THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code: AMA1007 **Subject Title:** Calculus and Linear Algebra

Session: Semester 1, 2016/2017

Date: Dec 22, 2016 **Time:** 12:30 - 14:30

Time Allowed: 2 hours

This question paper has 5 pages (including this page)

Instructions: This paper has **6** questions.

Attempt **ALL** questions in this paper.

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DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO.

1. Article 158 of The Basic Law allows the Hong Kong courts to seek interpretations (提請釋法) of the relevant provisions of the Basic Law from the Standing Committee of the National People's Congress (NPCSC, 全國人大常委會) through the Hong Kong Court of Final Appeal (CFA, 終審法院). In the past, out of the five times the NPCSC issued interpretations, only one of them was initiated by CFA (the other four were not initiated by Hong Kong courts). It has been heatedly argued by the Pan-Democracy Camp, that there are impacts on people's confidence in the rule of law in Hong Kong every time the NPCSC issue interpretations to the Basic Law not initiated by CFA. In view of this sentiment, we are proposing the following model to illustrate the idea. Suppose x_1 is the proportion of those having full confidence of the rule of law before an issue of an interpretation by the NPCSC not initiated by CFA and y_1 is the proportion of those after, and x_2 is the proportion of having no strong opinion to the rule of law before and y_2 is the proportion of those after, and x_3 is the proportion of those having pessimistic views before and y_3 is the proportion of those after. Thus, $x_1 + x_2 + x_3 = 1$ and $y_1 + y_2 + y_3 = 1$. Let $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Suppose we have $\boldsymbol{Px} = \boldsymbol{y}$ where $\boldsymbol{P} = \begin{bmatrix} 0.8 & 0 & 0 \\ 0.1 & 0.9 & 0 \\ 0.1 & 0.1 & 1 \end{bmatrix}$.

   This question is written by the Subject Lecturer Dr. Joseph Lee. It does not represent the political position of The Department of Applied Mathematics.

   (a) Initially, suppose all are having full confidence in the rule of law, i.e. $\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Compute $\boldsymbol{Pv}$ and $\boldsymbol{P}^2\boldsymbol{v}$. **[4 points]**

   (b) It can be shown that after infinitely many times of such issues of interpretations, $\lim_{n\to\infty} \boldsymbol{P}^n\boldsymbol{v}$ converges to a stationary vector $\boldsymbol{z}$, and $\boldsymbol{Pz} = \boldsymbol{z}$. In fact, $\boldsymbol{z}$ is also an eigenvector of $\boldsymbol{P}$. What is its associated eigenvalue λ? Give you answer with explanations. **[5 points]**

   (c) Find $\boldsymbol{z}$ (with column sum 1). **[11 points]**

2. Find $\lim_{x\to 1}\left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right)$, give detailed workings for your answer.
   [Hint : $\lim_{x\to a}\left(\frac{1}{f(x)} - \frac{1}{g(x)}\right) = \lim_{x\to a}\left(\frac{g(x)-f(x)}{f(x)g(x)}\right)$]. **[15 points]**

3. Consider the integral $\int x^3\sqrt{4-x^2}\,dx$.

   (a) Let $u = x^2$, and $dv = x\sqrt{4-x^2}dx$. Find v and du. **[7 points]**

   (b) Solve the integral using Integration by Parts. **[8 points]**

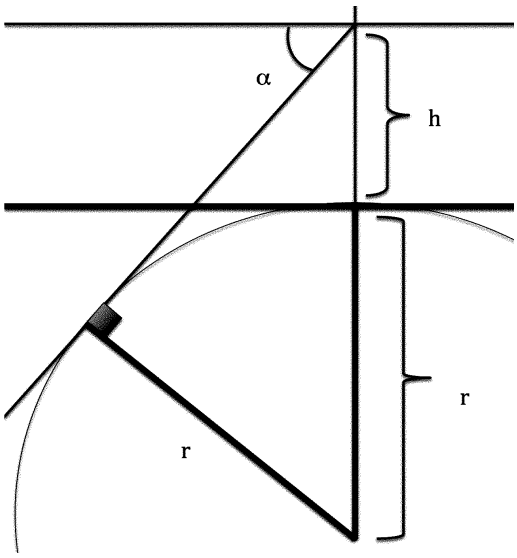
4. Suppose interest rate is r. The present value (value as of now) of any amount $\$A$ at the end of the nth period in the future is given by $\frac{A}{(1+r)^n}$. Consider a linearly increasing perpetuity, i.e., at the end of the nth period (e.g. end of the nth month, or end of the nth year), an amount of $\$n$ is given (that means, at the end of the 1st period, \$1 is given, at the end of the 2nd period, \$2 is given, ...etc). Thus, the present value of this linearly increasing perpetuity is given by $\sum_{n=1}^{\infty} \frac{n}{(1+r)^n}$. This can be thought of as the power series $\sum_{n=1}^{\infty} nx^n$, with $x = \frac{1}{1+r}$.

