THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code:	AMA1007/AMA1708	**Subject Title:**	Calculus and Linear Algebra
Session:	Semester 1, 2021/2022	**Assessment:**	Final Exam
Date:	11 Dec 2021	**Time:**	12:30-14:30

This set of question has 5 pages (including this cover page).

Instructions: This paper has 5 Questions.

Attempt **ALL** questions in this paper.

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DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO.

1. Consider the two non-intersecting parabolas:

$$\begin{aligned} y = f(x) &= x^2 - 4x + 4 \\ y = g(x) &= -x^2 - 12x - 37 \end{aligned}$$

Let u be the x-coordinate of a point on f, and v be the x-coordinate of a point on g.

(a) For any given x-coordinate u on parabola f, there exist a unique v on parabola g, so that the tangent line on f at u, and the tangent line on g at v, are parallel. (Similarly, we have the inverse, that for any given v, there is a unique u, so that the two tangent lines are parallel). Express v as a function of u under this condition. [**4 points**]

(b) Let D be the square of the distance between two points, one on each of the two parabolas, $(u, f(u))$ and $(v, g(v))$, and the tangent line at u on f and the tangent line at v on g are parallel. Use the result obtained in **(a)** to express D as a function of u only, i.e. $D(u)$. [**5 points**]

(c) Consider the following CoCalc work:

In [1]:
```
f(x)=x^2-4*x+4
g(x)=-x^2-12*x-37
fdash(x)=diff(f(x),x)
gdash(x)=diff(g(x),x)
var('u v')
parallel=solve(fdash(u)==gdash(v),v)
v=parallel[0].rhs()
D(u)=(v-u)^2+(g(v)-f(u))^2
show(factor(diff(D(u),u)))
```

Out[1]:

$$16\left(u^2 - 5u + 8\right)(u - 1)$$

In [2]:
```
show(expand(diff(D(u),u,2)))
```

Out[2]:

$$48u^2 - 192u + 208$$

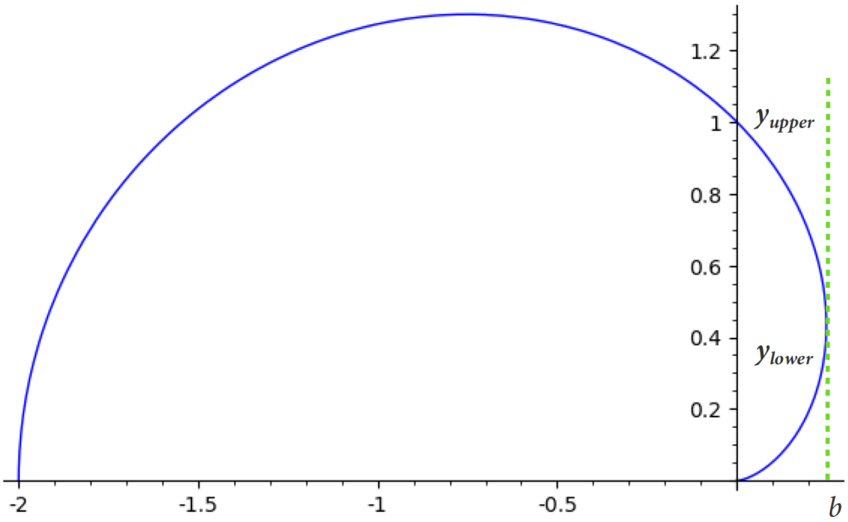
Use this result, or otherwise, find the minimum of $D(u)$. Include a second derivative test to confirm it is indeed a minimum. [**5 points**]

(d) State the minimum location u^* obtained in **(c)**, and the associated v^* according to the parallel condition obtained in **(a)**. Hence, list the points $(u^*, f(u^*))$ and $(v^*, g(v^*))$. [**4 points**]

(e) Briefly explain why the two points listed in **(d)** are the closest points between the two parabolas. Sketch a diagram if necessary. [**2 points**]

2. Recently, a pop song in Mandarin with the title *Fragile* has gone phenomenally viral over the internet world-wide. It is a duet, performed by a Malaysian hip-hop artist Namewee and an Australian singer Kimberley Chen. The music video of the song was premiered on 15 Oct 2021 on Youtube, and it has over 12 million views just after the first week, and over 20 million views after the second. Inspired by the song, this question aims to find the volume of a heart-shaped solid (or apple-shaped solid) made by fragile glass.

Consider the upper half of the well-known curve Cardioid given in polar coordinates $r(\theta) = 1 - \cos(\theta)$ where $0 \le \theta \le \pi$.



The Cardioid can also be expressed in Cartesian coordinates in parametric form

$$
\begin{aligned}
x(\theta) &= r(\theta)\cos(\theta) = (1-\cos(\theta))\cos(\theta), \\
y(\theta) &= r(\theta)\sin(\theta) = (1-\cos(\theta))\sin(\theta).
\end{aligned}
$$

(a) Explain why b can be found by solving $\frac{d}{d\theta}x(\theta) = 0$.
Show that $x = b = \frac{1}{4}$ when $\theta = \frac{\pi}{3}$ (or when $\cos(\theta) = \frac{1}{2}$). **[3 points]**

(b) Define $f(\theta) = (y(\theta))^2 \cdot \frac{d}{d\theta}x(\theta)$. Show that

$$
f(\theta) = \sin^2(\theta)(1-\cos(\theta))^2 \cdot (-\sin(\theta) + 2\cos(\theta)\sin(\theta))
$$

[3 points]

