inside the brackets. Find a suitable substitution of a new variable y so that the power series inside the brackets has the usual form
$\displaystyle\sum_{n=0}^{\infty}a_n y^n=a_0+a_1y+a_2y^2+a_3y^3+\ldots.$ **[2 points]**

   (b) Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}a_n y^n$, and thus, find the radius of convergence of the original power series of x, that is, $f(x)$. **[4 points]**

   (c) By differentiating $f(x)$ term by term, and by using the geometric series, and by using partial fraction expansion, show that the sum of the power series of $f'(x)$ is given by

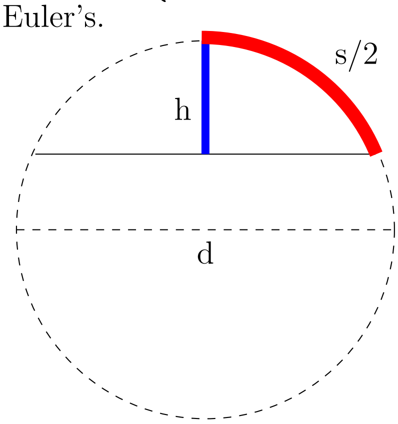
$$f'(x) = \frac{1/4}{1-x}+\frac{1/4}{1+x}+\frac{1/2}{1+x^2}.$$

   [6 points]

   (d) Note that $f(0)=0$. Find the sum of the power series of $f(x)$ by evaluating $\displaystyle f(x)=\int_0^x f'(t)\,dt$. **[8 points]**

6. [*Note: For those who would not like to read about the background of the question can skip it and go straight to the bottom paragraph*].

Background: In the middle of the *Edo period* 江戸時代 (1603 -1867 AD), ancient Japanese mathematicians were already mastering the art of ancient Chinese mathematics, and they were making their own great contributions. In 1722, a great mathematician Takebe Katahiro 建部賢弘 (1664 - 1739 AD) (a student of Seki Takakazu 關孝和) has published his famous work *Tetsujutsu Sankei* 綴術算經. (Note that, despite the similarity of its title in Chinese, one should not be confused this book with the lost ancient Chinese mathematics book entitled *Zhui Shu* 綴術 written by Zu Chongzhi 祖沖之 and his son at around 500 AD). (Takebe Katahiro was the mathematician commissioned by General Tokugawa Yoshimune 德川吉宗 to cartograph the map 《日本總繪圖》 from 1719 to 1723, one of the significant symbol of the Kyoho Reforms 享保改革). In Chapter 12 of *Tetsujutsu Sankei*, the square of half the arc with *sagitta* h in a circle of diameter d was considered, that is $f(h) = \left(\frac{s}{2}\right)^2 = \left(d\ \sin^{-1}\left(\sqrt{\frac{h}{d}}\right)\right)^2$. Takebe was calling this quantity the *definite half back arc squared* 定半背冪. A set of procedures was then presented in the book to give arbitrarily close numerical approximations to the value (square of arc-sine). Essentially, it was a power series. Rewriting the result using modern notation, we have $f(h) = dh\left\{1 + \sum_{n=1}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n+2)!}\left(\frac{h}{d}\right)^n\right\}$. Takebe's result was 15 years earlier than that of Euler's.



Note that h, the *sagitta* of a circular arc, is the distance from the center of the arc to the center of its base. The word *sagitta* came from Latin, and nowadays it is commonly used in geometry, architecture, and modern design. The etymology of this Latin word is *arrow*, and it is interesting to note that, in ancient Chinese mathematics text (and also the Japanese's), this quantity *sagitta* was called 矢, the meaning was also *arrow*.

Question: Consider the power series inside the braces, and let $x = \frac{h}{d}$, we have

$$\sum_{n=1}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n+2)!}x^n = \frac{1}{3}x + \frac{8}{45}x^2 + \frac{4}{35}x^3 + \frac{128}{1575}x^4 + \frac{128}{2079}x^5 + \frac{1024}{21021}x^5 + \frac{256}{6435}x^6 + \ldots.$$

Let $a_n = \frac{2^{2n+1}(n!)^2}{(2n+2)!}$ for $n = 1, 2, 3, \ldots$. Show that, the radius of convergence of this power series is 1. That is, show that $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 1$.

[10 points]

*** END ***