   (a) Consider the well known infinite geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ with radius of convergence 1. Obtain another power series by differentiating this geometric series, and state its radius of convergence. **[5 points]**

   (b) Given any power series formula $\sum_{n=1}^{\infty} a_n x^n = f(x)$ with radius of convergence R. We can generate another power series by multiplying both sides by x, that means $\sum_{n=1}^{\infty} a_n x^{n+1} = xf(x)$, and the radius of convergence would be the same. Using this result and what you have obtained above for $g'(x)$, obtain the power series formula for $\sum_{n=1}^{\infty} nx^n$. **[5 points]**

   (c) Assuming interest rate r can take any real value (possibly negative). Determine the range of r where the series $\sum_{n=1}^{\infty} \frac{n}{(1+r)^n}$ converges, and find the expression of the converged sum in terms of r. **[10 points]**

5. There were early historical records on mathematicians and scientists attempting to estimate the radius of the earth. Perhaps the first was the ancient Greek mathematician Eratosthenes (276-194 BC), who made use of the distance between the two cities Alexandria and Syene in his calculations (which could also be an extremely tedious task at the time). The second attempt was made a thousand years later, by the great Islamic mathematician and scientist al-Biruni (973-1048). It is believed that the unprecedented accuracy and precision of al-Biruni's technique was not known to the western world until the 16th century. Basically, al-Biruni's technique is as shown in the figure (standing on top of a mountain with known height (or a separately estimated height) h and made measurement on the angle α using an apparatus called astrolabe).



It can be shown that radius of the earth r can be computed by $r = \dfrac{h\cos(\alpha)}{1-\cos(\alpha)}$. al-Biruni's estimation of the radius of the earth was 6335725m. Suppose $\alpha = 0.01$ (radian), thus $\sin(\alpha) \approx 0.01$. Also, suppose $h = 305.1$m. How sensitive is the estimation of r with respect to the measurement of α? (i.e. compute the approximate value of $\dfrac{dr}{d\alpha}$. Hint : it could be easier to express $\cos(\alpha)$ in terms of h and r first.). Moreover, suppose there is a slight error in the measurement of α, say, $\triangle\alpha = 0.0001$, what is your approximate estimate of $\triangle r$? **[10 points]**

6. The 8th chapter of **The Nine Chapters of the Mathematical Art** (《九章算術》卷八) is dedicated to Rectangular Arrays (方程). The last problem (i.e. the 18th problem) (第十八問) in the chapter is as follows:

> 今有麻九斗、麥七斗、菽三斗、荅二斗、黍五斗，直錢一百四十；麻七斗、麥六斗、菽四斗、荅五斗、黍三斗，直錢一百二十八；麻三斗、麥五斗、菽七斗、荅六斗、黍四斗，直錢一百一十六；麻二斗、麥五斗、菽三斗、荅九斗、黍四斗，直錢一百一十二；麻一斗、麥三斗、菽二斗、荅八斗、黍五斗，直錢九十五。問一斗直幾何？

The problem was to find the cost per *dou* (斗, unit of volume) of each type of the grain of five different crops . The crops were namely, Hemp (麻), Wheat (麥), Soybean (菽), Adzuki bean (荅), and Broomtail Millet (黍). (For simplicity in the question, we give them code-names A, B, C, D, and E respectively). Also, for simplicity in the question, we denote the unit of measuring monetary value by the sign "$" (錢). Problem:

9 *dou* of A, 7 *dou* of B, 3 *dou* of C, 2 *dou* of D, 5 *dou* of E, cost \$140.
7 *dou* of A, 6 *dou* of B, 4 *dou* of C, 5 *dou* of D, 3 *dou* of E, cost \$128.
3 *dou* of A, 5 *dou* of B, 7 *dou* of C, 6 *dou* of D, 4 *dou* of E, cost \$116.
2 *dou* of A, 5 *dou* of B, 3 *dou* of C, 9 *dou* of D, 4 *dou* of E, cost \$112.
1 *dou* of A, 3 *dou* of B, 2 *dou* of C, 8 *dou* of D, 5 *dou* of E, cost \$95.
How much does it cost for a *dou* of each?

Let x_1 be the cost of A, x_2 be the cost of B, x_3 be the cost of C, x_4 be the cost of D, and x_5 be the cost of E.

(a) Briefly explain why you can obtain the following system of 5 linear equations with 5 unknowns, **[3 points]**

$$\begin{aligned} 9x_1 + 7x_2 + 3x_3 + 2x_4 + 5x_5 &= 140, \\ 7x_1 + 6x_2 + 4x_3 + 5x_4 + 3x_5 &= 128, \\ 3x_1 + 5x_2 + 7x_3 + 6x_4 + 4x_5 &= 116, \\ 2x_1 + 5x_2 + 3x_3 + 9x_4 + 4x_5 &= 112, \\ x_1 + 3x_2 + 2x_3 + 8x_4 + 5x_5 &= 95. \end{aligned}$$

(b) Suppose a student is solving the system using Gauss-Jordan elimination, and half way through obtained the following augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 3 & 2 & 8 & 5 & 95 \\ 0 & 1 & 1 & 7 & 6 & 78 \\ 0 & 0 & 1 & 14 & 16 & 169 \\ 0 & 0 & 0 & 1 & \frac{11}{8} & \frac{53}{4} \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

Using only the last two rows of this augmented matrix to find x_5, the cost of E, Broomtail Millet (黍), and x_4, the cost of D, Adzuki bean (荅). **[6 points]**

(c) Complete the Gauss-Jordan elimination of the above augmented matrix. You must keep your solutions in exact rational numbers (marks will not be given to solutions expressed in terms of truncated decimal numbers). **[10 points]**

(d) What is x_1, the cost of A, Hemp (麻)? **[1 points]**

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