THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code: AMA1007 **Subject Title:** Calculus and Linear Algebra

Session: Semester 1, 2013/2014

Date: December 12, 2013 **Time:** 12:30 - 14:30

Time Allowed: 2 hours

This question paper has 4 pages (including this page)

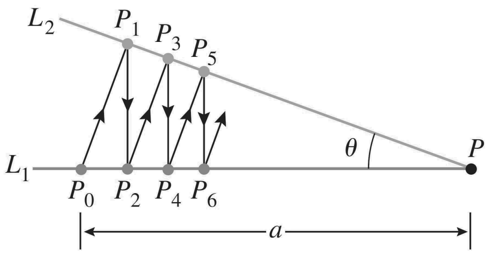
Instructions: This paper has **10** questions.

Attempt **ALL** questions in this paper.

Subject Examiners: Dr. LEE Heung Wing Joseph and Dr. LEE Yu Chung Eugene

DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO.

1. As shown in the accompanying figure below, suppose that lines L_1 and L_2 form an angle θ, $0 < \theta < \pi/2$, at their point of intersection P. A point P_0 is chosen that is on L_1 and a units from P. Starting from P_0 a zigzag path is constructed by successively going back and forth between L_1 and L_2, from one line onto the other perpendicularly.



Find the following three sums in terms of θ and a: **[10 points]**

$$P_0P_1 + P_1P_2 + P_2P_3 + \ldots$$

$$P_0P_1 + P_2P_3 + P_4P_5 + \ldots$$

$$P_1P_2 + P_3P_4 + P_5P_6 + \ldots$$

2. Consider each of the following improper integrals and determine their convergence (converge or diverge). Give reasons to your answers.

   (a) $\displaystyle\int_3^\infty \frac{dx}{\sqrt[5]{x^2 - x - 2}}$. **[5 points]**

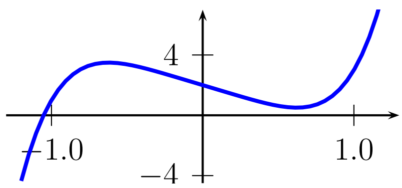
   (b) $\displaystyle\int_2^\infty \frac{5 + \sin x}{x - 1} dx$. **[5 points]**

3. (a) Find the Maclaurin Series (i.e., Taylor Series with $a = 0$) for e^{-x^2} using the Maclaurin Series for e^x. [Hint: replacing x with $-x^2$ in the series for e^x]. **[5 points]**

   (b) Evaluate $\displaystyle\int e^{-x^2} dx$ as an infinite series. [Hint: integrate the Maclaurin Series for e^{-x^2} obtained in **(a)** term by term]. **[5 points]**

   (c) Give an approximation for $\displaystyle\int_0^1 e^{-x^2} dx$ numerically using the first five terms obtained in **(b)**. **[5 points]**

4. Consider the polynomial $f(x) = 4x^5 - 3x + 2$. Note that $f(-1) = 1$, and $f(1) = 3$. Determine if there exist a number c in $(-1, 1)$ such that $f'(c) = 1$. Give reasons to your answer, and find all possible values of c if there exist such numbers. **[7 points]**



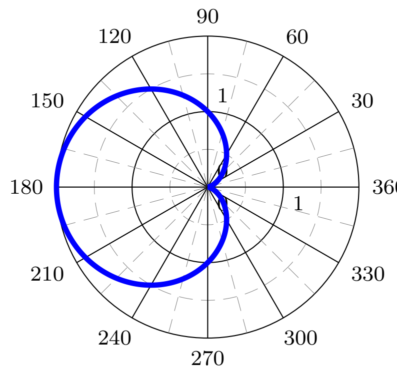
5. Suppose $x = r(\theta)\cos(\theta)$ and $y = r(\theta)\sin(\theta)$, where $r(\theta)$ is sufficiently smooth, and suppose $x = a$ when $\theta = \alpha$, and $x = b$ when $\theta = \beta$.

