THE HONG KONG POLYTECHNIC UNIVERSITY

DEPARTMENT OF APPLIED MATHEMATICS

Subject Code: AMA1007 Subject Title: Calculus and Linear Algebra

Session: Semester 2, 2016/2017

Date: May 15, 2017 Time: 12:30 p.m. – 2:30 p.m.

Time Allowed: TWO hours

This question paper has 4 pages (including this page).

Instructions to Candidates:

This question paper has 7 questions.
Attempt **ALL** questions.
Full marks will only be given to solutions with adequate explanations.

Subject Examiner: Dr. Lee Yu Chung, Eugene

DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO

Question 1 (10 marks)

Let $f(x)=\begin{cases} \dfrac{\cos x}{x^2-(\pi^2/4)} & \text{for } x\neq -\dfrac{\pi}{2} \text{ and } \dfrac{\pi}{2} \\ -\dfrac{1}{\pi} & \text{for } x=-\dfrac{\pi}{2} \text{ or } \dfrac{\pi}{2} \end{cases}$.

(a) Evaluate $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\cos x}{x^2-(\pi^2/4)}$ and determine whether $f(x)$ is continuous at $x=\dfrac{\pi}{2}$.

[4 marks]

(b) By computing the value of $f'\left(\dfrac{\pi}{6}\right)$, estimate the value of $f\left(\dfrac{7\pi}{36}\right)$, correct to 6 decimal places, using differentials (linear approximation). [6 marks]

Question 2 (15 marks)

Solve the system of linear equations by Gaussian elimination:

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 2 & 0 & 4 & 4 & 2 \\ 2 & -1 & 2 & 3 & 2 \\ 1 & 1 & 4 & 3 & 0 \end{bmatrix}\mathbf{x}=\begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix}.$$

Express the solution in vector form/parametric form and describe the solution space.

Question 3 (15 marks)

Find all the eigenvalues and the associated eigenvectors of $\mathbf{A}=\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

Question 4 (10 marks)

Consider a continuous, non-decreasing function $f(x)$, for $0\le x\le 4$ such that $f(0)=0$ and $f(4)=8$, and $2x\le f(x)\le 6x-x^2$. Consider the definite integral $I=\int_0^4 f(x)\,dx$.

(a) What is the smallest possible value for I? Write out the function $f(x)$ that yields this smallest value. [5 marks]

(b) What is the largest possible value for I? Write out the function $f(x)$ that yields this largest value. [5 marks]

Question 5 (10 marks)

Evaluate $\int \frac{1}{x(x^2+1)^2}\,dx$.

Question 6 (15 marks)

Find the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{5n^2 x^n}{4^n(n^3+2)}$ and justify whether the series is convergent at each of the end points.

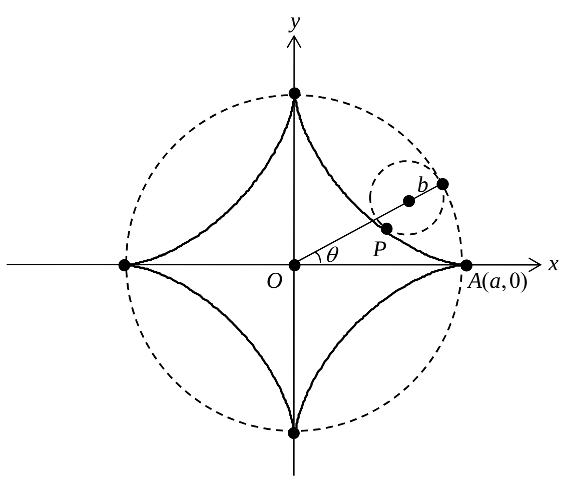
Question 7 (25 marks)

Historical note: Hypocycloids have a long history. According to the History of Mathematics Archive hosted by the University of St. Andrews, hypocycloids have been studied by many great mathematicians, including Girard Desargues, Gottfried Leibniz, Isaac Newton (both Leibniz and Newton were credited with the invention of calculus), Jacob Bernoulli, Johann Bernoulli, Daniel Bernoulli, and Leonhard Euler. Here we present one interesting theorem about hypocycloids, discovered by Daniel Bernoulli in 1725, called the **Double Generation Theorem**.

Theorem: Let *r* and *R* be positive numbers such that $r < R$. A circle of radius *r*, and a circle of radius $R - r$, when rolling inside a circle of radius *R*, generate the same hypocycloid.

The question

Consider a circle with radius b rolling inside a fixed circle of radius a, where $a > b$. A point P on the circumference of the rolling circle traces out a curve called a hypocycloid. If $b = \frac{a}{4}$, the parametric equations of the hypocycloid is given by $x = a\cos^3\theta$ and $y = a\sin^3\theta$ (θ in radian), and the shape traced out is called a four-cusped hypocycloid as shown in the diagram below. The Cartesian equation is given by $x^{2/3} + y^{2/3} = a^{2/3}$.



(a) Use implicit differentiation to obtain $\dfrac{dy}{dx}$ of the curve $x^{2/3} + y^{2/3} = a^{2/3}$. [4 marks]

(b) Using the result in part (a), set up and evaluate the integral for the perimeter of the four-cusped hypocycloid. (Hint: You may first consider the arc length traced by the hypocycloid in the first quadrant) [7 marks]

(c) By considering the parametric equations, find $\dfrac{dx}{d\theta}$ and $\dfrac{dy}{d\theta}$. [4 marks]

(d) The surface area S of the surface obtained by revolving the top half of the four-cusped hypocycloid about the x-axis is given by $S = \int_0^{\pi/2} 4\pi y \sqrt{\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2}\, d\theta$.

Show that S can be expressed in the form $\int_0^{\pi/2} 12\pi a^2 \cdot \cos^n \theta \cdot \sin^4 \theta \, d\theta$, and determine the value of n. [5 marks]

(e) Compute the surface area S as expressed in part (d). [5 marks]

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