THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code: AMA1007 **Subject Title:** Calculus and Linear Algebra

Session: Semester 2, 2013/2014

Date: May 07, 2014 **Time:** 12:30 - 14:30

Time Allowed: 2 hours

This question paper has 4 pages (including this page)

Instructions: This paper has **8** questions.

Attempt **ALL** questions in this paper.

Subject Examiners: Dr. LEE Heung Wing Joseph

DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO.

1. Find $\int \frac{x^3-4}{x^2-x-2}dx$. **[10 points]**

2. Consider part of the graph of $y = \frac{1}{x}$ for $x \geq 1$. By rotating the graph about the x-axis, the generated unbounded solid region is called the **Gabriel's Horn** (Archangel Gabriel who blows the horn to announce Judgment Day), or **Torricelli's trumpet** (named after an Italian physicist and mathematician Evangelista Torricelli 1608-1647).

   (a) Find the volume of the Gabriel's Horn. **[5 points]**

   (b) The surface area of an object generated by rotating part of the non-negative graph of $y = f(x)$ along the x-axis for $a \leq x \leq b$ is given by

$$A = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}dx.$$

   Show that the Gabriel's Horn has an infinite surface area, although it has a finite volume (obtained in **(a)** above). (This is known as the Painter's Paradox, that a finite volume of paint can 'filled-up' the Horn, but yet, never enough to paint the infinite inner surface). **[5 points]**

3. Consider the matrix $\mathbf{A} = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

   (a) Find the characteristic polynomial $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$, and show that it has one distinct real root $\lambda = 10$, and a double root $\lambda = 1$. **[5 points]**

   (b) Find a line such that any point $\boldsymbol{x}$ on the line would be mapped back onto the same line by the linear mapping $\mathbf{A}^k\boldsymbol{x}$, where k is a positive integer. **[5 points]**

   (c) Find a plane such that any point $\boldsymbol{x}$ on the plane would be mapped back onto the same plane by the linear mapping $\mathbf{A}^k\boldsymbol{x}$, where k is a positive integer. **[5 points]**

4. The **hyperbolic sine and cosine functions** are defined as $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$ respectively. In particular, the $\cosh(x)$ function is also known as the **Catenary function**, and it takes the shape of a hanging chain. It is also rumoured in Cambridge that the design for the chapel of King's College was based on an 'upside-down' hanging chain, eventhough the chapel was built 200 years before Calculus appeared.

   (a) Obtain the derivative and anti-derivative of $\cosh(x)$ in terms of $\sinh(x)$. **[4 points]**

(b) Obtain the Maclaurin Series of $\cosh(x)$. **[6 points]**

(c) Find the arc-length of the graph of $y = \cosh(x)$ from $x = 0$ to $x = 1$. **[10 points]**

5. Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{n2^n}$ **[10 points]**

6. (a) Suppose $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent. Is $\displaystyle\sum_{n=1}^{\infty} a_n^2$ convergent? If yes, give an explaination. If no, give a counter-example. **[5 points]**

   (b) Suppose $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Is $\displaystyle\sum_{n=1}^{\infty} a_n^2$ convergent? If yes, give an explaination. If no, give a counter-example. **[5 points]**

7. **Background information for those interested in the historical development (you may skip this first paragraph and go straight to the question in the second paragraph if you are not interested in the history):** The Cavalieri's quadrature formula gives an explicit solution to the integral $\displaystyle\int_0^b x^n dx$. For the case of the parabola ($n = 2$), it was proven by the ancient Greek mathematician Archimedes of Syracuse in the 3rd century BC in *The Quadrature of the Parabola* (written as a letter to his friend Dositheus). More precisely, Archimedes was using the method of exhaustion to find the area inside the parabola rather then the area under the graph. In the 11th century, Islamic mathematician Ibn al-Haytham computed the integrals of cubes ($n = 3$) and quartics ($n = 4$) using the method of mathematicial induction. In the 17th century, an Italian Jesuit mathematician Bonaventura Cavalieri used the method of indivisibles and computed the cases for integers n from 1 up to 9. The French lawyer/mathematician Pierre de Fermat had also obtained the areas of parabolas and hyperbolas of any order. By using an ingenious trick, Fermat was able to reduce the evaluation of the integral of general power to the sum of geometric series. The trick, however, was not considered systematic nor explicit. In 1656, an English mathematician John Wallis published his work *Arithmetica Infinitorum* in which the formula works for fractional and negative powers, although the case of the quadrature of the hyperbola ($n = -1$) was then interpreted incorrectly.

   Question: Compute the integral $\displaystyle\int_0^{\sqrt[n]{a}} x^n\, dx$ in terms of a, assuming n is a positive integer. Without performing further integration, find $\displaystyle\int_0^a \sqrt[n]{x}\, dx$ in terms of a. **[10 points]**

8. The 8th chapter of the ancient Chinese mathematics treatise of **The Nine Chapters of the Mathematical Art** (《九章算術》卷第八) is dedicated to Rectangular Arrays (方程). The 8th problem in the chapter is the following:

   Now sell 2 cows, 5 sheep, and buy 13 pigs, then there left 1000 coins; sell 3 cows, 3 pigs, and buy 9 sheep, then no coin is left nor deficient; sell 6 sheep, 8 pigs, and buy 5 cows, then there is 600 coins shortage. What is the price of a cow, a sheep, and a pig each?

   今有賣牛二、羊五，以買十三豕，有餘錢一千。賣牛三、豕三，以買九羊，錢適足。賣羊六、豕八，以買五牛，錢不足六百。問牛、羊、豕價各幾何？

   Assuming that there are no money (\$0) in hand before any buy and sell tradings. Formulate the problem into a set of linear equations, and solve the system using Gaussian elimination.

   [15 points]

*** END ***