THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code: AMA1007 **Subject Title:** Calculus and Linear Algebra

Session: Semester 1, 2015/2016

Date: Dec 15, 2015 **Time:** 12:30 - 14:30

Time Allowed: 2 hours

This question paper has 5 pages (including this page)

Instructions: This paper has **8** questions.

Attempt **ALL** questions in this paper.

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DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO.

1. During The Umbrella Movement (雨傘運動) occurred from late September to early December 2014, it was reported that access to the internet by all mobile devices were lost within the occupied areas (with speculations of possible government involvements), and in order to stay connected, protesters have adopted an alternative way. They were using the app FireChat to communicate and to broadcast messages among themselves, relying only on the Bluetooth capabilities of the smart phones. Suppose two phones, A and B, were far away from the rest of the crowd, and were found to be able to communicate with each other within $\sqrt{80}$ meters at most. Originally, phone A was at location $(0,0)$, and phone B was at $(1,1)$. While phone A was kept stationary, phone B was moving in a constant low speed following the path $y^2 = x^3$ in the increasing x direction.

This question is written by the Subject Lecturer Dr. Joseph Lee. It does not represent the political position of The Department of Applied Mathematics.

   (a) Find the location (x_f, y_f) on the path that the phones would stopped communicating with each other when phone B has gone passed beyond this location. [Hint: you may find the factorization $x^3 + x^2 - 80 = (x-4)(x^2+5x+20)$ useful.] **[3 points]**

   (b) Briefly explain why $\int_1^{x_f} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^{y_f} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$, i.e. briefly explain what these two terms (the LHS and the RHS) are trying to measure? **[2 points]**

   (c) Evaluate the total distance phone B has travelled while staying connected with phone A. **[8 points]**

   (d) Suppose phone B has been travelling at $\frac{1}{2}$ meter per second, how long has it been staying connected with phone A? **[2 points]**

2. Consider the graph of $f(x) = \dfrac{2x^2}{x^2-1}$.

   (a) Find the x and y intercepts. **[1 points]**

   (b) Determine if $f(x)$ is even or odd, or neither. **[1 points]**

   (c) Find the horizontal asymptotes, if there is any. **[2 points]**

   (d) Find the vertical asymptotes, if there is any. **[2 points]**

   (e) Find $f'(x)$ and determine the regions where $f(x)$ is increasing/decreasing. **[2 points]**

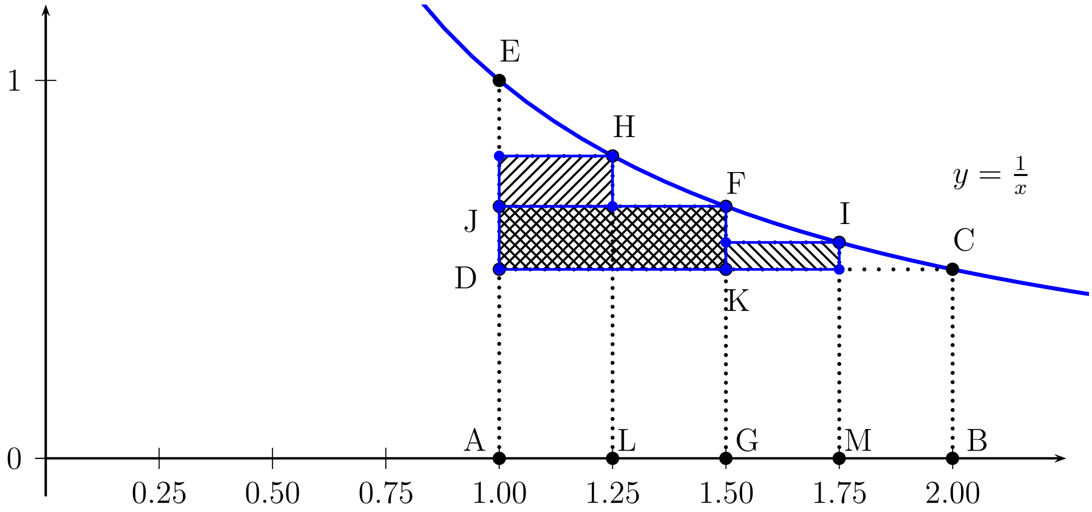
   (f) Find all local maximum and minimum points. **[2 points]**

   (g) Find $f''(x)$ and determine the regions where $f(x)$ is convex/concave. **[2 points]**

   (h) Sketch the graph of $y = f(x)$. **[3 points]**

3. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. **[10 points]**

4. In 1655 (published in 1668), William Brouncker obtained the area under the hyperbola $y = \frac{1}{x}$, bounded below by the x-axis, from $x = 1$ to $x = 2$, as an alternating series.



