THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code: AMA1007 **Subject Title:** Calculus and Linear Algebra

Session: Semester 2, 2014/2015

Date: May 06, 2015 **Time:** 12:30 - 14:30

Time Allowed: 2 hours

This question paper has 4 pages (including this page)

Instructions: This paper has **7** questions.

Attempt **ALL** questions in this paper.

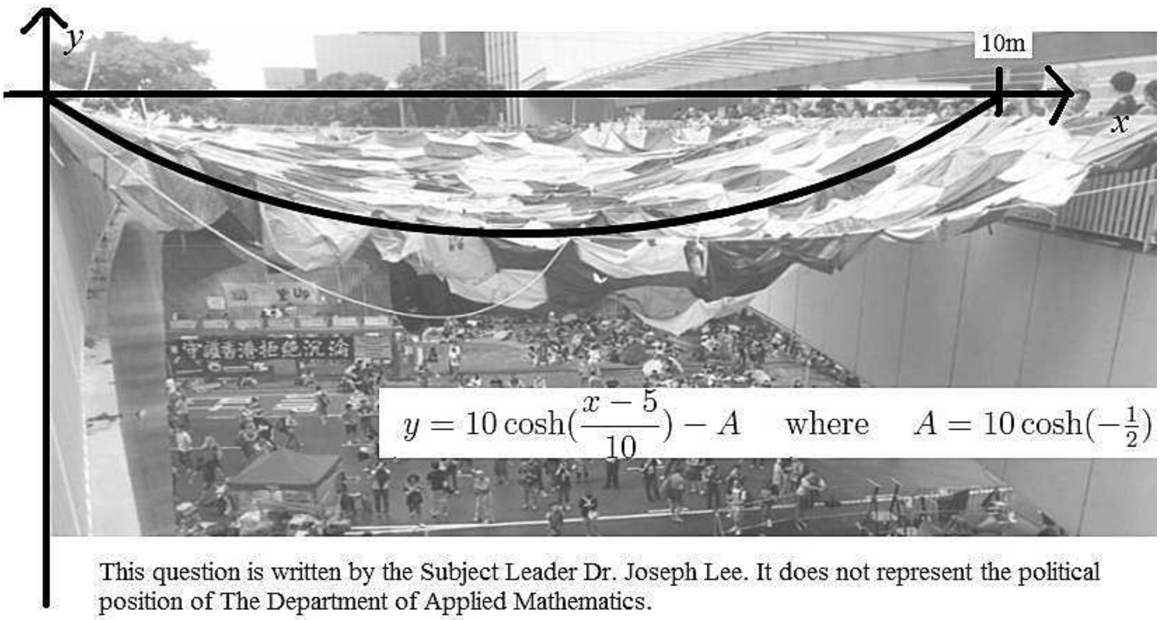
Subject Examiners: Dr. LEE Heung Wing Joseph

DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO.

1. On Friday 3 October 2014, a huge canopy named Umbrella Patchwork (百家傘) was installed between the footbridge gap over Harcourt Road, Admiralty. It was an installation art (裝置藝術) created by a group of students from The Academy of Visual Arts at The Hong Kong Baptist University. The canopy was made of tear gas stained umbrella fabric, stripped out from over 250 damaged umbrellas [which were collected after the police fired 87 canisters of tear gas into the crowd of protesters on Sunday 28 September 2014 in Harcourt Road, attempting to repress the unprecedented large scale civil disobedience movement (The Umbrella Movement 雨傘運動)]. Since the umbrella fabric was light, the supporting ropes of the canopy hung across the footbridge have taken the shape of a Catenary. For simplicity in this question, we assume the shape of a particular supporting rope is given by

$$y \quad = \quad 10\cosh(\frac{x-5}{10}) - A, \qquad 0 \le x \le 10,$$

where $A = 10\cosh(-\frac{1}{2})$. Note that $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$, and $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$.



(a) Show that $\frac{d}{dx}\cosh(x) = \sinh(x)$, and $\frac{d}{dx}\sinh(x) = \cosh(x)$. **[3 points]**

(b) Show that $\cosh^2(v) = 1 + \sinh^2(v)$. **[4 points]**

(c) Show that $\sqrt{1 + (y')^2} = \cosh\left(\frac{x-5}{10}\right)$. **[4 points]**

(d) Find the arc-length of the supporting rope, i.e. $\int_0^{10} \sqrt{1+(y')^2}dx$. **[4 points]**

2. Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -t & 0 \end{bmatrix}$ where t is real.

