THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code: AMA1007 **Subject Title:** Calculus and Linear Algebra

Session: Semester 1, 2014/2015

Date: Dec 08, 2014 **Time:** 12:30 - 14:30

Time Allowed: 2 hours

This question paper has 5 pages (including this page)

Instructions: This paper has **6** questions.

Attempt **ALL** questions in this paper.

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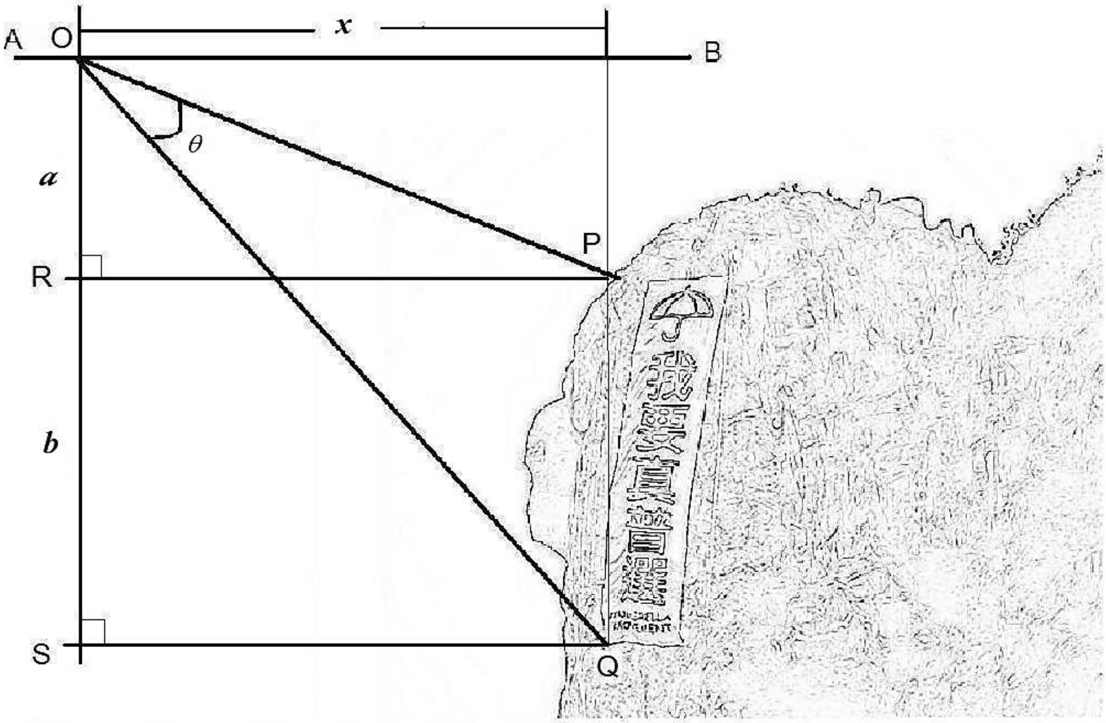
DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO.

1. On Thursday 23 October 2014, a huge banner was hung down from the highest point of Lion Rock（獅子山）by a group of rock climbers called *The Hong Kong Spidie*（香港蜘蛛仔）to show support for the democracy movement (the umbrella movement 雨傘運動). The size of the banner was said to be $6m$ by $28m$. The banner was sharp yellow in colour, and printed therein was a logo of the shape of an umbrella first, followed by 5 big Chinese characters 我要真普選 . At the bottom of the banner, a relatively smaller prints of the words *#umbrella movement* were shown.
For simplicity in this question, we assume the banner was hung perfectly vertical, and the vertical hang-down-length of the banner is $b = 30m$ instead of 28m. Assuming a flying camera is flying horizontally from A to B, passing through the point directly above the top point of banner P, and the vertical distance of this horizontal path is $a = 10m$ above P. Let x be the distance measured from the current position of the flying camera O to the point directly above P.



This question is written by the Subject Leader Dr. Joseph Lee. It does not represent the political position of The Department of Applied Mathematics.

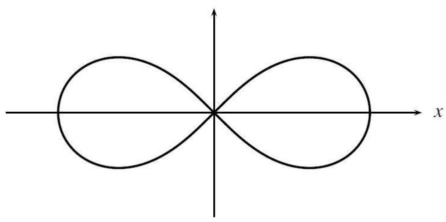
(a) Show that the view angle θ, i.e. $\angle POQ$, depends on x as

$$\theta(x) \;=\; \tan^{-1}\left(\frac{x}{a}\right) - \tan^{-1}\left(\frac{x}{a+b}\right) = \tan^{-1}\left(\frac{x}{10}\right) - \tan^{-1}\left(\frac{x}{40}\right).$$

[5 points]

(b) Find the best viewing location x along the flying path that maximize θ.
Hint: $\dfrac{d}{dy}\left[\ \tan^{-1}(y)\ \right] = \dfrac{1}{1+y^2}$. **[10 points]**

2. Consider the curve given in Cartesian coordinates $(x^2 + y^2)^2 = x^2 - y^2$, or in Polar coordinates $r^2 = \cos(2\theta)$. This curve is known as **The lemniscate of Bernoulli**. The curve cut the x-axis at the origin and $x = \pm 1$. Exactly at the crossing at the origin, the slope of the tangent of the curve is $\pm\frac{\pi}{4}$.



