**THE HONG KONG POLYTECHNIC UNIVERSITY**

**Department of Applied Mathematics**

**Subject Code:** AMA1007 **Subject Title:** Calculus and Linear Algebra

**Session:** Semester 1, 2018/2019

**Date:** Dec 21, 2018 **Time:** 12:30 - 14:30

**Time Allowed:** 2 hours

**This question paper has 6 pages** (including this page)

**Instructions:** This paper has **5** questions.

Attempt **ALL** questions in this paper.

**Subject Examiners:** Dr. LEE Heung Wing Joseph

**DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO.**

1. Suppose $f(x) = x^2 \sin(2x)$.

   (a) Consider the formula for the $k^{th}$ derivative of $\sin(2x)$:

$$\sin^{(k)}(2x) = 2^k \sin(2x + k(\frac{\pi}{2}))$$

   where $k$ is a positive integer, and

$$\sin(\theta + k(\frac{\pi}{2})) = \begin{cases} \cos(\theta), & \text{if } k = 4m+1 \text{ for some integer } m, \\ -\sin(\theta), & \text{if } k = 4m+2 \text{ for some integer } m, \\ -\cos(\theta), & \text{if } k = 4m+3 \text{ for some integer } m, \\ \sin(\theta), & \text{if } k = 4m \text{ for some integer } m. \end{cases}$$

   For example,

$$\text{if } k=1, \quad \frac{d}{dx}\sin(2x) = 2^1 \sin(2x + \frac{\pi}{2}) = 2\cos(2x), \text{ and}$$
$$\text{if } k=2, \quad \frac{d^2}{dx^2}\sin(2x) = 2^2 \sin(2x + \frac{2\pi}{2}) = -4\sin(2x), \text{ and}$$
$$\text{if } k=3, \quad \frac{d^3}{dx^3}\sin(2x) = 2^3 \sin(2x + \frac{3\pi}{2}) = -8\cos(2x), \text{ and}$$
$$\text{if } k=4, \quad \frac{d^4}{dx^4}\sin(2x) = 2^4 \sin(2x + \frac{4\pi}{2}) = 16\sin(2x).$$

   Using the above formula or otherwise, find $\sin^{(k)}(2x)$ for $k = 48, 49, 50$. Express your answers in terms of $\sin(2x)$ and $\cos(2x)$ only, and keep $2^{48}$, $2^{49}$, $2^{50}$ as they are without evaluating. Hint: $48 = 4 \times 12$, $49 = 4 \times 12 + 1$, $50 = 4 \times 12 + 2$. **[6 points]**

   (b) Find $f^{(50)}(x)$, i.e., find the $50^{th}$ derivative of $f(x)$. Express your answer in terms of $x^2\sin(2x)$, $x\sin(2x)$, $\sin(2x)$, $x^2\cos(2x)$, $x\cos(2x)$, $\cos(2x)$, only (not all the terms are needed), and keep $2^{48}$, $2^{49}$, $2^{50}$ as they are without evaluating. **[14 points]**

   Hint: First, consider the Leibniz rule $\frac{d}{dx}^{(n)}\{u(x)v(x)\} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)}(x)v^{(n-k)}(x)$.

   Let $u(x) = x^2$ and $v(x) = \sin(2x)$. Then, observe that $u^{(n)}(x) = 0$ for $n \geq 3$. Then, use the results obtained above.

2. Consider the improper rational function $f(x) = \dfrac{x^4 - 4x^2 + 6x + 6}{x^2 + x - 2}$.

   (a) Use long division to express $f(x)$ as a sum of a degree two polynomial and a proper rational function $g(x) = \dfrac{5x+4}{x^2+x-2}$. **[7 points]**

   (b) Since $x^2 + x - 2 = (x+2)(x-1)$, obtain the partial fraction expansion of $g(x)$, i.e. find $A$ and $B$ in the expression $g(x) = \dfrac{A}{x+2} + \dfrac{B}{x-1}$. **[7 points]**

