THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Subject Code: AMA1007 **Subject Title:** Calculus and Linear Algebra

Session: Semester 2, 2012/2013

Date: May 15, 2013 **Time:** 12:30 - 14:30

Time Allowed: 2 hours

This question paper has 4 pages (including this page)

Instructions: This paper has **12** questions.

Attempt **ALL** questions in this paper.

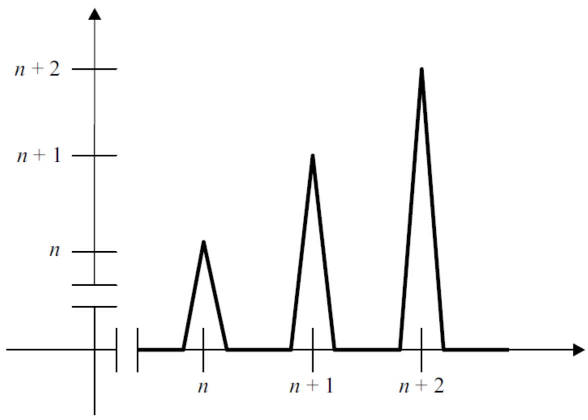
Subject Examiners: Dr. LEE Heung Wing Joseph

DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO.

1. (a) Let $f_1(x) = 2x + |x|$ and $f_2(x) = \frac{2}{3}x - \frac{1}{3}|x|$. Find $F(x) = f_1(f_2(x))$. **[3 points]**

   (b) Suppose $g_1(x)$ and $g_2(x)$ are continuous functions. If $g_2(x)$ is **not** differentiable at $x = a$ and $g_1(x)$ is **not** differentiable at $x = g_2(a)$. Determine if the function $G(x) = g_1(g_2(x))$ is also **not** differentiable at $x = a$. Provide a proof if your answer is 'yes', or a counter-example if your answer is 'no'. **[2 points]**

2. (a) Consider a continuous function $f(x)$ as depicted in the figure below. On every natural number $n \geq 2$, we construct a triangle of height n and base $\frac{2}{n^3}$. Except for the triangles, $f(x)$ is zero everywhere else.



   Determine if $\displaystyle\int_a^\infty f(x)\,dx$ is finite. Provide reasons for your answer. **[5 points]**

   (b) Suppose $g(x)$ is a continuous and non-negative function, and suppose the improper integral $\displaystyle\int_c^\infty g(x)\,dx$ converges. Determine if $\displaystyle\lim_{x\to\infty} g(x) = 0$. Provide a proof if your answer is 'yes', or a counter-example if your answer is 'no'. **[5 points]**

3. (a) Prove that, if f is continuous, then $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$. **[5 points]**

   (b) Show that,

$$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}dx \quad = \quad \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}dx$$

   for all positive integers n. **[5 points]**

   (c) Evaluate $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}dx$ for all positive integers n. **[5 points]**

4. Consider the function $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2 + \cos x}$ defined on $[0, 2\pi]$.
   Determine where $f(x)$ would be increasing/decreasing. **[5 points]**

5. Use Mean Value Theorem to show that $\tan x > x$ for all $x \in (0, \frac{\pi}{2})$. **[5 points]**

6. Find the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(x+5)^n$. State clearly if the series converges at the two end-points of the interval in your answer. **[5 points]**

7. Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$. **[8 points]**

8. Find the area of the region $S = \{\, (x, y) \mid x \geq 0,\ y \leq 1,\ x^2 + y^2 \leq 4y\}$. **[7 points]**

9. A town has three main industries: a coal-mining operation, an electric power-generating plant, and a local railroad. To mine \$1 of coal, the mining operation must purchase \$0.25 of electricity to run its equipment and \$0.25 of transportation for its shipping needs. To produce \$1 of electricity, the generating plant requires \$0.65 of coal for fuel, \$0.05 of its own electricity to run auxiliary equipment (this \$0.05 of electricity is part of the \$1 of electricity produced), and \$0.05 of transportation. To provide \$1 of transportation, the railroad requires \$0.55 of coal for fuel and \$0.10 of electricity for its auxiliary equipment. In a certain week the coal-mining operation receives orders for \$50,000 of coal from outside the town, and the generating plant receives orders for \$25,000 of electricity from outside. There is no outside demand for the local railroad. How much must each of the three industries produce in that week to exactly satisfy their own demand and the outside demand? **[10 points]**

10. Consider the homogeneous system

$$\begin{aligned} x + y + \alpha z &= 0 \\ x + y + \beta z &= 0 \\ \alpha x + \beta y + z &= 0. \end{aligned}$$

What is the condition for the system to have non-trivial solutions? **[10 points]**

11. Consider the linear map $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$.

   (a) Find all the straight lines that would be mapped onto itself (i.e. points on the line would only be mapped onto points of the same line). **[8 points]**

(b) Find all the straight lines that would be mapped onto itself if we apply the map 4 times,

i.e. $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$. **[2 points]**

12. Consider the **Triangular Pascal Matrix** defined by $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$.

Find its inverse. **[10 points]**

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