(c) Consider a heart-shaped solid of fragile glass generated by rotating the Cardioid about the x-axis. Explain why the volume of the fragile glass solid is given by

$$\begin{aligned} V &= \int_{x=-2}^{x=0} \pi\, y^2\, dx + \int_{x=0}^{x=b} \pi\, y_{upper}^2\, dx - \int_{x=0}^{x=b} \pi\, y_{lower}^2\, dx \\ &= \int_{\theta=\pi}^{\theta=\pi/2} \pi f(\theta)\, d\theta + \int_{\theta=\pi/2}^{\theta=\pi/3} \pi f(\theta)\, d\theta - \int_{0}^{\theta=\pi/3} \pi f(\theta)\, d\theta \\ &= \int_{\theta=\pi}^{\theta=\pi/2} \pi f(\theta)\, d\theta + \int_{\theta=\pi/2}^{\theta=\pi/3} \pi f(\theta)\, d\theta + \int_{\pi/3}^{\theta=0} \pi f(\theta)\, d\theta \\ &= \int_{\theta=\pi}^{\theta=0} \pi f(\theta)\, d\theta. \end{aligned}$$

[7 points]

(d) Note that the expression of $f(\theta)$ given in **(b)** can be reduced for the sake of integration. Consider the CoCalc reduction:

In [1]:
```
f(x)=sin(x)^2*(1-cos(x))^2*(2*cos(x)*sin(x)-sin(x))
show(f.reduce_trig())
```

Out[1]:

$$x \mapsto -\frac{1}{16}\sin(6\,x) + \frac{5}{16}\sin(5\,x) - \frac{1}{2}\sin(4\,x) - \frac{1}{16}\sin(3\,x) + \frac{19}{16}\sin(2\,x) - \frac{11}{8}\sin(x)$$

Using this, or otherwise, show that the volume obtained in **(c)** is given by $V = \frac{8}{3}\pi$. Keep your workings and answers in terms of π and simplified rational numbers only.

Hint: $\int_{\pi}^{0} \sin(kx)dx = 0$ when $k = 2, 4, 6, 8 ...$ **[7 points]**

3. Consider the rational function $f(x) = \dfrac{x - 100}{x^2 - 15x - 250}$.
In your answers to this question below, keep those values in exact and simplified rational number format only.

(a) Express $f(x)$ in partial fractions, i.e., $\dfrac{a}{x-\alpha} + \dfrac{b}{x-\beta}$. **[6 points]**

(b) Apart from using Taylor/Maclaurin series expansion, power series of $g(x) = \dfrac{d}{x-\gamma}$ at $x = 0$ can be obtained by letting $\dfrac{d}{x-\gamma} = \sum_{n=0}^{\infty} c_n\, x^n$ first. Then, multiply both sides by $x - \gamma$ to get

$$d = (x-\gamma)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 \dots) = -\gamma c_0 + \sum_{n=1}^{\infty} (c_{n-1} - \gamma c_n) x^n.$$

By comparing coefficients, we can get $c_0 = -\frac{d}{\gamma}$, and $c_n = \frac{c_{n-1}}{\gamma}$ for $n = 1, 2, ...$ The radius of convergence is given by $\lim_{n\to\infty} \dfrac{c_n}{c_{n+1}} = |\gamma|$.

Find the coefficients of the first three terms (i.e., x^0, x^1, and x^2) of the power series of each of the two partial fractions obtained in **(a)** at $x = 0$, and state the corresponding radius of convergence of each one. **[8 points]**

(c) Use the above, or otherwise, find the coefficients of the first three terms of the power series of $f(x)$ at $x = 0$, and state the radius of convergence. **[6 points]**

4. Consider the following system of linear equations:

$$\begin{aligned} x + y + \frac{1}{2}z &= 1997 \\ 166y + 247z &= 1984 \\ x + \frac{1}{2}y + 7z &= 2020 \end{aligned}$$

Use Gaussian-Jordan method only to solve the system, and state x, y, and z. Keep your workings and answers in simplified rational numbers format.

Hint: the answer x, y, and z are positive integers. **[20 points]**

5. Consider the function $f(x) = x \tan^{-1}(x)$.

(a) Show that $f'(x) = \dfrac{x}{x^2+1} + \tan^{-1}(x)$, and $f''(x) = \dfrac{2}{(x^2+1)^2}$.

Hint: $\dfrac{d}{dx} \tan^{-1}(x) = \dfrac{1}{x^2+1}$. **[3 points]**

(b) Show that $f'(x) \le 2\tan^{-1}(x)$ for $x > 0$.

That means, show that $\dfrac{x}{x^2+1} < \tan^{-1}(x) < x$ for $x > 0$.

Hint: Consider using the Mean Value Theorem for $\tan^{-1}(x)$ on $(0, x)$, and note that $\tan^{-1}(0) = 0$. **[3 points]**

(c) Note that $f'(x) = 0$ at $x = 0$. Explain why $x = 0$ is the only solution for $f'(x) = 0$.

Hint: Consider that fact that $f''(x)$ is strictly positive. **[2 points]**

(d) Explain why there is only one extremum, at $x = 0$, and it is a global minimum point. **[2 points]**

(e) Evaluate the indefinite integral $\displaystyle\int f(x)\,dx$.

Hint: Use Integration by Parts. **[6 points]**

(f) Use **(e)**, or otherwise, evaluate $\displaystyle\int_0^1 f(x)\,dx$. Keep your answer in terms of π and simplified rational numbers.

Hint: Note that $\tan^{-1}(1) = \dfrac{\pi}{4}$. **[4 points]**

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