   (c) Evaluate $\int f(x)dx$ using the results obtained above. **[6 points]**

3. The 8th chapter of the ancient Chinese text of **The Nine Chapters of the Mathematical Art** (《九章算術》卷八) is dedicated to Rectangular Arrays (方程). The 13th problem ( 第十三問 ) in the chapter is as follows:
今有五家共井。甲二綆不足如乙一綆，乙三綆不足如丙一綆，丙四綆不足如丁一綆，丁五綆不足如戊一綆，戊六綆不足如甲一綆，各得所不足一綆，皆逮。問井深、綆長，各幾何？ [Note: 綆= 繩]
English translation/descriptions to the problem :
There are five families sharing a well [*assuming the well is deep, i.e. from the ground level down to the water level, and no available single well drawing rope among the families would be long enough to reach; the well drawing ropes within each family are of the same length, but different families would have different lengths of ropes; the ropes can be joined together end to end to make extensions and the amount of ropes used in tiding knots when joining the ropes are of negligible length; and the size/length of the bucket is negligible*]. Joining two ropes of the first family is not long enough to draw water from the well, but if added one from the second family, it is then just enough. Joining three ropes of the second family is not long enough, but if added one from the third family, it is then just enough. Joining four ropes of the third family is not long enough, but if added one from the forth family, it is then just enough. Joining five ropes of the forth family is not long enough, but if added one from the fifth family, it is then just enough. Joining six ropes of the fifth family is not long enough, but if added one from the first family, it is then just enough. What is the depth of the well, and the different lengths of ropes from each family?

Let $x_i$ be the length of the rope of the $i$th family, for $i = 1, 2, 3, 4, 5$, and let $D$ be the depth the well. Thus,

$$\begin{aligned} 2x_1 + x_2 &= D \\ 3x_2 + x_3 &= D \\ 4x_3 + x_4 &= D \\ 5x_4 + x_5 &= D \\ 6x_5 + x_1 &= D \end{aligned}$$

Let $y_i = \frac{x_i}{D}$ for $i = 1, 2, 3, 4, 5$, we then have 5 linear equations with 5 unknowns

$$\begin{aligned} 2y_1 + y_2 &= 1 \\ 3y_2 + y_3 &= 1 \\ 4y_3 + y_4 &= 1 \\ 5y_4 + y_5 &= 1 \\ 6y_5 + y_1 &= 1 \end{aligned}$$

A student was attempting to solve the linear system using Gauss-Jordan elimination. Half way through, after performing Gaussian elimination but before performing Jordan back substitution, the student obtained the following augmented matrix:

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & | & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} & 0 & 0 & | & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{4} & 0 & | & \frac{1}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{5} & | & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 1 & | & \frac{76}{721} \end{bmatrix}$$

(a) Carry on with the process of Jordan back substitution from this augmented matrix to find $y_i$, $i = 1, 2, 3, 4, 5$. Suppose the depth of the well $D = 721$, find $x_i$ for $i = 1, 2, 3, 4, 5$. **[14 points]**

(b) Suggest another value of $D$ so that $x_i$ for $i = 1, 2, 3, 4, 5$ are also positive integers. **[6 points]**

4. James Gregory [1638-1675] was one of the mathematicians who contributed to The Fundamental Theorem of Calculus. In 1668, he published his work *Universal Part of Geometry*, in which he was discussing the work of Hendrik van Heuraet [1634-1660], relating slope of tangents of a curve and area under a curve. He stated a problem, that given a curve $y(t)$ is it possible to find another curve $u(t)$, so that the arc-length $u(t)$ from 0 to $x$ is of a constant ratio to the area under $y(t)$ from 0 to $x$. In other words, for some $c \neq 0$, given $y(t) \geq 0$, is it possible to find $u(t)$ so that for all $x \geq 0$,

$$c \int_0^x \sqrt{1 + \left(\frac{du}{dt}\right)^2}\, dt \quad = \quad \int_0^x y(t)\, dt\ ?$$

