THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code: AMA1007 **Subject Title:** Calculus and Linear Algebra

Session: Semester 2, 2015/2016

Date: Apr 29, 2016 **Time:** 12:30 - 14:30

Time Allowed: 2 hours

This question paper has 5 pages (including this page)

Instructions: This paper has **6** questions.

Attempt **ALL** questions in this paper.

Subject Examiners: Dr. LEE Heung Wing Joseph

DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO.

1. In the post- Umbrella Movement era (後雨傘時代), the political inclinations of general voters towards the Pan-Democracy Camp (泛民) have been refined into two groups, namely, **M**, the Moderate Democracy (傳統温和民主派), and **R**, and the Radical Democracy or Localism (激進民主派或本土派). Suppose 60% of the voters who voted for **M** in the last election would vote for **M** again in the next election, and 30% would vote for **R** instead. Moreover, suppose 10% of the voters who voted for **R** in the last election would vote for **M** in the next election instead, and 90% would vote for **R** again. Further more, 90% of those who voted for neither in the last election would remain voting for neither, and only 10% would vote for **M** instead in the next election. Let $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, where x_1, x_2, and x_3 are the numbers of people who voted for **M**, **R**, and neither, respectively, in the last election, and y_1, y_2, and y_3 are the numbers of people who would vote for **M**, **R**, and neither, respectively, in the next election. We are assuming a constant total voting population N, thus, $x_1+x_2+x_3 = y_1+y_2+y_3 = N$, and x_i or y_i may not be integers for $i = 1, 2, 3$.

This question is written by the Subject Lecturer Dr. Joseph Lee. It does not represent the political position of The Department of Applied Mathematics.

   (a) Briefly explain why we have $\boldsymbol{P}\boldsymbol{x} = \boldsymbol{y}$ where $\boldsymbol{P} = \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.3 & 0.9 & 0 \\ 0.1 & 0 & 0.9 \end{bmatrix}$. **[2 points]**

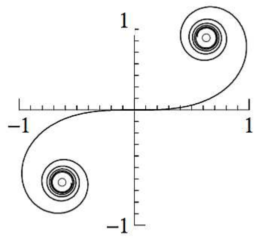
   (b) Consider a constant total voting population of 1000. Suppose $\boldsymbol{x} = \begin{bmatrix} 400 \\ 100 \\ 500 \end{bmatrix}$. Note that number of voters for $\boldsymbol{R}$ is the smallest. Find the number of voters for each group after one as well as after two elections, i.e. compute $\boldsymbol{P}\boldsymbol{x}$ and $\boldsymbol{P}^2\boldsymbol{x}$. **[2 points]**

   (c) It can be shown that after infinitely many elections (assuming a constant voting population), $\lim_{n\to\infty} \boldsymbol{P}^n\boldsymbol{x}$ converges to a stationary vector $\boldsymbol{z}$, and $\boldsymbol{P}\boldsymbol{z} = \boldsymbol{z}$. In fact, $\boldsymbol{z}$ is also an eigenvector of $\boldsymbol{P}$. What is its associated eigenvalue λ? **[5 points]**

   (d) Find $\boldsymbol{z}$ (with column sum 1000). Which voting group is the biggest? **[11 points]**

2. Consider the curve defined by $(x(t), y(t))$ for $t \in (-\infty, \infty)$, where $x(t)$ and $y(t)$ are given by the integrals $x(t) = \int_0^t \cos(u^2)\, du$, and $y(t) = \int_0^t \sin(u^2)\, du$. (These integrals are called **Fresnel Integrals**). According to the history given by Raphael Linus Levien in his PhD Thesis (entitled *From Spiral to Spline: Optimal Techniques in Interactive Curve Design*, UC Berkeley 2009), this curve has been called by different names in the literature, e.g. **Euler Spiral**, or **clothoid**, or **Cornu spiral**. In 1694, James Bernoulli posed a problem called elastica, and provided the geometric construction of this curve as a solution. However, James Bernoulli might not be aware of the shape of the curve,

as no figures of this curve were drawn then. In 1744, Leonhard Euler gave an accurate account of the details of the properties of the curve, including finding the limiting location of $(x(t), y(t))$, i.e. $\lim_{t\to\infty} x(t) = \sqrt{\frac{\pi}{8}}$ and $\lim_{t\to\infty} y(t) = \sqrt{\frac{\pi}{8}}$. Around 1818, Augustin Fresnel considered a totally different problem on light diffraction, and independently (unaware of Euler's work) re-derived these integrals. Continued with Fresnel's work, Alfred Marie Cornu plotted the spiral accurately in 1874. In 1899, Arthur Talbot, also unaware of Euler's work, arrived with the curve in his work in *The Railway Transition Spiral.*



