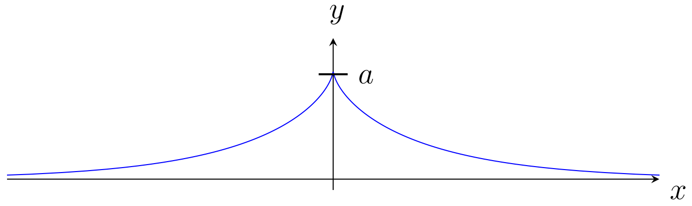
4. Consider the Tractrix curve (曳物線) given by

$$x \;=\; \pm\left\{a\ln\left(\frac{a+\sqrt{a^2-y^2}}{y}\right)-\sqrt{a^2-y^2}\right\}$$

where $a > 0$. This curve was first introduced by Claude Perrault in 1670, and later studied by Isaac Newton in 1676 and Christiaan Huygens in 1692 (also Leibniz and Johann Bernoulli in 1693).



The surface generated by rotating this curve about its asymptote forms a Tractricoid, it is used in a patented contour shape for speakers, and it is also a very popular shape for spinning tops nowadays (thanks to the movie *Inception*).

(a) Find the area bounded above by this curve and bounded below by the x-axix in terms of a and π only. **[13 points]**

Hint 1: $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)+C$

Hint 2: $\displaystyle\int \ln\left(\frac{a+\sqrt{a^2-u^2}}{u}\right)du = u\ln\left(\frac{a+\sqrt{a^2-u^2}}{u}\right)+a\sin^{-1}\left(\frac{u}{a}\right)+C$

(b) Now, consider only the right half plane, $x \geq 0$. Suppose $a \geq y \geq b > 0$. Find the arc-length from $y = a$ to $y = b$ in terms of a and b only. **[12 points]**

Hint: Use that fact that $\displaystyle\frac{dx}{dy} = -\frac{\sqrt{a^2-y^2}}{y}$.

5. Consider the following three series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}, \quad \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{n^2+n+1}\right), \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{\ln(n)}{n}.$$

Determine the convergence for each of them, and state your reasons. **[9 points]**

6. Suppose $\displaystyle f(x) = \frac{1+x}{1-x} = (1+x)\sum_{n=0}^{\infty}x^n = 1+2\sum_{n=1}^{\infty}x^n$ with radius of convergence R_1, and $\displaystyle g(x) = \frac{1-x}{1+x} = (1-x)\sum_{n=0}^{\infty}(-x)^n = 1+2\sum_{n=1}^{\infty}(-x)^n$ with radius of convergence R_2.

Let R be the radius of convergence of the power series of $f(x)g(x)$.

Evaluate R_1, R_2 and R. **[6 points]**