   (a) Show that $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (r'(\theta))^2 + (r(\theta))^2$. **[2 points]**

   (b) Show that the arc length from $\theta = \alpha$ to $\theta = \beta$ is given by **[2 points]**

$$L = \int_{\alpha}^{\beta} \sqrt{(r'(\theta))^2 + (r(\theta))^2}d\theta.$$

   (c) Consider the Cardioid $r(\theta) = 1 - \cos(\theta)$, $0 \leq \theta \leq 2\pi$. Use the above result to find its length. [Hint: $2 - 2\cos(\theta) = 4\sin^2\left(\frac{x}{2}\right)$]. **[4 points]**



6. One of the most important treatise in the history of ancient Chinese mathematics is **The Nine Chapters of the Mathematical Art** (九章算術). The first problem in the eighth chapter is the following:

   > *There are three classes of corn, of which three bundles of the first class, two of the second, and one of the third make 39 measures. Two of the first, three of the second, and one of the third make 34 measures. And one of the first, two of the second, and three of the third make 26 measures. How many measures of grain are contained in one bundle of each class?*

   Formulate the problem into a set of three linear equations with three unknowns, and solve the system using Gaussian elimination. **[10 points]**

7. Consider the upper triangular matrix $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. Find its inverse. **[10 points]**

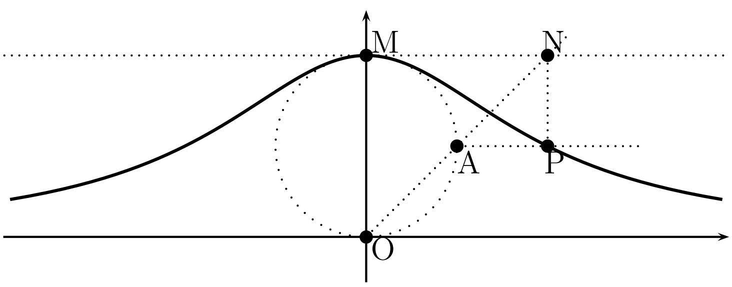
8. Check if the four points $A(1, 3, 2)$, $B(3, -1, 6)$, $C(5, 2, 0)$, and $D(3, 6, -4)$ lie on the same plane. **[5 points]**

9. Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find all the eigenvalues and the associated eigenspaces of $\mathbf{A}$.

[15 points]

10. **Background information for those interested in the historical development (you may skip this first paragraph and go straight to the question in the second paragraph if you are not interested in the history):** Consider a curve defined as follows: Draw a fixed circle of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$. Label the top point of the circle $(0, 1)$ as M, and the bottom point $(0, 0)$ as O. Draw the top horizontal tangent line to the circle at M. For any other point A on the circle, a secant line OA is drawn. The line OA intersects with the top horizontal tangent line at the point N. The verticle line through N and the horizontal line through A intersect at P. As the point A is varied along the circle, the path of P traces out a curve, for which its Cartesian equation is given by $y = \dfrac{1}{x^2+1}$. The is the famous **Witch of Agnesi**, named after a brilliant Italian mathematician Maria Gaetana Agnesi (1718-1799) who published her work *Instituzioni Analitiche* in 1748, which includes the study of this curve *versiera* (Latin *vertere*). *Instituzioni Analitiche* is the first surviving mathematicial work written by a woman, and it is over 1000 pages. It is believed that there was a mistranslation from the original text into English, that the word *versiera* was mistaken as *adversarius* (the wife of the devil), and thus, resulted in the association of the name *Witch*. Before Agnesi, the curve was first appeared in the works of Fermat in 1630. Later in 1824, the curve $y = \dfrac{1}{\pi(x^2+1)}$ (multiplied by a factor of $\frac{1}{\pi}$), was re-visited by Poisson as a probability density function, aiming to provide a counter-example to the earlier version of The Law of Large Numbers by Laplace in 1810. This is now commonly known as the Cauchy distribution, for which the mean, the variance, and any higher order moments do not exist.

Question: Find the unbounded area between the **Witch of Agnesi** $y = \dfrac{1}{x^2+1}$ and the x-axis. **[10 points]**.



*** END ***