   (a) State the condition for t that $\mathbf{A}$ will have two distinct real eigenvalues. **[5 points]**

   (b) Express the eigenvalues in terms of t. **[5 points]**

   (c) Find the rate of change of each eigenvalue with respect to t while the condition obtained in part **(a)** is satisfied. **[5 points]**

3. Consider the fact that 25875, 46552, 41354, 48691, and 95818 are all divisible by 23.

   Use this fact to determine if $\begin{vmatrix} 2 & 5 & 8 & 7 & 5 \\ 4 & 6 & 5 & 5 & 2 \\ 4 & 1 & 3 & 5 & 4 \\ 4 & 8 & 6 & 9 & 1 \\ 9 & 5 & 8 & 1 & 8 \end{vmatrix}$ is divisible by 23 or not, without directly evaluating the determinant. State your reasons.

   [Hint: $25875 = 2 \times 10000 + 5 \times 1000 + 8 \times 100 + 7 \times 10 + 5$.] **[15 points]**

4. In integral calculus, a technique called the **tangent half-angle substitution** (also known as the Weierstrass' half-angle substitution) is used to convert any rational function of $\sin(\theta)$ and $\cos(\theta)$ into an ordinary rational function of t, i.e. let $t = \tan(\theta/2)$ for $-\pi < \theta < \pi$. Today, the technique is named after the German mathematician Karl Weierstrass (1815-1897), although, it can be found as early as in the work of the Swiss mathematician Leonhard Euler (1707-1783).

   (a) Under the substitution, sketch a right-angled triangle and show that
   $\cos\left(\dfrac{\theta}{2}\right) = \dfrac{1}{\sqrt{1+t^2}}$ and $\sin\left(\dfrac{\theta}{2}\right) = \dfrac{t}{\sqrt{1+t^2}}$. **[4 points]**

   (b) Show that $\cos(\theta) = \dfrac{1-t^2}{1+t^2}$ and $\sin(\theta) = \dfrac{2t}{1+t^2}$.
   [Hint: $\cos(2A) = 2\cos^2(A) - 1$, and $\sin(2A) = 2\sin(A)\cos(A)$.] **[4 points]**

   (c) Show that $d\theta = \dfrac{2}{1+t^2}\,dt$. [Hint: $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$.] **[2 points]**

   (d) Using this substitution technique, evaluate $\displaystyle\int \frac{1}{3\sin(\theta) - 4\cos(\theta)}\,d\theta$. **[5 points]**

5. Consider the graph of $y = \dfrac{1-x}{1+x^2}$. Its 2nd derivative is given by $\dfrac{d^2y}{dx^2} = \dfrac{-2x^3+6x^2+6x-2}{(1+x^2)^3}$.
   One of the points of inflection for the graph of y is $(-1, 1)$.

   (a) Find $\dfrac{dy}{dx}$. **[4 points]**

   (b) Find all the other points of inflection. **[7 points]**

   (c) Determine if all the points of inflection lie on a straight line.
   If yes, find the equation of that line. **[4 points]**

6. Consider the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}$. Find the interval of convergence (you need to specify the interval and determine the convergence at the end-points). **[10 points]**

7. The 8th chapter of **The Nine Chapters of the Mathematical Art** (《九章算術》卷八) is dedicated to Rectangular Arrays (方程). The 16th problem (第十六問) in the chapter is as follows:

   今有令一人、吏五人、從者一十人，食雞一十；
   令一十人、吏一人、從者五人，食雞八；
   令五人、吏一十人、從者一人，食雞六。
   問令、吏、從者食雞各幾何？

   English translation:

   Suppose 1 prefect, 5 officers, 10 followers together they consume 10 poultry;
   10 prefects, 1 officer, 5 followers together they consume 8 poultry;
   5 prefects, 10 officers, 1 follower together they consume 6 poultry.
   What are the level of consumptions of poultry for each prefect, officer, and follower?

   The problem has made several assumptions:

   - all prefects consume the same amount of poultry,
   - all officers consume the same amount of poultry,
   - all followers consume the same amount of poultry.
   - consumption of poultry for any individual may or may not be an integer (not necessary an integer multiple of a whole chicken).

   Formulate the problem into a system of linear equations, and solve the system using Gaussian elimination. You must keep your solutions in exact rational numbers (marks will not be given to solutions expressed in terms of truncated decimal numbers).

   [15 points]

*** END ***