THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code: AMA1007 **Subject Title:** Calculus and Linear Algebra

Session: Semester 1, 2017/2018

Date: Dec 11, 2017 **Time:** 12:30 - 14:30

Time Allowed: 2 hours

This question paper has 4 pages (including this page)

Instructions: This paper has **6** questions.

Attempt **ALL** questions in this paper.

Subject Examiners: Dr. LEE Heung Wing Joseph

DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO.

1. It has been widely reported that the HKSAR Government has been working aggressively to push forward the so-called **co-location arrangement** of the Guangzhou-Shenzhen-Hong Kong Express Rail Link (廣深港高鐵香港段「一地兩檢」方案). Under the proposed arrangement, parts of the Hong Kong terminal of the express rail link and the compartments of both inbound and outbound trains have fallen within the jurisdiction of mainland authorities, and mainland law enforcement officers are allowed to exercise their jurisdiction within Hong Kong's territory in these areas.

   Suppose the proposed co-location arrangement is implemented as it is, and suppose one day a local university student is travelling in the outbound train towards mainland then. Moreover, suppose the train is travelling at a constant speed of 200 km per hour along the path $y = x^2 - \frac{1}{8}\ln(x)$ in the increasing x direction for $x > \dfrac{1}{4}$, where x and y are measured in km. While the train is still in Hong Kong but feeling a bit too warm, at location ($x = 1$, $y = 1$), the student removed his jacket showing underneath he is wearing a T-shirt clearly displaying a political slogan. Although it is perfectly acceptable and legal in Hong Kong for wearing such a T-shirt as it is the freedom of expression Hong Kong always enjoys, it may not be the case for Chinese mainland. Suppose the student is then apprehended by the mainland law enforcement officers at location ($x = 3$, $y = 9 - \frac{\ln(3)}{8}$) while the train is still within Hong Kong.

   (a) Show that $\left(1 + (y')^2\right) = \left(2x + \dfrac{1}{8x}\right)^2$. **[2 points]**

   (b) Find the arc-length of $y = x^2 - \frac{1}{8}\ln(x)$ from $x = 1$, to $x = 3$, i.e. compute $\displaystyle\int_1^3 \sqrt{1 + (y')^2}dx$. **[8 points]**

   (c) How long (in seconds) does it take from the removal of the jacket to the student's apprehension? **[5 points]**

   This question is written by the Subject Lecturer Dr. Joseph Lee. It does not represent the political position of The Department of Applied Mathematics.

2. Find $\displaystyle\lim_{x\to 0^+} x^x$, give detailed workings for your answer.

   [Hint : if $y = x^x$, then $y = e^{x\ln(x)}$, and $\displaystyle\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+}\left(\frac{\ln x}{\frac{1}{x}}\right)$]. **[15 points]**

3. Find the indefinite integral $\displaystyle\int \frac{x^3 + 1}{x(x-2)(x-3)}dx$. **[20 points]**

4. The 8th chapter of **The Nine Chapters of the Mathematical Art** (《九章算術》卷八) is dedicated to Rectangular Arrays (方程). The 12th problem (第十二問) in the chapter is as follows:

> 今有武馬一匹，中馬二匹，下馬三匹，皆載四十石至阪，皆不能上。武馬借中馬一匹，中馬借下馬一匹，下馬借武馬一匹，乃皆上。問武、中、下馬一匹各力引幾何？
>
> There are three groups of horses. The first group has only one military grade horse (武馬), the second group has two medium grade horses (中馬), and the third group has three low grade horses (下馬). Each group alone cannot perform the task of pulling a total of equivalent of 40 stones of weight. Suppose one extra medium grade horse join the military grade group, or one extra low grade horse join the medium grade group, or one extra military grade horse join the low grade group, then all of these groups with an additional horse can now just perform the task. How many stones of weight can each horse of each grade pull?

Let x_1 be the number of stones of weight a military grade horse can pull, x_2 be the number of stones of weight a medium grade horse can pull, and x_3 be the number of stones of weight a low grade horse can pull.

(a) Briefly explain why you can obtain the following system of 3 linear equations with 3 unknowns, **[3 points]**

$$\begin{aligned} x_1 + \quad x_2 \qquad &= 40, \\ 2x_2 + \quad x_3 &= 40, \\ x_1 \qquad + 3x_3 &= 40. \end{aligned}$$

(b) Solve the system using Gauss-Jordan elimination and find the number of stones of weight each horse of each grade can pull. **[17 points]**

5. Suppose it is not easy, or maybe even impossible, for a certain curve given by $y = f(x)$ to be expressed explicitly as $x = f^{-1}(y)$. Consider the solid obtained by rotating about the y-axis the region below the curve $y = f(x)$, and above the x-axis from $x = a$ to $x = b$ where $0 \leq a < b$. Without finding the explicit expression $x = f^{-1}(y)$, the volume of the solid can be expressed by $V = \displaystyle\int_a^b 2\pi x f(x) dx$. This is called the method of cylindrical shells. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curve $y = (x-1)(x-3)^2$ and the x-axis, from $x = 1$ to $x = 3$. **[15 points]**

6. Consider the commonly used geometric series

$$\frac{1}{1-x} \quad = \quad 1 + x + x^2 + x^3 + x^4 + \dots \quad \text{for} \quad |x| < 1,$$

and by replacing x with $-x$, one can easily obtain another expression :

$$\frac{1}{1+x} \quad = \quad 1 - x + x^2 - x^3 + x^4 - \dots.$$

In 1703, Luigi Guido Grandi, a professor at Pisa, considered this expression in his published work. By setting $x = 1$, he believed he has *obtained* 0 on one side, but $\frac{1}{2}$ instead on the other side, i.e.,

$$\begin{aligned} \frac{1}{2} = \frac{1}{1+1} \quad &= \quad 1 - 1 + 1 - 1 + 1 + \dots \\ &= \quad (1-1) + (1-1) + \dots \; = \; 0 + 0 + 0 + \dots. \end{aligned}$$

In the first draft of his work, he commented that this is an illustration, that the world could be created out of nothing (attempting to use mathematics *tricks* to back a religious view point on creation). The comment was later removed upon the editor's request before publication.

(a) Note that Maclaurin Series Expansion (Taylor Series expansion at $x = 0$) is given by

$$f(x) \quad = \quad f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \dots + \frac{f^{(n)}(0)}{n!}(x-0)^n + \dots$$

State the nth derivative of $\dfrac{1}{1+x}$, i.e. $f^{(n)}(x)$, and obtain the Maclaurin Series expansion for $\dfrac{1}{1+x}$ to confirm the above expression used by Grandi. **[10 points]**

(b) State what was wrong with Grandi's result on obtaining $\frac{1}{2}$ on one side and 0 on the other. **[5 points]**

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