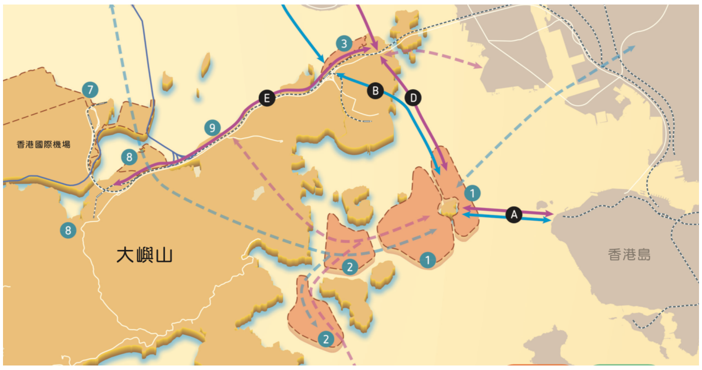
Consider the case of $c = 1$ and $f(t) = \left(\frac{1}{4}e^t\right) + e^{-t}$. In this question, we would like to show that if $y = f$, then $u = f$ satisfies the equation. Let $L(x) = \displaystyle\int_0^x \sqrt{1 + \left(f'(t)\right)^2}\, dt$, the arc-length of $f(t)$ from 0 to $x$, and let $F(x) = \displaystyle\int_0^x f(t)\, dt$, the area under $f(t)$ from 0 to $x$.

(a) Evaluate $\left(L'(x)\right)^2$ in terms of $e^{2x}$ and $e^{-2x}$ only. **[6 points]**

(b) Evaluate $\left(F'(x)\right)^2$ in terms of $e^{2x}$ and $e^{-2x}$ only. **[6 points]**

(c) Explain why we can say that $L(x) = F(x)$ for $x \geq 0$. **[8 points]**
Hint: use the results obtained above. Also check if $L(0) = F(0)$.

5. On 7 August 2018, a foundation published a research report proposing an unprecedented massive scale of reclamation of land to build a man-made island to the east of Lantau. Quoting from a consultant firm, the report claimed that the height of the sea waves around the region would not exceed 2m high. Shortly after the publication of the report, on 10 October 2018 during her policy address, the Chief Executive of Hong Kong also announced a similar plan reclaiming 1,700 hectares of land in that same location. The plan was entitled Lantau Tomorrow (明日大嶼), and the magic number 1,700 was not even given as an option to The Task Force on Land Supply (土地供應專責小組) while they were conducting the The Big Debate (土地大辯論) earlier in April soliciting public opinion on future land use. It was reported that the reclamation plan is expected to cost HK$500 billion. Lam Chiu Ying (林超英), former Director of The Hong Kong Observatory and a highly respected meteorologist, has been openly giving warnings about the idea of building a man-made island. He has raised numerous valid questions to the plan, from the projection of population and land demand, funds required, possible economic downturns, environmental impact, to the most important point of safety in the age of global climate change. As to the claim that the height of the sea waves around the region would not exceed 2m, Lam Chiu Ying's reaction was 「傻人才會信」(only a fool would believe that). Since then, and not forgetting the related devastating wave-hit damages brought by Typhoon Hato (天鴿) and Typhoon Mangkhut (山竹), the science of sea waves has gained a bit more public interests.




A conceptual picture of the reclamation plan, taken from the promotion leaflet of Lantau Tomorrow, The Chief Executive's 2018 Policy Address.

Suppose the velocity $v$ (in $m/s$) of idealized ocean wave is given by $v = \sqrt{\frac{g\lambda}{2\pi}\tanh(2\pi\frac{d}{\lambda})}$ where $\lambda$ is the wavelength measured from crest to crest (in $m$), $d$ is the depth of water (in $m$), $g = 10m/s^2$, and hyperbolic tangent is given by $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Consider the Maclaurin series $\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \dots$ where $|x| < \frac{\pi}{2}$. Note that if $|x|$ is sufficiently small, all 3rd or higher order terms would be negligible, and only the linear term (the first term) is dominating. Suppose wave enters the shallow water region, say, $d < \frac{\lambda}{20}$. Using Maclaurin series of $\tanh(x)$, explain why $v \approx \sqrt{gd}$. **[20 points]**

*This question is written by the Subject Lecturer Dr. Joseph Lee. It does not represent the political position of The Department of Applied Mathematics.*

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