(a) Recall the cosine series $\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n}}{(2n)!}$. Use this result to obtain a series for $\cos(u^2)$ in terms of u. **[3 points]**

(b) Recall the sine series $\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+1}}{(2n+1)!}$. Use this result to obtain a series for $\sin(u^2)$ in terms of u. **[3 points]**

(c) Obtain a series for $x(t) = \int_0^t \cos(u^2)\,du$ in terms of t. [Hint: you might like to consider term-by-term integration]. **[7 points]**

(d) Obtain a series for $y(t) = \int_0^t \sin(u^2)\,du$ in terms of t. [Hint: you might like to consider term-by-term integration]. **[7 points]**

3. Consider the graph of $f(x) = \dfrac{x^2}{(x-2)(x-1)(x+1)(x+2)}$.

(a) Find the x and y intercepts. **[1 points]**

(b) Determine if $f(x)$ is even or odd, or neither. **[1 points]**

(c) Find the horizontal asymptotes, if there is any. **[1 points]**

(d) Find the vertical asymptotes, if there is any. **[4 points]**

(e) Find $f'(x)$ and determine the regions where $f(x)$ is increasing/decreasing. **[4 points]**

(f) Find all local maximum and minimum points. **[3 points]**

(g) Given that $f''(x) = \dfrac{2(3x^8 + 5x^6 - 48x^4 + 60x^2 + 16)}{(x^2-1)^3(x^2-4)^3}$. Determine the regions where f is convex/concave. [Hint: $3x^8 + 5x^6 - 48x^4 + 60x^2 + 16 > 0$ for all x.] **[3 points]**

(h) Sketch the graph of $y = f(x)$. **[3 points]**

4. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = x^2 - 1$ from -1 to 1. **[10 points]**

5. The 8th chapter of **The Nine Chapters of the Mathematical Art** (《九章算術》卷八) is dedicated to Rectangular Arrays (方程). The 14th problem (第十四問) in the chapter is as follows:

   今有白禾二步、青禾三步、黃禾四步、黑禾五步，實各不滿斗。
   白取青、黃，青取黃、黑，黃取黑、白，黑取白、青，各一步，而實滿斗。
   問白、青、黃、黑禾實一步各幾何？

   English translation:

   There are 2 units (步) of white grain rice, 3 units of green grain rice, 4 units of yellow grain rice, 5 units of black grain rice, and and the volume of each is less than 1 *dou* (斗, volume unit).
   Adding 1 unit each of green and yellow grain rice into the white grain rice, or adding 1 unit each of yellow and black grain rice into the green grain rice, or adding 1 unit each of black and white grain rice into the yellow grain rice, or adding 1 unit each of white and green grain rice into the black grain rice, makes 1 *dou* each.
   What is the volume (in *dou*) of 1 unit of each colour grain rice?

   Let x_1 be the volume of 1 unit of white grain rice, x_2 be the volume of 1 unit of green grain rice, x_3 be the volume of 1 unit of yellow grain rice, and x_4 be the volume of 1 unit of black grain rice. At the beginning, the problem states that $2x_1 < 1$, $3x_2 < 1$, $4x_3 < 1$, and $5x_4 < 1$.

   (a) Briefly explain why you can obtain the following system of 4 linear equations with 4 unknowns, **[5 points]**

$$
\begin{aligned}
2x_1 + x_2 + x_3 \qquad &= 1, \\
3x_2 + x_3 + x_4 &= 1, \\
x_1 \qquad + 4x_3 + x_4 &= 1, \\
x_1 \ + x_2 \qquad + 5x_4 &= 1.
\end{aligned}
$$

   (b) Solve the system using Gauss-Jordan elimination. You must keep your solutions in exact rational numbers (marks will not be given to solutions expressed in terms of truncated decimal numbers). **[12 points]**

   (c) Use your answer obtained in **(b)** and verify that indeed $2x_1 < 1$, $3x_2 < 1$, $4x_3 < 1$, and $5x_4 < 1$.

   [3 points]

6. Suppose a bouncing ball is dropped from a height of 6 meters and it begins to bounce. The height of each bounce is $\frac{3}{4}$ that of the preceding bounce. Find the total distance travelled by the ball (assuming perfectly vertical and infinitely many bounces). **[10 points]**

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