Background information for those interested in the historical development (you may skip this paragraph and go straight to the question if you are not interested in the history): In 1694, Jakob Bernoulli considered this curve, and called it *lemniscus* (Latin for a *pendant ribbon*), and showed that its total arc length is given by $4\int_0^1 \frac{dr}{\sqrt{1-r^4}}$, which the integral is known as the **lemniscatic integral**. This is the simpliest of a kind of integral called **elliptic integral**. Jakob Bernoulli was not aware that the lemniscate curve was actually a special case of the Cassini ovals, which had been described by Cassini earlier in 1680. Subsequent to Jakob Bernoulli, many other mathematicians have made contributions to the topic, including John Bernoulli, Fagnano, Euler, Legendre, Abel, Jacobi, Liouville, and Weierstrass. In particular, in 1833, Liouville proved that elliptic integrals cannot be evaluated in terms of the elementary functions (closed form).

Question:

(a) Let $x(\theta) = r(\theta)\cos(\theta)$, and $y(\theta) = r(\theta)\sin(\theta)$. Show that $\frac{dx}{d\theta} = r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)$, and $\frac{dy}{d\theta} = r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)$. **[2 points]**

(b) Show that $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$. **[3 points]**

(c) By taking $ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$, show that one quarter of the arc length of the lemniscate of Bernoulli is given by $\int_0^{\pi/4} \frac{d\theta}{r}$. **[5 points]**

(d) Explain why $\frac{d\theta}{r} = -\frac{dr}{\sin(2\theta)}$, and $\frac{1}{\sin(2\theta)} = \frac{1}{\sqrt{1-r^4}}$. Note that when $\theta = 0$, $r = 1$; and when $\theta = \frac{\pi}{4}$, $r = 0$. Show that $\int_0^{\pi/4} \frac{d\theta}{r} = \int_0^1 \frac{dr}{\sqrt{1-r^4}}$. **[5 points]**

(e) Use a proper change of variables, show that the integral can be written as $\frac{1}{4}\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du$ for some value of α and β. [Note that this integral is known as the Beta function $B(\alpha, \beta)$, or the Euler integral. Beta function was studied by Wallis (1655), Euler (1730), Stirling (1730) and Legendre (1826). The name Beta was given by Jacques P.M. Binet (1839).] **[5 points]**

3. The 8th chapter of the ancient Chinese mathematics treatise of **The Nine Chapters of the Mathematical Art** (《九章算術》卷八) is dedicated to Rectangular Arrays (方程). The 17th problem (第十七問) in the chapter is the following problem:

> *Now, we have*
> *5 sheep, 4 dogs, 3 chickens and 2 rabbits together worth \$1496;*
> *4 sheep, 2 dogs, 6 chickens and 3 rabbits together worth \$1175;*
> *3 sheep, 1 dog, 7 chickens and 5 rabbits together worth \$958;*
> *2 sheep, 3 dogs, 5 chickens and 1 rabbit together worth \$861.*
> *What is the price of a sheep, a dog, a chicken and a rabbit each?*
> 今有
> 五羊、四犬、三雞、二兔，直錢一千四百九十六；
> 四羊、二犬、六雞、三兔直錢一千一百七十五；
> 三羊、一犬、七雞、五兔，直錢九百五十八；
> 二羊、三犬、五雞、一兔，直錢八百六十一。
> 問羊、犬、雞、兔價各幾何？

Formulate the problem into a set of linear equations, and solve the system using Gaussian elimination to **find the price of rabbit only** (there is no need to find the price of sheep, dog and chicken for this question). **[15 points]**

4. Consider the graph of $y = f(x) = x^2$ for $1 \le x \le 2$.

   (a) Sketch f on the first quadrant of the x-y plane. **[2 points]**

   (b) Let A be the region bounded by $y = f(x)$, the x-axis, $x = 1$ and $x = 2$. Indicate A on the sketch, and find its area by integration. **[5 points]**

   (c) Let B be the region bounded by $x = f^{-1}(y)$, the y-axis, $y = 1$ and $y = 4$. Indicate B on the sketch. **[3 points]**

   (d) Deduce from the sketch the sum of the area of A and the area of B. **[2 points]**

   (e) Without performing direct integration again, find the area of B. Note that this is the geometric interpretation of integration by parts: Suppose when $x = a$, $y = \alpha$; and when $x = b$, $y = \beta$. Then

   $$\int_a^b y dx = \Big[\, xy \,\Big]_{x=a,y=\alpha}^{x=b,y=\beta} - \int_\alpha^\beta x dy = [b\beta - a\alpha] - \int_\alpha^\beta x dy \ .$$ **[3 points]**

5. Given a quadratic form $f(x,y) = ax^2 + 2bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Suppose λ_1 and λ_2 are the eigenvalues of $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ with the order $\lambda_1 \leq \lambda_2$, and if x, y are constrained on the unit circle $x^2 + y^2 = 1$, then, the maximum value of f is given by λ_2 and the minimum value of f is given by λ_1.
Find the maximum value and minimum value of $f(x,y) = x^2 + 4xy + y^2$ subject to the constraint $x^2 + y^2 = 1$. [Note that you are not required to find the locations (i.e. x and y) on where the extreme values of f occur.] **[15 points]**

6. Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$.

   (a) Find the interval of convergence without considering the end points (α, β) of this power series. **[7 points]**

   (b) Consider the end point α. Determine if the series converges at $x = \alpha$. **[7 points]**

   (c) Consider the end point β. Determine if the series converges at $x = \beta$. [Note that when $x = \beta$, this is called the p-Series, or the Hyperharmonic Series.] **[6 points]**

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