Refer to the figure, G is the midpoint of AB, L is the midpoint of AG, and M is the midpoint of GB. Lines AE, LH, GF, MI, and BC are vertical. Point K is the perpendicular intersection of lines DC and GF.

(a) Show that the area of rectangle ABCD is $\frac{1}{2} = \frac{1}{1\times 2}$. **[1 points]**

(b) Show that the area of rectangle DKFJ is $\frac{1}{12} = \frac{1}{3\times 4}$. **[1 points]**

(c) Show that the area of the rectangle with opposite corners JH is $\frac{1}{30} = \frac{1}{5\times 6}$. **[1 points]**

(d) Show that the area of the rectangle with opposite corners KI is $\frac{1}{56} = \frac{1}{7\times 8}$. **[1 points]**

(e) Using partial fraction decomposition $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$, we can rewrite the terms obtained. Brouncker conjectured that the desired area is given by

$$\begin{aligned}\frac{1}{1\times 2} + \frac{1}{3\times 4} + \frac{1}{5\times 6} + \frac{1}{7\times 8} + \dots &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \dots \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.\end{aligned}$$

This series is commonly known as the **alternating harmonic series**. Determine if the series is convergent. Give reasons. **[6 points]**

(f) Find the desired area (keep your answer in terms of natural logarithm). **[5 points]**

5. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. **[10 points]**

6. The 8th chapter of **The Nine Chapters of the Mathematical Art** (《九章算術》卷八) is dedicated to Rectangular Arrays (方程). The 3rd problem (第三問) in the chapter is as follows:

今有上禾二秉、中禾三秉、下禾四秉，實皆不滿斗;
上取中、中取下、下取上、各一秉，而實滿斗。
問上、中、下禾實一秉各幾何?

- 秉 [in modern Chinese: 束, 捆紮]= sheaf (for singluar), sheaves (for plural);
- 上禾= top grade rice;
- 中禾= middle grade rice;
- 下禾= low grade rice;
- 斗=*dou*, ancient unit of volume (10 *dou* = 10.354688 Liters).

English translation:

There are 2 sheaves of top grade rice, 3 sheaves of middle grade rice, and 4 sheaves of low grade rice, and the volume of each is less than 1 *dou*.
Adding 1 sheaf of middle grade rice to the top grade rice, 1 sheaf of low grade rice to the middle grade rice, 1 sheaf of top grade rice to the low grade rice, makes 1 *dou* each.
What is the volume (in unit of *dou*) of 1 sheaf of rice of each grade?

Let x be the volume of 1 sheaf of top grade rice, y be the volume of 1 sheaf of middle grade rice, and z be the volume of 1 sheaf of low grade rice. At the beginning, the problem states that $2x < 1$, $3y < 1$, and $4z < 1$.

(a) Briefly explain why you can obtain the following system of 3 linear equations with 3 unknowns, **[3 points]**

$$\begin{aligned} 2x + y &= 1, \\ 3y + z &= 1, \\ x + 4z &= 1. \end{aligned}$$

(b) Solve the system using Gauss-Jordan elimination. You must keep your solutions in exact rational numbers (marks will not be given to solutions expressed in terms of truncated decimal numbers). **[9 points]**

(c) Use your answer obtained in **(b)** and verify that indeed $2x < 1$, $3y < 1$, and $4z < 1$. **[3 points]**

7. Consider the argument: That there is no such point on the parabola $4y = x^2$ closest to $(0, 1)$. The reason is that, the squared distance from $(0, 1)$ to any point (x, y) on the parabola is $D^2 = (x - 0)^2 + (y - 1)^2$. Since $4y = x^2$, thus, $D^2 = 4y + (y - 1)^2$. Since D^2 is now a function of y, let $D^2 = f(y)$. By differentiating it with respect to y, $\frac{df}{dy} = 2y + 2$, there is only one critical point, which is at $y = -1$; however, there is no x such that $(x, -1)$ is on the parabola. Hence, there is no shortest distance! What is wrong with the argument? If there is indeed a closest point on the parabola to $(0, 1)$, find it. [**5 points**]

8. Consider the homogeneous system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ where $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{bmatrix}$.

   Note that this is a system of 3 equations with 4 unknowns. Find the solution space, and describe its nature. [**15 points**]

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